Introduction to Mathematical Finance

Solution sheet 6

Solution 6.1 Solutions are included in the R-file.

Solution 6.2

(a) Since the notion of absence of arbitrage is invariant under any change of numéraire (see Exercise 5.3 (c)), we see that

$$\mathbb{P}(X) \neq \varnothing \iff \text{Both markets are arbitrage-free} \iff \mathbb{P}(Y) \neq \varnothing.$$

(b) Denote the collection on the RHS by $\mathcal{C}$. First we observe the following relation bewteen X and Y:

$$Y_k = X_k \frac{S_k^0}{S_k^1} = \frac{X_k}{X_k^1}, \quad \forall k = 0, 1, ..., T.$$

Suppose that $Q \in \mathbb{P}(X)$. Then $(X_k^1/X_0^1)_{k=0}^T$ is clearly a positive martingale under Q with Q-expectation 1. So $D := d\tilde{Q}/dQ = X_T^1/X_0^1$ induces a probability measure. Moreover, under $\tilde{Q}$, we compute for $j \leq k$ by using the Bayes rule that

$$\begin{aligned}
E_{\tilde{Q}}\left[Y_k \,|\, \mathcal{F}_j\right] &= \frac{E_Q\left[Y_k E_Q\left[D \,|\, \mathcal{F}_k\right] \,|\, \mathcal{F}_j\right]}{E_Q\left[D \,|\, \mathcal{F}_j\right]} \\
&= \frac{E_Q\left[Y_k X_k^1/X_0^1 \,|\, \mathcal{F}_j\right]}{X_j^1/X_0^1} \\
&= \frac{E_Q\left[X_k \,|\, \mathcal{F}_j\right]}{X_j^1} \\
&= \frac{X_j}{X_j^1} = Y_j.
\end{aligned}$$

Thus Y is a martingale under $\tilde{Q}$ and $\mathcal{C} \subseteq \mathbb{P}(Y)$. Reversing the roles of X and Y yields

$$\left\{Q : \frac{dQ}{d\tilde{Q}} = \frac{Y_T^0}{Y_0^0} \text{ for some } \tilde{Q} \in \mathbb{P}(Y)\right\} \subset \mathbb{P}(X).$$

Finally, observe $Y_T^0/Y_0^0 = (X_T^1/X_0^1)^{-1}$. Therefore, for any $\tilde{Q} \in \mathbb{P}(Y)$, we have

$$d\tilde{Q} = \frac{X_T^1}{X_0^1} \underbrace{\frac{Y_T^0}{Y_0^0} d\tilde{Q}}_{\in \mathbb{P}(X)}.$$

This shows $\mathbb{P}(Y) \subset \mathcal{C}$ and the equality follows.

(c) Since the function $x \mapsto 1/x$ is strictly convex on $(0, \infty)$, we obtain

$$E_{\tilde{Q}}[Y_T^0] = Y_0^0 = \frac{1}{X_0^1} = \frac{1}{E_Q[X_T^1]} < E_Q\left[\frac{1}{X_T^1}\right] = E_Q[Y_T^0], \quad \forall Q \in \mathbb{P}(X), \tilde{Q} \in \mathbb{P}(Y).$$

So if there is $Q_0 \in \mathbb{P}(X) \cap \mathbb{P}(Y)$, then we obtain from above that $E_{Q_0}[Y_T^0] < E_{Q_0}[Y_T^0]$, which is a contradiction.

Solution 6.3

(a) Suppose the R_k are independent with mean 0. Clearly X is adapted to its own filtration. Also using the independence of R_k and $E[R_k] = 0$, we have

$$E[X_k] = X_0 \prod_{i=1}^{k} E[1 + R_i] = X_0 < \infty.$$

We now check the martingale property:

$$E\left[X_k \,|\, \mathcal{F}_{k-1}\right] = X_{k-1} E[1 + R_k] = X_{k-1}, \quad k = 1, 2, ..., T.$$

Thus X is a martingale.

(b) The process X is clearly adapted to its natural filtration. So we only need to impose conditions on (R_k) so that X is integrable and X satisfies the martingale property.

Claim. X is a martingale if and only if $R_k > -1$ P-a.s., $E\left[R_k \,|\, \mathcal{F}_{k-1}\right] = 0$ and $E[|R_k|] \leq 2$ for $k = 1, \ldots, T$.

Proof of Claim. "$\Longrightarrow$" Suppose that X is a martingale. Then $E\left[\frac{X_k}{X_{k-1}} \middle| \mathcal{F}_{k-1}\right] = 1$ and $\frac{X_k}{X_{k-1}} = 1 + R_k \geq 0$; so $R_k \geq -1$, $R_k^- \leq 1 \in L^1$ and $E[1 + R_k] = 1$ or $E[R_k] = 0$. But then $0 = E[R_k] = E[R_k^+ - R_k^-] = E[R_k^+] - E[R_k^-]$ so that $E[R_k^+] = E[R_k^-] \leq 1$ and $E[|R_k^+|] \leq 2$.

"$\Longleftarrow$" Suppose that the converse is true. Then $E[X_k] \leq X_0^1(1 + 2)^T < \infty$ for $k = 1, ..., T$. So X is integrable. Then we compute

$$E\left[\frac{X_k}{X_{k-1}} \middle| \mathcal{F}_{k-1}\right] = E\left[1 + R_k \,|\, \mathcal{F}_{k-1}\right] = 1,$$

which shows $E\left[X_k \,|\, \mathcal{F}_{k-1}\right] = X_{k-1}$ for $k = 1, ..., T$. Therefore, X is a martingale.

(c) Note that in Exercise 5.2, we constructed such a multiplicative model. There, the returns are independent iff $u = d = 0$. But the discounted price process is a P-martingale as long as $r = 0$ and $u = -d$.

Solution 6.4

(a) Suppose that ψ is not a-admissible. Then there exists a time point k for which the event $A = \{V_k(\psi) < -a\}$ has strictly positive probability, i.e., $P[A] > 0$. Let k_0 be the largest such time point. Construct the self-financing strategy $\psi' = (\psi'^0, \vartheta')$ described by $V_0 = 0$ and

$$\vartheta'_k = \begin{cases} 0 & \text{if } k \neq k_0 + 1, \\ \vartheta_k 1_A & \text{if } k = k_0 + 1. \end{cases}$$

Note that this process is indeed predictable since ϑ is predictable and $A \in \mathcal{F}_{k_0}$, and that ψ' is well defined by Proposition II.1.2. We first compute, using that $\vartheta \triangle X = \triangle G(\vartheta) = \triangle V(\psi)$ that

$$V_k(\psi') = V_0 + G_k(\vartheta') = 0 + \mathbb{1}_{\{k \geq k_0+1\}} \mathbb{1}_A \vartheta_k \triangle X_k = \mathbb{1}_{\{k \geq k_0+1\}} \mathbb{1}_A \triangle V_{k_0+1}(\psi).$$

By definition of A and k_0, this is nonnegative P-a.s., and strictly positive on A, hence with positive probability. So ψ' is 0-admissible and $V_0(\psi') = 0$, $V_T(\psi') \in L^0_+ \setminus \{0\}$ so that ψ' is an arbitrage opportunity.

(b) By assumption, X is a local Q-martingale. Therefore, by Proposition C.4, $G(\vartheta)$ (hence $V(\psi)$) is a local Q-martingale. Furthermore,

$$E_Q[|V_0(\psi)|] < \infty$$

and $E_Q[V_T^-(\psi)] \leq a$, so from Theorem C.5 we conclude that $V(\psi)$ is a (true) Q-martingale.

By the martingale property,

$$V_k(\psi) = E[V_T(\psi)|\mathcal{F}_k] \geq -a \quad Q\text{-a.s.},$$

for all $t = 0, \dots, T$, thus also P-a.s., which is what we wanted to show.

Solution 6.5

(a) At the end of the t-th trading period, the investor's wealth in a given asset is the sum of the ex-dividend price of the asset and the dividend payout. The investor's total wealth at time t is thus $\psi_t \cdot (S_t + \delta_t)$. Under self-financing assumption, the investor uses this money to rebalance his portfolio for the next period so that

$$\psi_t \cdot (S_t + \delta_t) = \psi_{t+1} \cdot S_t$$

as required.

(b) Let

$$M_t = Z_t S_t + \sum_{s=1}^{t} Z_s \delta_s$$

and consider the consumption stream

$$
\begin{aligned}
&c_0 = -\psi_1 \cdot S_0 \\
&c_t = 0 \quad \forall 0 < t < T \quad \text{(no consumption between t=0 and t=T)} \\
&c_T = \psi_T \cdot S_T
\end{aligned}
$$

Define

$$N_t = Z_t \psi_{t+1} \cdot S_t + \sum_{s=0}^{t} Z_s c_s$$

By calculating the increments $N_t - N_{t-1}$ and $M_t - M_{t-1}$, one can easily see that N can be written as a discrete time stochastic integral of ψ with respect to M:

$$N_t = \sum_{s=1}^{t} \psi_s (M_s - M_{s-1})$$

Assuming that M is a martingale implies that N is a local martingale. Take an arbitrage candidate, i.e. a strategy ψ such that $c_0 \geq 0$ and $c_T \geq 0$. We need to show that ψ cannot be an arbitrage, i.e. that we must have $c_0 = c_T = 0$. For such an arbitrage candidate ψ, note that $N_T = c_0 Z_0 + c_T Z_t \geq 0$ (since the process Z is positive). Hence N is a true martingale by the hint. Therefore $E[N_T] = N_0 = 0$. By the pigeonhole principle, $N_T = 0$ almost surely, and hence $c_t = 0 \quad \forall 0 \leq t \leq T$ since Z is supposed to be a positive process. In particular ψ cannot be an arbitrage and hence the market is arbitrage-free.

(c) Let

$$\psi_t = \frac{\tilde{V}_{t-1}}{S_{t-1}}$$

Notice that ψ is clearly predictable. This portfolio consists of holding buying one share of the asset at time $t = 0$ and then reinvesting the dividend payments at all time. In particular no external funding is required and hence we expect this strategy to be self-financing. More precisely, the strategy is self-financing (without consumption) since

$$\psi_t (S_t + \delta_t) = \tilde{V}_t = \psi_{t+1} S_t$$