Introduction to Mathematical Finance

Exercise sheet 7

Exercise 7.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space. Consider an infinite discrete time model with a numéraire S^0 and a risky asset S^1. As usual let $X = S^1/S^0$ denote the discounted price process of the risky asset. Assume that there exists $\epsilon > 0$ and $\delta > 0$ such that for all time steps $k \geq 0$ we have

$$P\left(X_{k+1} \geq X_k + \epsilon \mid \mathcal{F}_k\right) \geq \delta, \quad \text{a.s.}$$

An example could be our usual binomial model, where the returns are defined such that for all time steps $k \geq 0$, the conditional transition probabilities are given by

$$P\left(X_{k+1} = X_k + \epsilon \mid \mathcal{F}_k\right) = 1 - P\left(X_{k+1} = -100X_k \mid \mathcal{F}_k\right) > 0$$

We equip our probability space with the natural filtration of X, and consider the stopping time τ given by

$$\tau := \inf\{k \geq 1 : X_k \geq X_{k-1} + \epsilon\}.$$

Consider the self-financing strategy $(\psi_k)_{k\geq 1} = (\psi_k^0, \vartheta_k)_{k\geq 1}$ defined by $\vartheta_1 = 1$, $\vartheta_{k+1} = 1 - G_k(\vartheta)/\epsilon$ until τ and $\vartheta_{k+1} = 0$ for $k > \tau$. The holdings ψ_k^0 at time k in the numéraire for $k \geq 1$ are defined such that ψ becomes a self financing strategy with zero initial wealth.

(a) Show that τ is a P-almost surely finite stopping time.

(b) Derive the holdings in the numéraire that make ψ a self-financing strategy with zero initial value.

(c) Show that the corresponding gains and loss process satisfies $G_k(\vartheta) \geq \epsilon$ for all $k > \tau$.

(d) Conclude that ψ is a generalized arbitrage opportunity.

(e) Is the strategy ψ an arbitrage opportunity, i.e. is ψ admissible?

(f) Explain why such strategies are called "doubling strategies".

Solution 7.1

(a) We first show that τ is a stopping time. The event $\{\tau \leq m\}$ for some $m \geq 1$ can be written

$$\begin{aligned}\{\tau \leq m\} &= \{\exists k \leq m : X_k \geq X_{k-1} + \epsilon\} \\ &= \cup_{k \leq m}\{X_k \geq X_{k-1} + \epsilon\} \in \mathcal{F}_m\end{aligned}$$

Remains to prove that $P(\tau < \infty) = 1$. This follows directly from our assumption $P(X_{k+1} \geq X_k + \epsilon \mid \mathcal{F}_k) \geq \delta > 0$ for all $k \geq 0$.

(b) The initial value is given by $V_0 = \psi_1^0 + \vartheta_1 \cdot X_0$. In order to have $V_0 = 0$, we therefore take $\psi_1^0 = -X_0$. Moreover, the trading strategy ψ is self-financing if and only if

$$\psi_{k+1}^0 - \psi_k^0 + (\vartheta_{k+1} - \vartheta_k) \cdot X_k = 0$$

for all $k \geq 1$. Hence we define recursively $\psi_1^0 = -X_0$ and

$$\psi_{k+1}^0 = \psi_k^0 - (\vartheta_{k+1} - \vartheta_k) \cdot X_k$$

for $k \geq 1$. Note that ψ^0 (respectively ϑ) is adapted (respectively predictable) with respect to the natural filtration of X, and hence $\psi = (\psi^0, \vartheta)$ indeed defines a self-financing trading strategy with zero initial wealth.

(c) Note that on the event $\{\tau = m + 1\}$ we have

$$\begin{aligned}G_{m+1}(\vartheta) &= G_m(\vartheta) + \vartheta_{m+1} \cdot (X_{m+1} - X_m) \\ &\geq G_m(\vartheta) + \vartheta_{m+1}\epsilon \\ &= G_m(\vartheta) + \left(1 - \frac{G_m(\vartheta)}{\epsilon}\right)\epsilon \\ &= \epsilon\end{aligned}$$

This proves $G_\tau(\vartheta) \geq \epsilon$. Moreover, for $k > \tau$, we have by construction $\vartheta_k = 0$ and hence

$$G_k(\vartheta) = G_\tau(\vartheta) \geq \epsilon.$$

(d) Since the market horizon is infinite and $P(\tau < \infty) = 1$, the strategy ψ guarantees almost surely a value of ϵ if we wait long enough. As the initial value of the portfolio is $0 < \epsilon$, we conclude that ψ is a generalized arbitrage opportunity.

(e) No, ψ is not an arbitrage. Indeed, they may not exist a uniform lower bound on the value process. To see this, note that even though $\tau = k$ will eventually happen for some k large enough, the losses occured until then are not bounded. Indeed, it is possible that we loose an arbitrarly large amount of amount before finally gaining enough to have an overall gain of at least ϵ. In particular, one most have access to an infinite credit line to set up this strategy. Note that as the losses accumulate before finally making a gain large enough, the holdings in the risky asset have to be increased exponentially.

(f) Consider the particular case when the discounted stock price follows a simple random walk:

$$X_k = \sum_{i=0}^{k} \xi_i$$

where ξ_i are i.i.d random variables satisfying $P(\xi_i = 1) = 1 - P(\xi_i = -1) = p > 0$. This example fits in the assumptions of the exercise: it suffices to take $\epsilon = 1$ and $\delta = p > 0$. The strategy ψ corresponds to buying one share of the risky stock at time 0 ($\vartheta_1 = 1$), and doubling the size of the holdings after each loss until a gain is finally obtained. Suppose for simplicity that the initial price of the stock is 1 CHF and assume that the first gain occurs at the n^{th} time period (i.e. $\tau = n$). Then the gains and loss process at time $n-1$ is given by

$$G_{n-1}(\psi) = \sum_{i=1}^{n-1} 2^i(-1) = 1 - 2^n$$

At time $\tau = n$, we make 1 CHF gain on each our our 2^n shares of the risky asset and hence $G_n(\psi) = G_{n-1}(\psi) + 2^n = 1$. As we started with zero initial value and did not use external cash flow to re-balance our portfolio (we financed our investment by borrowing money from the bank account), ψ is a generalized arbitrage opportunity because $\tau < \infty$ P-almost surely. In such a situation, the strategy offers a sure way to make money. Unfortunately, an investor using this strategy must be prepared to incur arbitrary large losses before eventually making an almost sure profit of 1 CHF.

Remark: In discrete time, we have used that the time horizon was infinite. Otherwise, it may be possible to never observe a gain large enough to quit our positions and make a profit. In continuous time however, the analogue of the doubling strategy can be implemented on a finite time interval. Indeed a technical problem with continuous time models is that events that will happen eventually can be made to happen in bounded time by speeding up the clock.

Exercise 7.2 Consider the space $(\Omega, \mathcal{F})$ with filtration $(\mathcal{F}_k)_{k \in N_0}$ and two locally equivalent measures P and Q.

(a) Show that if $Z^{Q;P}$ is the density process of Q with respect to P, i.e.,

$$Z_k^{Q;P} = \frac{\mathrm{d}Q|_{\mathcal{F}_k}}{\mathrm{d}P|_{\mathcal{F}_k}}$$

for all $k \in \mathbb{N}_0$, where $Q|_{\mathcal{F}_k}$ denotes the restriction of Q to $\mathcal{F}_k$, then

$$Z^{P;Q} = \frac{1}{Z^{Q;P}},$$

i.e., $\frac{1}{Z^{Q;P}}$ is the density process of P with respect to Q.

(b) Show that $Z^{Q;P}$ is a P-martingale and $\frac{1}{Z^{Q;P}}$ is a Q-martingale.

Solution 7.2

(a) We want to show that for every $k \in \mathbb{N}_0$ and every $A \in \mathcal{F}_k$,

$$P|_{\mathcal{F}_k}[A] = E_{Q|_{\mathcal{F}_k}}\left[I_A \frac{1}{Z_k^{Q;P}}\right].$$

Using the Radon–Nikodým derivative $Z_k^{Q;P}$ of $Q|_{\mathcal{F}_k}$ with respect to $P|_{\mathcal{F}_k}$, we obtain

$$E_{Q|_{\mathcal{F}_k}}\left[I_A \frac{1}{Z_k^{Q;P}}\right] = E_{P|_{\mathcal{F}_k}}\left[I_A \frac{Z_k^{Q;P}}{Z_k^{Q;P}}\right] = E_{P|_{\mathcal{F}_k}}[I_A] = P|_{\mathcal{F}_k}[A].$$

Hence,

$$\frac{1}{Z_k^{Q;P}} = \frac{\mathrm{d}P|_{\mathcal{F}_k}}{\mathrm{d}Q|_{\mathcal{F}_k}} = Z_k^{P;Q},$$

which is what we wanted to show.

(b) $A \in \mathcal{F}_k \subseteq \mathcal{F}_\ell$ for $k \leq \ell$ implies that

$$E_P[Z_\ell^{Q;P} \mathbb{1}_A] = Q|_{\mathcal{F}_\ell}[A] = Q|_{\mathcal{F}_k}[A] = E_P[Z_k^{Q;P} \mathbb{1}_A]$$

and therefore $E_P[Z_\ell^{Q;P} | \mathcal{F}_k] = Z_k^{Q;P}$. Adaptedness and integrability are clear from the Radon–Nikodým theorem.

Exercise 7.3 Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1\ldots,T}$ and let $\mathcal{F}_0$ be trivial.

Recall that the (conditional) Fatou's Lemma tells that given a sequence $Z = (Z_n)_{n\geq 0}$ of non-negative (i.e $Z_n \geq 0 \quad \forall n$) random variables on $(\Omega, \mathcal{F}, P)$ and a sigma algebra $\mathcal{G} \subseteq \mathcal{F}$, we have

$$E\left[\liminf_{n\to\infty} Z_n|\mathcal{G}\right] \leq \liminf_{n\to\infty} E\left[Z_n|\mathcal{G}\right]$$

Our trading strategies are usually assumed to be a-admissible but not necessarily 0-admissible. We thus would like to extend Fatou's lemma from non-negative random variables to random variables bounded from below.

(a) Let $(Y_n)_{n\geq 0}$ be a sequence of random variables on $(\Omega, \mathcal{F}, P)$ such that there exists some $a \in \mathbb{R}$ such that $Y_n \geq -a \quad \forall n$. Show that Fatou's lemma still holds for the sequence $(Y_n)_{n\geq 0}$.

Recall that one of the key steps in the proof of the easy direction of the FTAP was to realize that a local martingale bounded from below is a supermartingale. Let $X = (X_k)_{k=0,1\ldots,T}$ be a local martingale with $E[|X_0|] < \infty$ and $X_k \geq -a$ for some $a \geq 0$ for all $k = 0, 1 \ldots, T$.

(b) Show that X is a supermartingale.
Hint: Fatou's Lemma.

(c) Is the stochastic integral with respect to a supermartingale always a supermartingale? Why or why not?

Solution 7.3

(a) Follows directly by applying Fatou's lemma to the non-negative random variable $\tilde{Y}_n := Y_n + a$.

(b) By the definition of a local martingale, there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ increasing to T such that the stopped process $X^{\tau_n} = (X_{\tau_n\wedge k})_{k=0,1\ldots,T}$ is a martingale for each $n \in \mathbb{N}$. By the martingale property, we have

$$E[X_{\tau_n\wedge k}|\mathcal{F}_j] = X_{\tau_n\wedge j}$$

for every $0 \leq j \leq k \leq T$. Since we clearly have that for all $j = 0, 1, \ldots, T$, $\lim_{n\to\infty} X_{\tau_n\wedge j} = X_j$ P-a.s. and X is bounded below, Fatou's lemma gives us that

$$E[X_k|\mathcal{F}_j] = E[\lim X_{\tau_n\wedge k}|\mathcal{F}_j] \leq \liminf_{n\to\infty} E[X_{\tau_n\wedge k}|\mathcal{F}_j] = \liminf_{n\to\infty} X_{\tau_n\wedge j} = X_j, \quad (1)$$

which is the supermartingale property. Adaptedness of X is clear from the fact that X is a local martingale by assumption. For integrability, using the supermartingale property (1) that we have shown already, we have

$$E[X_k|\mathcal{F}_0] \leq X_0 \tag{2}$$

Since X is bounded from below by assumption we also have that $E[|X_k|] < \infty$ for all $0 \leq k \leq T$. Indeed,

$$\begin{aligned} E[|X_k|] = E[|X_k + a - a|] &\leq E[|X_k + a|] + |-a| \\ &\leq E[X_k] + 2a \\ &= E\left[E[X_k|\mathcal{F}_0]\right] + 2a \\ &\leq E[X_0] + 2a \quad \textit{by (2)} \\ &\leq E[|X_0|] + 2a < \infty \end{aligned}$$

where the last inequality uses the assumption that $E[|X_0|] < \infty$. So X is indeed a supermartingale.

(c) In order to conclude that this statement is in general not true it is enough to consider the trivially predictable process $\vartheta \equiv -1$. In that case we obtain for a supermartingale X that

$$E[(\vartheta \bullet X)_k - (\vartheta \bullet X)_{k-1}|\mathcal{F}_{k-1}] = E[-(X_k - X_{k-1})|\mathcal{F}_{k-1}] = -E[X_k - X_{k-1}|\mathcal{F}_{k-1}] \geq 0,$$

since $E[X_k - X_{k-1}|\mathcal{F}_{k-1}] \leq 0$ by assumption. The above, however, implies that $\vartheta \bullet X$ is a submartingale since integrability and adaptedness of $\vartheta \bullet X$ are clear. So if X is not a martingale (in which case it would be both a supermartingale and a submartingale), $\vartheta \bullet X$ is not a supermartingale.

If we assume ϑ to be additionally positive and bounded, then the stochastic integral process can be shown to be a supermartingale. This is left as a bonus exercise.

Exercise 7.4 Consider an undiscounted financial market in finite discrete time with two assets S^0, S^1 which are both strictly positive. Suppose that the market is arbitrage-free and denote by $\mathbb{P}(S^i)$ for $i = 0, 1$ the set of all equivalent martingale measures for S^i-discounted prices.

(a) Take any $Q \in \mathbb{P}(S^0)$ and define R by $\frac{\mathrm{d}R}{\mathrm{d}Q} := \frac{S^1_T}{S^0_T} / \frac{S^1_0}{S^0_0}$. Prove that $R \in \mathbb{P}(S^1)$.

(b) Take any $Q =: Q^{S^0} \in \mathbb{P}(S^0)$ and define $Q^{S^1} := R$ as in (a). For any $H \in L^0_+(\mathcal{F}_T)$, prove the *change of numéraire* formula

$$S^0_k E_{Q^{S^0}}\left[\frac{H}{S^0_T}\middle|\mathcal{F}_k\right] = S^1_k E_{Q^{S^1}}\left[\frac{H}{S^1_T}\middle|\mathcal{F}_k\right]$$

for $k = 0, 1, ..., T$.

Solution 7.4

(a) R defined by the Radon–Nikodým derivative $\frac{\mathrm{d}R}{\mathrm{d}Q} := \frac{S^1_T}{S^0_T} / \frac{S^1_0}{S^0_0}$ defines a measure equivalent to Q and therefore equivalent to P. Indeed

$$\frac{\mathrm{d}R}{\mathrm{d}Q} := \frac{S^1_T S^0_0}{S^0_T S^1_0} > 0.$$

Moreover R is a probability measure since

$$\mathbb{E}_R[\mathbb{1}_\Omega] = \mathbb{E}_Q\left[\tfrac{\mathrm{d}R}{\mathrm{d}Q}\mathbb{1}_\Omega\right] = \mathbb{E}_Q\left[\frac{S^1_T S^0_0}{S^0_T S^1_0}\right] = \mathbb{E}_Q\left[\mathbb{E}_Q\left[\frac{S^1_T}{S^0_T} \mid \mathcal{F}_0\right]\frac{S^0_0}{S^1_0}\right] = 1$$

Finally the S^1 discounted prices are martingales under R:

$$\mathbb{E}_R\left[\frac{S^0_T}{S^1_T} \mid \mathcal{F}_k\right] = \frac{S^0_k}{S^1_k}$$

where the last equality uses the change of numéraire formula from b).

(b) We use Bayes rule to compute

$$\begin{aligned}
S^1_k E_{Q^{S^1}}\left[\frac{H}{S^1_T}\middle|\mathcal{F}_k\right] &= S^1_k \frac{E_{Q^{S^0}}[H\frac{dQ^{S^1}}{dQ^{S^0}}/S^1_T|\mathcal{F}_k]}{E_{Q^{S^0}}\left[\frac{dQ^{S^1}}{dQ^{S^0}}\middle|\mathcal{F}_k\right]} \\
&= S^1_k \frac{E_{Q^{S^0}}[H\frac{dQ^{S^1}}{dQ^{S^0}}/S^1_T|\mathcal{F}_k]}{\frac{S^1_k}{S^0_k}/\frac{S^1_0}{S^0_0}} \\
&= S^0_k \frac{E_{Q^{S^0}}\left[H\frac{S^1_T}{S^0_T}\frac{S^0_0}{S^1_0}\frac{1}{S^1_T}|\mathcal{F}_k\right]}{1/\frac{S^1_0}{S^0_0}} \\
&= S^0_k E_{Q^{S^0}}\left[\frac{H}{S^0_T}\middle|\mathcal{F}_k\right].
\end{aligned}$$

Exercise 7.5 This exercise guides you through an alternative proof of the "hard" direction of the Dalang-Morton-Willinger Theorem. In this exercise we will focus on the basic one-period model, i.e we suppose that $T = 1$. The proof for the multi-period case is very similar but is a little more difficult because of some technicalities involving measurability. For simplicity, we also assume that $\mathcal{F}_0$ is (P-) trivial, so θ predictable means $\theta \in \mathbb{R}^d$. Moreover we also suppose that there exists a numéraire asset.

Let S_0 (respectively S_T) denote the vector of initial prices (respectively terminal prices), and let $(1, X)$ be the discounted prices with respect to the numéraire asset.

(a) Define a *pricing kernel* (also called *stochastic discount factor* or *state price density*) as a strictly positive random variable ρ satisfying

$$S_0 = \mathbb{E}_P [\rho S_T]$$

where P is the objective (or historical or statistical) measure of our filtered probability space $(\Omega, \mathcal{F}, P)$. When the market has a numéraire, we can characterize pricing kernels in terms of the discounted prices $(1, X)$: the pricing kernel ρ is a positive random variable $\rho > 0$ in $L^\infty(P)$ satisfying

$$E_P [\rho \Delta X_1] = 0$$

Show that when the market has a numéraire, the notion of a pricing kernel and that of an EMM are essentially the same. More precisely, show that the measure Q defined by

$$\frac{dQ}{dP} = \frac{\rho}{E_P[\rho]}$$

gives an EMM.

Since we suppose the existence of a numéraire, by Proposition II.2.1, the market is arbitrage free iff there is no arbitrage of the first kind. Moreover by question (a) the existence of a pricing kernel is equivalent to the existence of an EMM. We thus have to show that no arbitrage (of first kind) implies the existence of a pricing kernel ρ.

(b) Consider the function $F\colon \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ defined by

$$F(\theta) = \mathbb{E}_P \left[e^{-\theta \cdot \Delta X_1 - \frac{1}{2}||\Delta X_1||^2} \right]$$

Show that F is finite valued and smooth (C^1).

(c) Suppose that there exists a minimiser θ^* of F. Construct a pricing kernel ρ and show that the corresponding EMM Q has a bounded Radon–Nikodým derivative, i.e. $\frac{dQ}{dP} \in L^\infty$.

(d) In this question we show that the no arbitrage (of first kind) assumption implies the existence of a minimiser θ^* of F.

- Let $(\theta_k)_k$ be a minimizing sequence, i.e a sequence that satisfies

$$\lim_{k\to\infty} F(\theta_k) = \inf_{\theta\in\mathbb{R}^d} F(\theta)$$

Suppose that $(\theta_k)_k$ is bounded. Show that in this case F admits a minimiser θ^*.

It remains to show that no arbitrage (of first kind) implies the existence of a bounded minimising sequence $(\theta_k)_k$.

Let $\mathcal{U} = \{\theta \in \mathbb{R}^d : \theta \cdot \Delta X_1 = 0 \ P\text{-}a.s\} \subseteq \mathbb{R}^d$ and $\mathcal{V} = \mathcal{U}^\perp$ the orthogonal complement of $\mathcal{U}$.

- Show that if $u \in \mathcal{U}$ and $v \in \mathcal{V}$ then $F(u+v) = F(v)$

Choose a minimising sequence $(\theta_k)_k$. By the previous result we can assume without loss of generality that $\theta_k \in \mathcal{V}$ for all k (otherwise we project the sequence $(\theta_k)_k$ on $\mathcal{V}$ without changing the value of the function $F(\cdot)$ since $F(u+v) = F(v)$ if $u \in \mathcal{U}$ and $v \in \mathcal{V}$). Assume for contradiction that $(\theta_k)_k$ is unbounded, i.e after passing to a subsequence (again we continue to denote it by $(\theta_k)_k$), $||\theta_k|| \to \infty$. The goal of the next questions is to use the No Arbitrage assumption to get a contradiction.

- Since $(\theta_k)_k$ is unbounded, we can pass to a subsequence such that $||\theta_k|| \to \infty$. Define $\hat{\theta}_k = \frac{\theta_k}{||\theta_k||}$.

  Show that $\hat{\theta}_k \in \mathcal{V}$ and $||\hat{\theta}_k|| = 1$.

By Bolzano-Weierstrass Theorem, the bounded sequence $(\hat{\theta}_k)_k$ admits a converging subsequence. Let $\hat{\theta}_k$ denote this converging subsequence and let $\hat{\theta}$ be the limit of $\hat{\theta}_k$.

- Show that $\hat{\theta} \in \mathcal{V}$ and has unit norm.

- Show that the sequence $(F(\theta_k))_k$ is bounded.
- By showing that

$$F(\theta_k) = E_P\left[\left(e^{-\hat{\theta}_k\cdot\Delta X_1}\right)^{||\theta_k||} e^{-\frac{||\Delta X_1||^2}{2}}\right]$$

  conclude that we must have $\hat{\theta} \cdot \Delta X_1 \geq 0$ a.s.

- By using the no arbitrage assumption find a contradiction. Conclude that $(\theta_k)_k$ is bounded.

Solution 7.5

(a) This question simply tests your understanding of the definitions. Note that by Bayes formula we have

$$\begin{aligned}\mathbb{E}_Q[X_1] &= \mathbb{E}_P\left[\frac{dQ}{dP}X_1\right] \\ &= \mathbb{E}_P\left[\frac{\rho}{E_P[\rho]}X_1\right] \\ &= \frac{\mathbb{E}_P[\rho X_1]}{E_P[\rho]}\end{aligned}$$

If ρ is a pricing kernel then we get $\mathbb{E}_P[\rho X_1] = X_0 E_P[\rho]$ and so

$$\mathbb{E}_Q[X_1] = X_0$$

Moreover since $\rho > 0$, we conclude that Q is indeed an EMM.

On the other hand if Q is an EMM then $\mathbb{E}_Q[X_1] = X_0$ and so by the relation we derived above, the Radon–Nikodým derivative $\frac{dQ}{dP}$ defines a pricing kernel up to normalization.

(b) $F(\cdot)$ is clearly finite valued since the integrand is bounded. Indeed

$$\begin{aligned}e^{-\theta\cdot\Delta X_1-\frac{1}{2}||\Delta X_1||^2} &\le e^{-\theta\cdot\Delta X_1-\frac{1}{2}||\Delta X_1||^2}e^{-\frac{||\theta||^2}{2}}e^{\frac{||\theta||^2}{2}} \\ &= e^{-\frac{||\theta+\Delta X_1||^2}{2}}e^{\frac{||\theta||^2}{2}} \\ &\le e^{\frac{||\theta||^2}{2}}\end{aligned}$$

where in the last equality we used that $||\theta + \Delta X_1||^2 \ge 0$ so $e^{-\frac{||\theta+\Delta X_1||^2}{2}} \le 1$.

For the C^1 property of $F(\cdot)$, consider

$$f(\theta) := -\Delta X_1 \exp\left(-\theta\cdot\Delta X_1 - \frac{1}{2}||\Delta X_1||^2\right)$$

We need to show that $f(\cdot)$ is locally bounded in θ. Indeed in that case we can exchange the gradient and expectation operators and we get $F(\cdot) \in C^1$ with

$$\nabla F(\theta) = E_P\left[-\Delta X_1 \exp\left(-\theta\cdot\Delta X_1 - \frac{1}{2}||\Delta X_1||^2\right)\right]$$

We have by the Cauchy Schwarz inequality

$$||f(\theta)|| \le ||\Delta X_1|| \exp\left(||\theta||\,||\Delta X_1|| - \frac{1}{2}||\Delta X_1||^2\right)$$
$$\le \sup_{\lambda \ge 0}\left[\lambda \exp\left(\lambda||\theta|| - \frac{\lambda^2}{2}\right)\right]$$

The latter can be bounded as a function of θ hence $f(\cdot)$ is locally bounded in θ and thus $F(\cdot) \in C^1$.

(c) We have seen in question (b), that we can exchange the gradient and expectation operations to get

$$\nabla F(\theta) = E_P\left[-\Delta X_1 \exp\left(-\theta \cdot \Delta X_1 - \frac{1}{2}||\Delta X_1||^2\right)\right]$$

Let θ^* be a minimiser of F. By the first order condition for a minimum, we have

$$\nabla F(\theta^*) = 0 = E_P\left[-\Delta X_1 \exp\left(-\theta^* \cdot \Delta X_1 - \frac{1}{2}||\Delta X_1||^2\right)\right]$$

and hence $\rho = \exp\left(-\theta^* \cdot \Delta X_1 - \frac{1}{2}||\Delta X_1||^2\right)$ is a pricing kernel.

Note that using the bound $e^{-\theta\cdot\Delta X_1 - \frac{1}{2}||\Delta X_1||^2} \le e^{\frac{||\theta||^2}{2}}$ (obtained in b), we have

$$\rho = \exp\left(-\theta^* \cdot \Delta X_1 - \frac{1}{2}||\Delta X_1||^2\right)$$
$$\le \exp\left(\frac{||\theta^*||^2}{2}\right)$$

and hence the Radon–Nikodýmderivative of the corresponding EMM Q, $\frac{dQ}{dP} = \frac{\rho}{E_P[\rho]}$, is in L^∞.

(d)
- If $(\theta_k)_k$ is bounded, then Bolzano Weierstrass Theorem gives us the existence of a converging subsequence. For notational simplicity we will continue to denote this sequence by $(\theta_k)_k$ and write θ^* for the limit of this converging subsequence. Since F is continuous,

$$\lim_{k\to\infty} F(\theta_k) = F(\theta^*) = \inf_{\theta\in\mathbb{R}^d} F(\theta)$$

where the last equality comes from the fact that $(\theta_k)_k$ is a minimizing sequence. Hence θ^* is a minimiser of F.

- Direct from the definition of F and $\mathcal{U}$:

$$\begin{aligned} F(u+v) &= \mathbb{E}_P\left[e^{-(u+v)\cdot\Delta X_1 - \frac{1}{2}||\Delta X_1||^2}\right] \\ &= \mathbb{E}_P\left[e^{-v\cdot\Delta X_1 - \frac{1}{2}||\Delta X_1||^2}\right] \\ &= F(v) \end{aligned}$$

  where the last equality uses that $u \in \mathcal{U}$.
- $||\hat{\theta}_k|| = 1$ by definition. Morever $\hat{\theta}_k \in \mathcal{V}$ since $\theta_k \in \mathcal{V}$.
- By a standard result on finite-dimensional linear subspaces, $\mathcal{V}$ is closed. Hence $\hat{\theta} \in \mathcal{V}$ as the limit of the sequence $\hat{\theta}_k \in \mathcal{V}$. Another standard result tells that the inner product is a continuous map which implies that $||\hat{\theta}|| = ||\lim \hat{\theta}_k|| = \lim ||\hat{\theta}_k|| = 1$.

  You were not required to prove these standard results, but here is a proof of them. Let X be an inner product space (in our case $X = \mathbb{R}^d$). First we show that the inner product is a continuous map. Let $x_1, x_2, y_1, y_2 \in X$, by linearity of the inner product and Cauchy-Schwarz inequality we get,

$$\begin{aligned} |x_1 \cdot y_1 - x_2 \cdot y_2| &= |(x_1 - x_2)\cdot y_1 + x_2 \cdot (y_1 - y_2)| \\ &\leq ||x_1 - x_2||\ ||y_1|| + ||x_2||\ ||y_1 - y_2|| \end{aligned}$$

  This implies the continuity of inner products.

  Now let $A \subset X$. To show that $A^\perp$ is closed, consider a converging sequence (y_n) of elements of $A^\perp$ that converges to $y \in X$. We have to show that $y \in A^\perp$. Since the sequence (y_n) takes values in $A^\perp$, we have for all n

$$y_n \cdot a = 0 \ \forall a \in A$$

  hence

$$\lim_{n\to\infty}(y_n \cdot a) = 0 \ \forall a \in A$$

  But by continuity of the inner product,

$$\lim_{n\to\infty}(y_n \cdot a) = (\lim_{n\to\infty} y_n)\cdot a$$

  which shows that $A^\perp$ is closed.
- Since θ_k is a minimising sequence, there exists an index k_0 such that

$$F(\theta_k) \leq F(0) + 1 \quad \forall k \geq k_0$$

  So

$$F(\theta_k) \leq (F(0)+1) \vee \max_{k\leq k_0} F(\theta_k) \quad \forall k$$

  Hence the sequence $(F(\theta_k))_k$ is bounded.

- By definition
$$F(\theta) = \mathbb{E}_P\left[e^{-\theta\cdot\Delta X_1 - \frac{1}{2}||\Delta X_1||^2}\right]$$
Using that $\theta_k = \hat{\theta}_k||\theta_k||$ we directly get
$$F(\theta_k) = E_P\left[\left(e^{-\hat{\theta}_k\cdot\Delta X_1}\right)^{||\theta_k||} e^{-\frac{||\Delta X_1||^2}{2}}\right]$$
Since $F(\theta_k)$ is bounded, we must have $\hat{\theta}_k \cdot \Delta X_1 \geq 0$ a.s (as otherwise the right-hand side of the above expression would blow up). By taking the limit, $\hat{\theta} \cdot \Delta X_1 \geq 0$ a.s.
- No arbitrage implies $\hat{\theta} \cdot \Delta X_1 = 0$ which means that $\hat{\theta} \in \mathcal{U}$. But we already saw that $\hat{\theta} \in \mathcal{V}$ and hence $\hat{\theta} \in \mathcal{U} \cap \mathcal{V} = \{0\}$. Note that the last equality comes from the fact that $\mathcal{V}$ is the orthogonal complement of $\mathcal{U}$. So in particular we must have $\hat{\theta} = 0$. But this contradicts the fact that $||\hat{\theta}|| = 1$.