Introduction to Mathematical Finance

Exercise sheet 7

Exercise 7.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space. Consider an infinite discrete time model with a numéraire S^0 and a risky asset S^1. As usual let $X = S^1/S^0$ denote the discounted price process of the risky asset. Assume that there exists $\epsilon > 0$ and $\delta > 0$ such that for all time steps $k \geq 0$ we have

$$P\left(X_{k+1} \geq X_k + \epsilon \mid \mathcal{F}_k\right) \geq \delta, \quad \text{a.s.}$$

An example could be our usual binomial model, where the returns are defined such that for all time steps $k \geq 0$, the conditional transition probabilities are given by

$$P\left(X_{k+1} = X_k + \epsilon \mid \mathcal{F}_k\right) = 1 - P\left(X_{k+1} = -100X_k \mid \mathcal{F}_k\right) > 0$$

We equip our probability space with the natural filtration of X, and consider the stopping time τ given by

$$\tau := \inf\{k \geq 1 : X_k \geq X_{k-1} + \epsilon\}.$$

Consider the self-financing strategy $(\psi_k)_{k\geq 1} = (\psi_k^0, \vartheta_k)_{k\geq 1}$ defined by $\vartheta_1 = 1$, $\vartheta_{k+1} = 1 - G_k(\vartheta)/\epsilon$ until τ and $\vartheta_{k+1} = 0$ for $k > \tau$. The holdings ψ_k^0 at time k in the numéraire for $k \geq 1$ are defined such that ψ becomes a self financing strategy with zero initial wealth.

(a) Show that τ is a P-almost surely finite stopping time.

(b) Derive the holdings in the numéraire that make ψ a self-financing strategy with zero initial value.

(c) Show that the corresponding gains and loss process satisfies $G_k(\vartheta) \geq \epsilon$ for all $k > \tau$.

(d) Conclude that ψ is a generalized arbitrage opportunity.

(e) Is the strategy ψ an arbitrage opportunity, i.e. is ψ admissible?

(f) Explain why such strategies are called "doubling strategies".

Exercise 7.2 Consider the space $(\Omega, \mathcal{F})$ with filtration $(\mathcal{F}_k)_{k\in N_0}$ and two locally equivalent measures P and Q.

(a) Show that if $Z^{Q;P}$ is the density process of Q with respect to P, i.e.,

$$Z_k^{Q;P} = \frac{\mathrm{d}Q|_{\mathcal{F}_k}}{\mathrm{d}P|_{\mathcal{F}_k}}$$

for all $k \in \mathbb{N}_0$, where $Q|_{\mathcal{F}_k}$ denotes the restriction of Q to $\mathcal{F}_k$, then

$$Z^{P;Q} = \frac{1}{Z^{Q;P}},$$

i.e., $\frac{1}{Z^{Q;P}}$ is the density process of P with respect to Q.

(b) Show that $Z^{Q;P}$ is a P-martingale and $\frac{1}{Z^{Q;P}}$ is a Q-martingale.

Exercise 7.3 Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1\dots,T}$ and let $\mathcal{F}_0$ be trivial.

Recall that the (conditional) Fatou's Lemma tells that given a sequence $Z = (Z_n)_{n\geq 0}$ of non-negative (i.e $Z_n \geq 0 \quad \forall n$) random variables on $(\Omega, \mathcal{F}, P)$ and a sigma algebra $\mathcal{G} \subseteq \mathcal{F}$, we have

$$E\left[\liminf_{n\to\infty} Z_n|\mathcal{G}\right] \leq \liminf_{n\to\infty} E\left[Z_n|\mathcal{G}\right]$$

Our trading strategies are usually assumed to be a-admissible but not necessarily 0-admissible. We thus would like to extend Fatou's lemma from non-negative random variables to random variables bounded from below.

(a) Let $(Y_n)_{n\geq 0}$ be a sequence of random variables on $(\Omega, \mathcal{F}, P)$ such that there exists some $a \in \mathbb{R}$ such that $Y_n \geq -a \quad \forall n$. Show that Fatou's lemma still holds for the sequence $(Y_n)_{n\geq 0}$.

Recall that one of the key steps in the proof of the easy direction of the FTAP was to realize that a local martingale bounded from below is a supermartingale. Let $X = (X_k)_{k=0,1\dots,T}$ be a local martingale with $E[|X_0|] < \infty$ and $X_k \geq -a$ for some $a \geq 0$ for all $k = 0, 1 \dots, T$.

(b) Show that X is a supermartingale.
Hint: Fatou's Lemma.

(c) Is the stochastic integral with respect to a supermartingale always a supermartingale? Why or why not?

Exercise 7.4 Consider an undiscounted financial market in finite discrete time with two assets S^0, S^1 which are both strictly positive. Suppose that the market is arbitrage-free and denote by $\mathbb{P}(S^i)$ for $i = 0, 1$ the set of all equivalent martingale measures for S^i-discounted prices.

(a) Take any $Q \in \mathbb{P}(S^0)$ and define R by $\frac{\mathrm{d}R}{\mathrm{d}Q} := \frac{S^1_T}{S^0_T} / \frac{S^1_0}{S^0_0}$. Prove that $R \in \mathbb{P}(S^1)$.

(b) Take any $Q =: Q^{S^0} \in \mathbb{P}(S^0)$ and define $Q^{S^1} := R$ as in (a). For any $H \in L^0_+(\mathcal{F}_T)$, prove the *change of numéraire* formula

$$S^0_k E_{Q^{S^0}}\left[\frac{H}{S^0_T}\middle|\mathcal{F}_k\right] = S^1_k E_{Q^{S^1}}\left[\frac{H}{S^1_T}\middle|\mathcal{F}_k\right]$$

for $k = 0, 1, ..., T$.

Exercise 7.5 This exercise guides you through an alternative proof of the "hard" direction of the Dalang-Morton-Willinger Theorem. In this exercise we will focus on the basic one-period model, i.e we suppose that $T = 1$. The proof for the multi-period case is very similar but is a little more difficult because of some technicalities involving measurability. For simplicity, we also assume that $\mathcal{F}_0$ is (P-) trivial, so θ predictable means $\theta \in \mathbb{R}^d$. Moreover we also suppose that there exists a numéraire asset.

Let S_0 (respectively S_T) denote the vector of initial prices (respectively terminal prices), and let $(1, X)$ be the discounted prices with respect to the numéraire asset.

(a) Define a *pricing kernel* (also called *stochastic discount factor* or *state price density*) as a strictly positive random variable ρ satisfying

$$S_0 = \mathbb{E}_P\left[\rho S_T\right]$$

where P is the objective (or historical or statistical) measure of our filtered probability space $(\Omega, \mathcal{F}, P)$. When the market has a numéraire, we can characterize pricing kernels in terms of the discounted prices $(1, X)$: the pricing kernel ρ is a positive random variable $\rho > 0$ in $L^\infty(P)$ satisfying

$$E_P\left[\rho \Delta X_1\right] = 0$$

Show that when the market has a numéraire, the notion of a pricing kernel and that of an EMM are essentially the same. More precisely, show that the measure Q defined by

$$\frac{dQ}{dP} = \frac{\rho}{E_P[\rho]}$$

gives an EMM.

Since we suppose the existence of a numéraire, by Proposition II.2.1, the market is arbitrage free iff there is no arbitrage of the first kind. Moreover by question (a) the existence of a pricing kernel is equivalent to the existence of an EMM. We thus have to show that no arbitrage (of first kind) implies the existence of a pricing kernel ρ.

(b) Consider the function $F\colon \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ defined by

$$F(\theta) = \mathbb{E}_P\left[e^{-\theta\cdot\Delta X_1 - \frac{1}{2}||\Delta X_1||^2}\right]$$

Show that F is finite valued and smooth (C^1).

(c) Suppose that there exists a minimiser θ^* of F. Construct a pricing kernel ρ and show that the corresponding EMM Q has a bounded Radon–Nikodým derivative, i.e. $\frac{dQ}{dP} \in L^\infty$.

(d) In this question we show that the no arbitrage (of first kind) assumption implies the existence of a minimiser θ^* of F.

- Let $(\theta_k)_k$ be a minimizing sequence, i.e a sequence that satisfies

$$\lim_{k\to\infty} F(\theta_k) = \inf_{\theta\in\mathbb{R}^d} F(\theta)$$

  Suppose that $(\theta_k)_k$ is bounded. Show that in this case F admits a minimiser θ^*.

It remains to show that no arbitrage (of first kind) implies the existence of a bounded minimising sequence $(\theta_k)_k$.

Let $\mathcal{U} = \{\theta \in \mathbb{R}^d : \theta \cdot \Delta X_1 = 0 \ P\text{-}a.s\} \subseteq \mathbb{R}^d$ and $\mathcal{V} = \mathcal{U}^\perp$ the orthogonal complement of $\mathcal{U}$.

- Show that if $u \in \mathcal{U}$ and $v \in \mathcal{V}$ then $F(u + v) = F(v)$

Choose a minimising sequence $(\theta_k)_k$. By the previous result we can assume without loss of generality that $\theta_k \in \mathcal{V}$ for all k (otherwise we project the sequence $(\theta_k)_k$ on $\mathcal{V}$ without changing the value of the function $F(\cdot)$ since $F(u + v) = F(v)$ if $u \in \mathcal{U}$ and $v \in \mathcal{V}$). Assume for contradiction that $(\theta_k)_k$ is unbounded, i.e after passing to a subsequence (again we continue to denote it by $(\theta_k)_k$), $||\theta_k|| \to \infty$. The goal of the next questions is to use the No Arbitrage assumption to get a contradiction.

- Since $(\theta_k)_k$ is unbounded, we can pass to a subsequence such that $||\theta_k|| \to \infty$. Define $\hat{\theta}_k = \frac{\theta_k}{||\theta_k||}$.
  Show that $\hat{\theta}_k \in \mathcal{V}$ and $||\hat{\theta}_k|| = 1$.

By Bolzano-Weierstrass Theorem, the bounded sequence $(\hat{\theta}_k)_k$ admits a converging subsequence. Let $\hat{\theta}_k$ denote this converging subsequence and let $\hat{\theta}$ be the limit of $\hat{\theta}_k$.

- Show that $\hat{\theta} \in \mathcal{V}$ and has unit norm.

- Show that the sequence $(F(\theta_k))_k$ is bounded.
- By showing that

$$F(\theta_k) = E_P\left[\left(e^{-\hat{\theta}_k \cdot \Delta X_1}\right)^{||\theta_k||} e^{-\frac{||\Delta X_1||^2}{2}}\right]$$

  conclude that we must have $\hat{\theta} \cdot \Delta X_1 \geq 0$ a.s.

- By using the no arbitrage assumption find a contradiction. Conclude that $(\theta_k)_k$ is bounded.