Introduction to Mathematical Finance

Solution sheet 7

Solution 7.1

(a) We first show that τ is a stopping time. The event $\{\tau \leq m\}$ for some $m \geq 1$ can be written

$$\begin{aligned} \{\tau \leq m\} &= \{\exists k \leq m : X_k \geq X_{k-1} + \epsilon\} \\ &= \cup_{k \leq m} \{X_k \geq X_{k-1} + \epsilon\} \in \mathcal{F}_m \end{aligned}$$

Remains to prove that $P(\tau < \infty) = 1$. This follows directly from our assumption $P(X_{k+1} \geq X_k + \epsilon \mid \mathcal{F}_k) \geq \delta > 0$ for all $k \geq 0$.

(b) The initial value is given by $V_0 = \psi_1^0 + \vartheta_1 \cdot X_0$. In order to have $V_0 = 0$, we therefore take $\psi_1^0 = -X_0$. Moreover, the trading strategy ψ is self-financing if and only if

$$\psi_{k+1}^0 - \psi_k^0 + (\vartheta_{k+1} - \vartheta_k) \cdot X_k = 0$$

for all $k \geq 1$. Hence we define recursively $\psi_1^0 = -X_0$ and

$$\psi_{k+1}^0 = \psi_k^0 - (\vartheta_{k+1} - \vartheta_k) \cdot X_k$$

for $k \geq 1$. Note that ψ^0 (respectively ϑ) is adapted (respectively predictable) with respect to the natural filtration of X, and hence $\psi = (\psi^0, \vartheta)$ indeed defines a self-financing trading strategy with zero initial wealth.

(c) Note that on the event $\{\tau = m + 1\}$ we have

$$\begin{aligned} G_{m+1}(\vartheta) &= G_m(\vartheta) + \vartheta_{m+1} \cdot (X_{m+1} - X_m) \\ &\geq G_m(\vartheta) + \vartheta_{m+1}\epsilon \\ &= G_m(\vartheta) + \left(1 - \frac{G_m(\vartheta)}{\epsilon}\right)\epsilon \\ &= \epsilon \end{aligned}$$

This proves $G_\tau(\vartheta) \geq \epsilon$. Moreover, for $k > \tau$, we have by construction $\vartheta_k = 0$ and hence

$$G_k(\vartheta) = G_\tau(\vartheta) \geq \epsilon.$$

(d) Since the market horizon is infinite and $P(\tau < \infty) = 1$, the strategy ψ guarantees almost surely a value of ϵ if we wait long enough. As the initial value of the portfolio is $0 < \epsilon$, we conclude that ψ is a generalized arbitrage opportunity.

(e) No, ψ is not an arbitrage. Indeed, they may not exist a uniform lower bound on the value process. To see this, note that even though $\tau = k$ will eventually happen for some k large enough, the losses occured until then are not bounded. Indeed, it is possible that we loose an arbitrarly large amount of amount before finally gaining enough to have an overall gain of at least ϵ. In particular, one most have access to an infinite credit line to set up this strategy. Note that as the losses accumulate before finally making a gain large enough, the holdings in the risky asset have to be increased exponentially.

(f) Consider the particular case when the discounted stock price follows a simple random walk:

$$X_k = \sum_{i=0}^{k} \xi_i$$

where ξ_i are i.i.d random variables satisfying $P(\xi_i = 1) = 1 - P(\xi_i = -1) = p > 0$. This example fits in the assumptions of the exercise: it suffices to take $\epsilon = 1$ and $\delta = p > 0$. The strategy ψ corresponds to buying one share of the risky stock at time 0 ($\vartheta_1 = 1$), and doubling the size of the holdings after each loss until a gain is finally obtained. Suppose for simplicity that the initial price of the stock is 1 CHF and assume that the first gain occurs at the n^{th} time period (i.e. $\tau = n$). Then the gains and loss process at time $n-1$ is given by

$$G_{n-1}(\psi) = \sum_{i=1}^{n-1} 2^i(-1) = 1 - 2^n$$

At time $\tau = n$, we make 1 CHF gain on each our our 2^n shares of the risky asset and hence $G_n(\psi) = G_{n-1}(\psi) + 2^n = 1$. As we started with zero initial value and did not use external cash flow to re-balance our portfolio (we financed our investment by borrowing money from the bank account), ψ is a generalized arbitrage opportunity because $\tau < \infty$ P-almost surely. In such a situation, the strategy offers a sure way to make money. Unfortunately, an investor using this strategy must be prepared to incur arbitrary large losses before eventually making an almost sure profit of 1 CHF.

Remark: In discrete time, we have used that the time horizon was infinite. Otherwise, it may be possible to never observe a gain large enough to quit our positions and make a profit. In continuous time however, the analogue of the doubling strategy can be implemented on a finite time interval. Indeed a technical problem with continuous time models is that events that will happen eventually can be made to happen in bounded time by speeding up the clock.

Solution 7.2

(a) We want to show that for every $k \in \mathbb{N}_0$ and every $A \in \mathcal{F}_k$,

$$P|_{\mathcal{F}_k}[A] = E_{Q|_{\mathcal{F}_k}}\left[I_A \frac{1}{Z_k^{Q;P}}\right].$$

Using the Radon–Nikodým derivative $Z_k^{Q;P}$ of $Q|_{\mathcal{F}_k}$ with respect to $P|_{\mathcal{F}_k}$, we obtain

$$E_{Q|_{\mathcal{F}_k}}\left[I_A \frac{1}{Z_k^{Q;P}}\right] = E_{P|_{\mathcal{F}_k}}\left[I_A \frac{Z_k^{Q;P}}{Z_k^{Q;P}}\right] = E_{P|_{\mathcal{F}_k}}[I_A] = P|_{\mathcal{F}_k}[A].$$

Hence,

$$\frac{1}{Z_k^{Q;P}} = \frac{\mathrm{d}P|_{\mathcal{F}_k}}{\mathrm{d}Q|_{\mathcal{F}_k}} = Z_k^{P;Q},$$

which is what we wanted to show.

(b) $A \in \mathcal{F}_k \subseteq \mathcal{F}_\ell$ for $k \leq \ell$ implies that

$$E_P[Z_\ell^{Q;P}\mathbb{1}_A] = Q|_{\mathcal{F}_\ell}[A] = Q|_{\mathcal{F}_k}[A] = E_P[Z_k^{Q;P}\mathbb{1}_A]$$

and therefore $E_P[Z_\ell^{Q;P}|\mathcal{F}_k] = Z_k^{Q;P}$. Adaptedness and integrability are clear from the Radon–Nikodým theorem.

Solution 7.3

(a) Follows directly by applying Fatou's lemma to the non-negative random variable $\tilde{Y}_n := Y_n + a$.

(b) By the definition of a local martingale, there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ increasing to T such that the stopped process $X^{\tau_n} = (X_{\tau_n \wedge k})_{k=0,1\dots,T}$ is a martingale for each $n \in \mathbb{N}$. By the martingale property, we have

$$E[X_{\tau_n\wedge k}|\mathcal{F}_j] = X_{\tau_n \wedge j}$$

for every $0 \leq j \leq k \leq T$. Since we clearly have that for all $j = 0, 1, \dots, T$, $\lim_{n\to\infty} X_{\tau_n\wedge j} = X_j$ P-a.s. and X is bounded below, Fatou's lemma gives us that

$$E[X_k|\mathcal{F}_j] = E[\lim X_{\tau_n\wedge k}|\mathcal{F}_j] \leq \liminf_{n\to\infty} E[X_{\tau_n\wedge k}|\mathcal{F}_j] = \liminf_{n\to\infty} X_{\tau_n\wedge j} = X_j, \quad (1)$$

which is the supermartingale property. Adaptedness of X is clear from the fact that X is a local martingale by assumption. For integrability, using the supermartingale property (1) that we have shown already, we have

$$E[X_k|\mathcal{F}_0] \leq X_0 \quad (2)$$

Since X is bounded from below by assumption we also have that $E[|X_k|] < \infty$ for all $0 \leq k \leq T$. Indeed,

$$\begin{aligned} E[|X_k|] = E[|X_k + a - a|] &\leq E[|X_k + a|] + |-a| \\ &\leq E[X_k] + 2a \\ &= E\left[E[X_k|\mathcal{F}_0]\right] + 2a \\ &\leq E[X_0] + 2a \quad \textit{by } (2) \\ &\leq E[|X_0|] + 2a < \infty \end{aligned}$$

where the last inequality uses the assumption that $E[|X_0|] < \infty$. So X is indeed a supermartingale.

(c) In order to conclude that this statement is in general not true it is enough to consider the trivially predictable process $\vartheta \equiv -1$. In that case we obtain for a supermartingale X that

$$E[(\vartheta \bullet X)_k - (\vartheta \bullet X)_{k-1}|\mathcal{F}_{k-1}] = E[-(X_k - X_{k-1})|\mathcal{F}_{k-1}] = -E[X_k - X_{k-1}|\mathcal{F}_{k-1}] \geq 0,$$

since $E[X_k - X_{k-1}|\mathcal{F}_{k-1}] \leq 0$ by assumption. The above, however, implies that $\vartheta \bullet X$ is a submartingale since integrability and adaptedness of $\vartheta \bullet X$ are clear. So if X is not a martingale (in which case it would be both a supermartingale and a submartingale), $\vartheta \bullet X$ is not a supermartingale.

If we assume ϑ to be additionally positive and bounded, then the stochastic integral process can be shown to be a supermartingale. This is left as a bonus exercise.

Solution 7.4

(a) R defined by the Radon–Nikodým derivative $\frac{\mathrm{d}R}{\mathrm{d}Q} := \frac{S_T^1}{S_T^0} / \frac{S_0^1}{S_0^0}$ defines a measure equivalent to Q and therefore equivalent to P. Indeed

$$\frac{\mathrm{d}R}{\mathrm{d}Q} := \frac{S_T^1 S_0^0}{S_T^0 S_0^1} > 0.$$

Moreover R is a probability measure since

$$\mathbb{E}_R\left[\mathbb{1}_\Omega\right] = \mathbb{E}_Q\left[\tfrac{\mathrm{d}R}{\mathrm{d}Q}\mathbb{1}_\Omega\right] = \mathbb{E}_Q\left[\frac{S_T^1 S_0^0}{S_T^0 S_0^1}\right] = \mathbb{E}_Q\left[\mathbb{E}_Q\left[\frac{S_T^1}{S_T^0} \mid \mathcal{F}_0\right]\frac{S_0^0}{S_0^1}\right] = 1$$

Finally the S^1 discounted prices are martingales under R:

$$\mathbb{E}_R\left[\frac{S_T^0}{S_T^1} \mid \mathcal{F}_k\right] = \frac{S_k^0}{S_k^1}$$

where the last equality uses the change of numéraire formula from b).

(b) We use Bayes rule to compute

$$
\begin{aligned}
S_k^1 E_{Q^{S^1}}\left[\frac{H}{S_T^1}\Big|\mathcal{F}_k\right] &= S_k^1 \frac{E_{Q^{S^0}}[H\frac{dQ^{S^1}}{dQ^{S^0}}/S_T^1|\mathcal{F}_k]}{E_{Q^{S^0}}\left[\frac{dQ^{S^1}}{dQ^{S^0}}\Big|\mathcal{F}_k\right]} \\
&= S_k^1 \frac{E_{Q^{S^0}}[H\frac{dQ^{S^1}}{dQ^{S^0}}/S_T^1|\mathcal{F}_k]}{\frac{S_k^1}{S_k^0}/\frac{S_0^1}{S_0^0}} \\
&= S_k^0 \frac{E_{Q^{S^0}}\left[H\frac{S_T^1}{S_T^0}\frac{S_0^0}{S_0^1}\frac{1}{S_T^1}|\mathcal{F}_k\right]}{1/\frac{S_0^1}{S_0^0}} \\
&= S_k^0 E_{Q^{S^0}}\left[\frac{H}{S_T^0}\Big|\mathcal{F}_k\right].
\end{aligned}
$$

Solution 7.5

(a) This question simply tests your understanding of the definitions. Note that by Bayes formula we have

$$
\begin{aligned}
\mathbb{E}_Q[X_1] &= \mathbb{E}_P\left[\frac{dQ}{dP}X_1\right] \\
&= \mathbb{E}_P\left[\frac{\rho}{E_P[\rho]}X_1\right] \\
&= \frac{\mathbb{E}_P[\rho X_1]}{E_P[\rho]}
\end{aligned}
$$

If ρ is a pricing kernel then we get $\mathbb{E}_P[\rho X_1] = X_0 E_P[\rho]$ and so

$$
\mathbb{E}_Q[X_1] = X_0
$$

Moreover since $\rho > 0$, we conclude that Q is indeed an EMM.

On the other hand if Q is an EMM then $\mathbb{E}_Q[X_1] = X_0$ and so by the relation we derived above, the Radon–Nikodým derivative $\frac{dQ}{dP}$ defines a pricing kernel up to normalization.

(b) $F(\cdot)$ is clearly finite valued since the integrand is bounded. Indeed

$$
\begin{aligned}
e^{-\theta\cdot\Delta X_1 - \frac{1}{2}||\Delta X_1||^2} &\leq e^{-\theta\cdot\Delta X_1 - \frac{1}{2}||\Delta X_1||^2} e^{-\frac{||\theta||^2}{2}} e^{\frac{||\theta||^2}{2}} \\
&= e^{-\frac{||\theta+\Delta X_1||^2}{2}} e^{\frac{||\theta||^2}{2}} \\
&\leq e^{\frac{||\theta||^2}{2}}
\end{aligned}
$$

where in the last equality we used that $||\theta + \Delta X_1||^2 \geq 0$ so $e^{-\frac{||\theta+\Delta X_1||^2}{2}} \leq 1$.
For the C^1 property of $F(\cdot)$, consider

$$f(\theta) := -\Delta X_1 \exp\left(-\theta \cdot \Delta X_1 - \frac{1}{2}||\Delta X_1||^2\right)$$

We need to show that $f(\cdot)$ is locally bounded in θ. Indeed in that case we can exchange the gradient and expectation operators and we get $F(\cdot) \in C^1$ with

$$\nabla F(\theta) = E_P\left[-\Delta X_1 \exp\left(-\theta \cdot \Delta X_1 - \frac{1}{2}||\Delta X_1||^2\right)\right]$$

We have by the Cauchy Schwarz inequality

$$\begin{aligned} ||f(\theta)|| &\leq ||\Delta X_1|| \exp\left(||\theta||\,||\Delta X_1|| - \frac{1}{2}||\Delta X_1||^2\right) \\ &\leq \sup_{\lambda \geq 0}\left[\lambda \exp\left(\lambda||\theta|| - \frac{\lambda^2}{2}\right)\right] \end{aligned}$$

The latter can be bounded as a function of θ hence $f(\cdot)$ is locally bounded in θ and thus $F(\cdot) \in C^1$.

(c) We have seen in question (b), that we can exchange the gradient and expectation operations to get

$$\nabla F(\theta) = E_P\left[-\Delta X_1 \exp\left(-\theta \cdot \Delta X_1 - \frac{1}{2}||\Delta X_1||^2\right)\right]$$

Let θ^* be a minimiser of F. By the first order condition for a minimum, we have

$$\nabla F(\theta^*) = 0 = E_P\left[-\Delta X_1 \exp\left(-\theta^* \cdot \Delta X_1 - \frac{1}{2}||\Delta X_1||^2\right)\right]$$

and hence $\rho = \exp\left(-\theta^* \cdot \Delta X_1 - \frac{1}{2}||\Delta X_1||^2\right)$ is a pricing kernel.

Note that using the bound $e^{-\theta\cdot\Delta X_1 - \frac{1}{2}||\Delta X_1||^2} \leq e^{\frac{||\theta||^2}{2}}$ (obtained in b), we have

$$\begin{aligned} \rho &= \exp\left(-\theta^* \cdot \Delta X_1 - \frac{1}{2}||\Delta X_1||^2\right) \\ &\leq \exp\left(\frac{||\theta^*||^2}{2}\right) \end{aligned}$$

and hence the Radon–Nikodýmderivative of the corresponding EMM Q, $\frac{dQ}{dP} = \frac{\rho}{E_P[\rho]}$, is in L^∞.

(d)
- If $(\theta_k)_k$ is bounded, then Bolzano Weierstrass Theorem gives us the existence of a converging subsequence. For notational simplicity we will continue to denote this sequence by $(\theta_k)_k$ and write θ^* for the limit of this converging subsequence. Since F is continuous,

$$\lim_{k\to\infty} F(\theta_k) = F(\theta^*) = \inf_{\theta\in\mathbb{R}^d} F(\theta)$$

where the last equality comes from the fact that $(\theta_k)_k$ is a minimizing sequence. Hence θ^* is a minimiser of F.

- Direct from the definition of F and $\mathcal{U}$:

$$\begin{aligned} F(u+v) &= \mathbb{E}_P\left[e^{-(u+v)\cdot\Delta X_1-\frac{1}{2}||\Delta X_1||^2}\right] \\ &= \mathbb{E}_P\left[e^{-v\cdot\Delta X_1-\frac{1}{2}||\Delta X_1||^2}\right] \\ &= F(v) \end{aligned}$$

where the last equality uses that $u\in\mathcal{U}$.

- $||\hat{\theta}_k|| = 1$ by definition. Morever $\hat{\theta}_k \in \mathcal{V}$ since $\theta_k\in\mathcal{V}$.

- By a standard result on finite-dimensional linear subspaces, $\mathcal{V}$ is closed. Hence $\hat{\theta}\in\mathcal{V}$ as the limit of the sequence $\hat{\theta}_k\in\mathcal{V}$. Another standard result tells that the inner product is a continuous map which implies that $||\hat{\theta}|| = ||\lim\hat{\theta}_k|| = \lim||\hat{\theta}_k|| = 1$.

You were not required to prove these standard results, but here is a proof of them. Let X be an inner product space (in our case $X=\mathbb{R}^d$). First we show that the inner product is a continuous map. Let $x_1,x_2,y_1,y_2\in X$, by linearity of the inner product and Cauchy-Schwarz inequality we get,

$$\begin{aligned} |x_1\cdot y_1 - x_2\cdot y_2| &= |(x_1-x_2)\cdot y_1 + x_2\cdot(y_1-y_2)| \\ &\le ||x_1-x_2||\;||y_1|| + ||x_2||\;||y_1-y_2|| \end{aligned}$$

This implies the continuity of inner products.

Now let $A\subset X$. To show that $A^\perp$ is closed, consider a converging sequence (y_n) of elements of $A^\perp$ that converges to $y\in X$. We have to show that $y\in A^\perp$. Since the sequence (y_n) takes values in $A^\perp$, we have for all n

$$y_n\cdot a = 0\ \forall a\in A$$

hence

$$\lim_{n\to\infty}(y_n\cdot a) = 0\ \forall a\in A$$

But by continuity of the inner product,

$$\lim_{n\to\infty}(y_n \cdot a) = (\lim_{n\to\infty} y_n) \cdot a$$

which shows that $A^{\perp}$ is closed.

- Since θ_k is a minimising sequence, there exists an index k_0 such that

$$F(\theta_k) \leq F(0) + 1 \quad \forall k \geq k_0$$

So

$$F(\theta_k) \leq (F(0)+1) \vee \max_{k\leq k_0} F(\theta_k) \quad \forall k$$

Hence the sequence $(F(\theta_k))_k$ is bounded.

- By definition

$$F(\theta) = \mathbb{E}_P\left[e^{-\theta\cdot\Delta X_1 - \frac{1}{2}||\Delta X_1||^2}\right]$$

Using that $\theta_k = \hat{\theta}_k ||\theta_k||$ we directly get

$$F(\theta_k) = E_P\left[\left(e^{-\hat{\theta}_k\cdot\Delta X_1}\right)^{||\theta_k||} e^{-\frac{||\Delta X_1||^2}{2}}\right]$$

Since $F(\theta_k)$ is bounded, we must have $\hat{\theta}_k \cdot \Delta X_1 \geq 0$ a.s (as otherwise the right-hand side of the above expression would blow up). By taking the limit, $\hat{\theta}\cdot\Delta X_1 \geq 0$ a.s.

- No arbitrage implies $\hat{\theta}\cdot\Delta X_1 = 0$ which means that $\hat{\theta} \in \mathcal{U}$. But we already saw that $\hat{\theta} \in \mathcal{V}$ and hence $\hat{\theta} \in \mathcal{U}\cap\mathcal{V} = \{0\}$. Note that the last equality comes from the fact that $\mathcal{V}$ is the orthogonal complement of $\mathcal{U}$. So in particular we must have $\hat{\theta} = 0$. But this contradicts the fact that $||\hat{\theta}|| = 1$.