Introduction to Mathematical Finance

Exercise sheet 8

Exercise 8.1 In Exercise 5.4, we have introduced a multiperiod binomial market. In a similar fashion, we construct a trinomial market: Fix $r > -1$ and let $S_k^0 = (1+r)^k$. Now define $S_0^1 = 1$ and

$$S_k^1 = \prod_{i=1}^{k} R_i^1, \quad k = 1, \dots, T,$$

where the R_k^1 are i.i.d. and

$$P[R_k^1 = 1+u] = p^u, \quad P[R_k^1 = 1+m] = p^m, \quad P[R_k^1 = 1+d] = p^d,$$

all > 0, for $u > m > d$ and $u > r > d$. Note that the superscripts do not indicate powers.

Describe the set of *all* equivalent martingale measures for the S^0-discounted prices. You may provide an answer with, e.g., $T = 2$.
Hint: Use $\Omega = \{u, m, d\}^T$.

Solution 8.1 We use $\Omega = \{u, m, d\}^T$, and define the random variables $R_k^1(\omega) = 1 + \omega_k$.

Begin by introducing the notation $I_k = \{u, m, d\}^k$ for the set of outcomes until time k and $J_k = \{u, m, d\}^{T-k}$ for the set of future outcomes. Then set $X := S^1/S^0$. By rewriting the martingale condition $X_k = E_Q[X_{k+1}|\mathcal{F}_k]$, we obtain

$$1 + r = E_Q\left[\left.\frac{S_{k+1}^1}{S_k^1}\right|\mathcal{F}_k\right] = E_Q[R_{k+1}^1|\mathcal{F}_k] = \sum_{\omega^k \in I_k} E_Q[R_{k+1}^1|\{\omega^k\} \times J_k] 1_{\{\omega^k\} \times J_k},$$

for $k = 0, 1, \dots, T-1$. With the notation $Q[R_{k+1}^1 = 1 + v|\{\omega^k\} \times J_k] = q_{\omega^k}^v$, for $v \in \{u, m, d\}$ and $\omega^k \in I_k$, this condition reduces to

$$q_{\omega^k}^u u + q_{\omega^k}^m m + q_{\omega^k}^d d = r, \quad \forall \omega^k \in I_k,$$

which is precisely the equation solved in Exercise 2.4. In the case $k = 0$, we have

$$q_{\omega^0}^u u + q_{\omega^0}^m m + q_{\omega^0}^d d = r,$$

where $q_{\omega^0}^v = Q[R_1^1 = 1 + v]$. The solution to this case is analogous to the other cases handled below.

With the parameter λ_{ω^k}, we write the set of solutions as

$$(q_{\omega^k}^u, q_{\omega^k}^m, q_{\omega^k}^d) = \left(\frac{(r-d) - (m-d)\lambda_{\omega^k}}{u-d}, \lambda_{\omega^k}, \frac{(u-r) - (u-m)\lambda_{\omega^k}}{u-d}\right),$$

where

$$\lambda_{\omega^k} \in \left(0, \min\left\{\frac{r-d}{m-d}, \frac{u-r}{u-m}\right\}\right).$$

For any sequence of λ_{ω^k}, $\omega^k \in I_k$ for $k = 0, \dots, T-1$ as above, we get an (equivalent) martingale measure Q, namely

$$Q[\{\omega\}] = \prod_{k=1}^{T} q_{\omega^{k-1}}^{\omega_k},$$

where $\omega = (\omega_1, \dots, \omega_k, \dots, \omega_T) \in \Omega$ and $\omega^{k-1} = (\omega_1, \dots, \omega_{k-1}) \in I_{k-1}$ and $q_{\omega^{k-1}}^{\omega_k}$, as defined earlier, is the conditional probability under Q that R_k^1 takes the value $1 + \omega_k$, given that we are in the node ω^{k-1} at time $k-1$, for $k = 1, \dots, T$.

Exercise 8.2

(a) Consider a market without arbitrage. Show that for every (European) contingent claim $H \in L^0(\Omega, \mathcal{F}_T, P)$, there exists an equivalent martingale measure Q such that $H \in L^1(\Omega, \mathcal{F}_T, Q)$.

(b) Construct an example for a family of uniformly bounded random variables whose pointwise supremum is not a random variable. This illustrates why the *essential supremum* is needed in probability and measure theory, rather than the (usual) supremum.

Solution 8.2

(a) Note that

$$E_P\left[\frac{1}{1+|H|}\right] \in (0,1].$$

Hence, P' defined by

$$\frac{\mathrm{d}P'}{\mathrm{d}P} = \frac{1}{1+|H|}\bigg/ E_P\left[\frac{1}{1+|H|}\right]$$

is well-defined. Furthermore,

$$E_{P'}\left[|H|\right] = E_P\left[\frac{|H|}{1+|H|}\right]\bigg/ E_P\left[\frac{1}{1+|H|}\right] \le 1 \bigg/ E_P\left[\frac{1}{1+|H|}\right] < \infty,$$

showing that $H \in L^1(\Omega, \mathcal{F}_T, P')$. Since $P \approx P'$, the market is free of arbitrage also under P'. Thus by Theorem II.3.1, there exists an EMM Q with $\frac{\mathrm{d}Q}{\mathrm{d}P'} \in L^\infty$. Therefore, H is also Q-integrable.

(b) Let $\Omega = [0,1]$ with the Borel sigma-algebra and P the Lebesgue measure. Let $V \subseteq \Omega$ be any set which is not Lebesgue-measurable, for example the Vitali set, and $(X_v)_{v\in V}$ the family of random variables X_v defined by

$$X_v = 1_{\{v\}}.$$

Note that every X_v is indeed a random variable because $\{v\}$ is closed, hence a Borel set. However, the pointwise supremum is

$$\sup_{v\in V} X_v = 1_V,$$

which is not measurable, by construction.

Exercise 8.3 Let $Q \sim R$ be two equivalent probability measures on a filtered measurable space $(\Omega, (\mathcal{F}_k)_{0 \le k \le T}, \mathcal{F})$, and let $\sigma : \Omega \to \{0, \ldots, T\}$ be a stopping time. We define the pasting $\tilde{Q}$ of Q and R at σ as

$$\tilde{Q}(A) := \mathbb{E}_Q \left[\mathbb{E}_R \left[\mathbb{1}_A \mid \mathcal{F}_\sigma \right] \right].$$

(a) Show that $\tilde{Q}$ gives a probability measure on $\mathcal{F}_T$.

(b) Prove that the density process $\tilde{Z} = Z^{\tilde{Q};Q}$ of $\tilde{Q}$ with respect to Q is given by

$$\tilde{Z}_k = \mathbb{1}_{k \le \sigma} + \frac{Z_k}{Z_\sigma} \mathbb{1}_{k > \sigma},$$

where $Z = Z^{R;Q}$ is the density process of R with respect to Q.

We say that a set $\mathcal{Q}$ of equivalent probability measures on $(\Omega, \mathcal{F})$ is *m-stable* if for any $Q_1, Q_2 \in \mathcal{Q}$, and any stopping time $\sigma : \Omega \to \{0, \ldots, T\}$, the pasting of Q_1 and Q_2 at σ is also in $\mathcal{Q}$.

(c) Prove that the set $\mathbb{P}_{loc}(X)$ is m-stable.
Hint: The monotone convergence theorem guarantees that, for all $Y \ge 0$,

$$\mathbb{E}_{\tilde{Q}}[Y] = \mathbb{E}_Q \left[\mathbb{E}_R \left[Y \mid \mathcal{F}_\sigma \right] \right].$$

One can show (left as bonus exercise–see Lemma 6.41 in Hans Föllmer and Alexander Schied, "Stochastic Finance: An Introduction in Discrete Time", de Gruyter) that for all stopping times $\tau : \Omega \to \{0, \ldots, T\}$, and $\mathcal{F}_T$ measurable random variable $Y \ge 0$ we have

$$\mathbb{E}_{\tilde{Q}}[Y \mid \mathcal{F}_\tau] = \mathbb{E}_Q \left[\mathbb{E}_R \left[Y \mid \mathcal{F}_{\sigma \vee \tau} \right] \mid \mathcal{F}_\tau \right].$$

Solution 8.3

(a) We first check that $\tilde{Q}$ is indeed a probability measure.

- $\tilde{Q}(\emptyset) = 0$
- $\tilde{Q}(A) \ge 0$ for all $A \in \mathcal{F}_T$
- Let $A_1, A_2, \ldots$ be pairwise disjoint elements of $\mathcal{F}_T$. We then have:

$$\begin{aligned} \tilde{Q}\left(\cup_{k=1}^\infty A_k\right) &= \mathbb{E}_Q \left[\mathbb{E}_R \left[\mathbb{1}_{\cup_{k=1}^\infty A_k} \mid \mathcal{F}_\sigma \right] \right] \\ &= \mathbb{E}_Q \left[\mathbb{E}_R \left[\sum_{k=1}^\infty \mathbb{1}_{A_k} \mid \mathcal{F}_\sigma \right] \right] \\ &= \sum_{k=1}^\infty \tilde{Q}(A_k), \end{aligned}$$

where in line 2 and 3 we have used that the sets A_k are disjoint.

Finally, $\tilde{Q}$ is probability measure since

$$\tilde{Q}(\Omega) = \mathbb{E}_Q\left[\mathbb{E}_R\left[\mathbb{1}_\Omega \mid \mathcal{F}_\sigma\right]\right] = \mathbb{E}_Q\left[\mathbb{1}_\Omega\right] = 1.$$

(b) Let $A \in \mathcal{F}_T$. We need to show that

$$\tilde{Q}(A) := \mathbb{E}_Q\left[\mathbb{E}_R\left[\mathbb{1}_A \mid \mathcal{F}_\sigma\right]\right] = \mathbb{E}_Q\left[\frac{d\tilde{Q}}{dQ}\mathbb{1}_A\right],$$

where

$$\begin{aligned}\frac{d\tilde{Q}}{dQ} = \tilde{Z}_T &= \mathbb{1}_{T\le\sigma} + \frac{Z_T}{Z_\sigma}\mathbb{1}_{T>\sigma} \\ &= \mathbb{1}_{\sigma=T} + \frac{Z_T}{Z_\sigma}\mathbb{1}_{\sigma<T}.\end{aligned}$$

Note that the inner conditional expectation appearing in the definition of the pasting measure can be written as

$$\begin{aligned}\mathbb{E}_R\left[\mathbb{1}_A \mid \mathcal{F}_\sigma\right] &= \mathbb{E}_R\left[\mathbb{1}_A(\mathbb{1}_{\sigma=T} + \mathbb{1}_{\sigma<T}) \mid \mathcal{F}_\sigma\right] \\ &= \mathbb{E}_R\left[\mathbb{1}_A\mathbb{1}_{\sigma=T} \mid \mathcal{F}_T\right] + \mathbb{E}_R\left[\mathbb{1}_A\mathbb{1}_{\sigma<T} \mid \mathcal{F}_\sigma\right] \\ &= \mathbb{1}_A\mathbb{1}_{\sigma=T} + \frac{\mathbb{E}_Q\left[Z_T\mathbb{1}_A\mathbb{1}_{\sigma<T} \mid \mathcal{F}_\sigma\right]}{Z_\sigma}\end{aligned}$$

where in the last line we used $\mathbb{1}_A\mathbb{1}_{\sigma=T} \in \mathcal{F}_T$ as well as Bayes rule to express the conditional expectation under R in terms of the measure Q. Hence, using the definition of the pasting measure $\tilde{Q}$, we have:

$$\begin{aligned}\tilde{Q}(A) := \mathbb{E}_Q\left[\mathbb{E}_R\left[\mathbb{1}_A \mid \mathcal{F}_\sigma\right]\right] &= \mathbb{E}_Q\left[\mathbb{1}_A\mathbb{1}_{\sigma=T} + \frac{\mathbb{E}_Q\left[Z_T\mathbb{1}_A\mathbb{1}_{\sigma<T} \mid \mathcal{F}_\sigma\right]}{Z_\sigma}\right] \\ &= \mathbb{E}_Q\left[\left(\mathbb{1}_{\sigma=T} + \frac{Z_T}{Z_\sigma}\mathbb{1}_{\sigma<T}\right)\mathbb{1}_A\right] \\ &= \mathbb{E}_Q\left[\frac{d\tilde{Q}}{dQ}\mathbb{1}_A\right],\end{aligned}$$

which proves that the density process $\tilde{Z} = Z^{\tilde{Q};Q}$ of $\tilde{Q}$ with respect to Q is given by

$$\tilde{Z}_k = \mathbb{1}_{k\le\sigma} + \frac{Z_k}{Z_\sigma}\mathbb{1}_{k>\sigma},$$

where $Z = Z^{R;Q}$ of R with respect to Q.

(c) Let $Q, R \in \mathbb{P}_{loc}(X)$ be two ELMMs, and $\sigma : \Omega \to \{0, 1, \dots, T\}$ be a stopping time. We need to show that the pasting of Q and R at σ is also an ELMM for the discounted price process X. To show that $\tilde{Q} \in \mathbb{P}_{loc}(X)$, we need to find a localizing sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that X^{σ_n} is a $\tilde{Q}$ martingale for all $n \geq 0$. Let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence for X and ZX so that X^{τ_n} and $(ZX)^{\tau_n}$ are Q-martingales (or equivalently, using Bayes theorem, X^{τ_n} is a martingale under both Q and R). The following lines show that $(\tau_n)_{n\in\mathbb{N}}$ is also a localizing sequence for $\tilde{Z}X$, i.e $\tilde{Q} \in \mathbb{P}_{loc}(X)$.

$$\begin{aligned}
\mathbb{E}_{\tilde{Q}}\left[X_{k\wedge\tau_n} \mid \mathcal{F}_j\right] &= \mathbb{E}_Q\left[\mathbb{E}_R\left[X_{k\wedge\tau_n} \mid \mathcal{F}_{\sigma\vee j}\right] \mid \mathcal{F}_j\right] \\
&= \mathbb{E}_Q\left[X_{(k\wedge\tau_n)\wedge(\sigma\vee j)} \mid \mathcal{F}_j\right] \\
&= X_{j\wedge\tau_n}
\end{aligned}$$

In the first equality, we used the hint with $Y = X_k$ and $\tau := j$ for $j \leq k$. The other equalities follow from Doob's Optional Stopping theorem and the fact that X^{τ_n} is a martingale under both Q and R.