Introduction to Mathematical Finance

Exercise sheet 8

Exercise 8.1 In Exercise 5.4, we have introduced a multiperiod binomial market. In a similar fashion, we construct a trinomial market: Fix $r > -1$ and let $S_k^0 = (1+r)^k$. Now define $S_0^1 = 1$ and

$$S_k^1 = \prod_{i=1}^{k} R_i^1, \quad k = 1, \dots, T,$$

where the R_k^1 are i.i.d. and

$$P[R_k^1 = 1+u] = p^u, \quad P[R_k^1 = 1+m] = p^m, \quad P[R_k^1 = 1+d] = p^d,$$

all > 0, for $u > m > d$ and $u > r > d$. Note that the superscripts do not indicate powers.

Describe the set of *all* equivalent martingale measures for the S^0-discounted prices. You may provide an answer with, e.g., $T = 2$.
Hint: Use $\Omega = \{u, m, d\}^T$.

Exercise 8.2

(a) Consider a market without arbitrage. Show that for every (European) contingent claim $H \in L^0(\Omega, \mathcal{F}_T, P)$, there exists an equivalent martingale measure Q such that $H \in L^1(\Omega, \mathcal{F}_T, Q)$.

(b) Construct an example for a family of uniformly bounded random variables whose pointwise supremum is not a random variable. This illustrates why the *essential supremum* is needed in probability and measure theory, rather than the (usual) supremum.

Exercise 8.3 Let $Q \sim R$ be two equivalent probability measures on a filtered measurable space $(\Omega, (\mathcal{F}_k)_{0 \le k \le T}, \mathcal{F})$, and let $\sigma : \Omega \to \{0, \dots, T\}$ be a stopping time. We define the pasting $\tilde{Q}$ of Q and R at σ as

$$\tilde{Q}(A) := \mathbb{E}_Q\left[\mathbb{E}_R\left[\mathbb{1}_A \mid \mathcal{F}_\sigma\right]\right].$$

(a) Show that $\tilde{Q}$ gives a probability measure on $\mathcal{F}_T$.

(b) Prove that the density process $\tilde{Z} = Z^{\tilde{Q};Q}$ of $\tilde{Q}$ with respect to Q is given by

$$\tilde{Z}_k = \mathbb{1}_{k \le \sigma} + \frac{Z_k}{Z_\sigma}\mathbb{1}_{k > \sigma},$$

where $Z = Z^{R;Q}$ is the density process of R with respect to Q.

We say that a set $\mathcal{Q}$ of equivalent probability measures on $(\Omega, \mathcal{F})$ is *m-stable* if for any $Q_1, Q_2 \in \mathcal{Q}$, and any stopping time $\sigma : \Omega \to \{0, \dots, T\}$, the pasting of Q_1 and Q_2 at σ is also in $\mathcal{Q}$.

(c) Prove that the set $\mathbb{P}_{loc}(X)$ is m-stable.
Hint: The monotone convergence theorem guarantees that, for all $Y \geq 0$,

$$\mathbb{E}_{\tilde{Q}}[Y] = \mathbb{E}_Q\left[\mathbb{E}_R\left[Y \mid \mathcal{F}_\sigma\right]\right].$$

One can show (left as bonus exercise–see Lemma 6.41 in Hans Föllmer and Alexander Schied, "Stochastic Finance: An Introduction in Discrete Time", de Gruyter) that for all stopping times $\tau : \Omega \to \{0, \dots, T\}$*, and* $\mathcal{F}_T$ *measurable random variable* $Y \geq 0$ *we have*

$$\mathbb{E}_{\tilde{Q}}[Y \mid \mathcal{F}_\tau] = \mathbb{E}_Q\left[\mathbb{E}_R\left[Y \mid \mathcal{F}_{\sigma \vee \tau}\right] \mid \mathcal{F}_\tau\right].$$