Introduction to Mathematical Finance

Exercise sheet 9

Exercise 9.1 Let $(Y_t)_{0\leq t\leq T}$ be a given integrable adapted discrete-time process. Define an adapted process $(U_t)_{0\leq t\leq T}$ by the recursion

$$U_T = Y_T$$
$$U_t = \max\left(Y_t, E[U_{t+1}|\mathcal{F}_t]\right) \quad \text{for } 0 \leq t \leq T-1$$

The process $(U_t)_{0\leq t\leq T}$ is called the Snell envelope of $(Y_t)_{0\leq t\leq T}$. For simplicity, we suppose in this exercise that $\mathcal{F}_0$ is the trivial σ-algebra.

(a) Show that the Snell envelope of a process is the smallest supermartingale dominating that process.

(b) Show that if Y is a supermartingale then $U_t = Y_t$ for all t, and if Y is submartingale, then $U_t = E[Y_T|\mathcal{F}_t]$.

(c) Let τ be any stopping time taking values in $\{0,...,T\}$. Show that the process $(U_{t\wedge\tau})_{0\leq t\leq T}$ is a supermartingale.

Define the random time τ^* by

$$\tau^* = \min\{t \in \{0,...,T\} \text{ such that } U_t = Y_t\}$$

(d) Show that τ^* is a stopping time. Furthermore, show that the process $(U_{t\wedge\tau^*})_{0\leq t\leq T}$ is a martingale and, in particular, $U_0 = E[Y_{\tau^*}]$

(e) Show that $U_0 = \sup\{E[Y_\tau] : \text{stopping times } 0 \leq \tau \leq T\}$

(f) Conclude that τ^* is an optimal stopping time, i.e. a solution to the problem of finding a stopping time $\tau \leq T$ that achieves the supremum in $\sup_{\tau\leq T} E[Y_\tau]$.

(g) Give a financial example where this result could be used.

Solution 9.1

(a) The integrability and adaptedness of U follow from the same properties of Y. Moreover, by definition, $U_t \geq Z_t$ and $U_t \geq E[U_{t+1}|\mathcal{F}_t]$ for all $0 \leq t \leq T$ hence the Snell envelope U of the process Y is a supermartingale dominating the process Y. Remains to show that U is the smallest such process. Let $V = (V_n)$ be any other supermartingale dominating Y, i.e. $V_n \geq Y_n$ for all n. We have to show that V dominates U as well. We do this by (backwards) induction. First, since $U_T = Y_T$ and V dominates Y, we have $V_T \geq U_T$. Assume inductively that $V_t \geq U_t$. Then

$$\begin{aligned} V_{t-1} &\geq E[V_t|\mathcal{F}_{t-1}] \quad \textit{as } V \textit{ is a supermartingale} \\ &\geq E[U_t|\mathcal{F}_{t-1}] \quad \textit{by the induction hypotheses} \end{aligned}$$

and also $V_{t-1} \geq Y_{t-1}$ as V dominates Y. Combining these two observations and using the definition of Snell enveleppe, we have

$$V_{t-1} \geq U_{t-1}$$

as required.

(b) In both cases we proceed by induction.

First suppose that Y is a supermartingale, and that we have proved $U_{t+1} = Y_{t+1}$ for some $t < T$. Then

$$U_t = \max\left(Y_t, E[U_{t+1}|\mathcal{F}_t]\right) = \max\left(Y_t, E[Y_{t+1}|\mathcal{F}_t]\right) = Y_t$$

where in the second equality we used the induction hypotheses and the last equality holds since Y is a supermartingale by assumption.

Now suppose that Y is a submartingale and that we have proved $U_{t+1} = E[Y_T|\mathcal{F}_{t+1}]$ for some $t < T$. Then

$$U_t = \max\left(Y_t, E[U_{t+1}|\mathcal{F}_t]\right) = \max\Big(Y_t, E\left[E[Y_T|\mathcal{F}_{t+1}]|\mathcal{F}_t\right]\Big) = E[Y_T|\mathcal{F}_t]$$

where in the second equality we used the induction hypotheses and the last equality holds by the tower property and the assumption that Y is a submartingale.

(c) Note that $U_{(t+1)\wedge\tau} - U_{t\wedge\tau} = \mathbb{1}_{t+1\leq\tau}(U_{t+1} - U_t)$. The supermartingale property of $(U_{t\wedge\tau})_{0\leq t\leq T}$ now immediately follows from the supermartingale property of U

$$\begin{aligned} E\left[U_{(t+1)\wedge\tau} - U_{t\wedge\tau}|\mathcal{F}_t\right] &= E\left[\mathbb{1}_{t+1\leq\tau}(U_{t+1} - U_t)|\mathcal{F}_t\right] \\ &= \mathbb{1}_{t+1\leq\tau} E\left[(U_{t+1} - U_t)|\mathcal{F}_t\right] \\ &\leq 0 \end{aligned}$$

where in the second line we used that $\mathbb{1}_{t+1\leq\tau} = 1 - \mathbb{1}_{\tau\leq t}$ is $\mathcal{F}_t$ measurable and the last inequality holds since U is a supermartingale.

(d) The event

$$\{\tau^* > t\} = \{Y_0 < U_0, ..., Y_t < U_t\}$$

is $\mathcal{F}_t$ measurable since both U and Y are adapted processes, hence τ^* is indeed a stopping time.

Now note that on the event $\{t+1 \leq \tau^*\}$, $U_t = E[U_{t+1}|\mathcal{F}_t]$ by definition of the Snell envelope U. Hence using the same observation as in (c), we have

$$\begin{aligned} U_{(t+1)\wedge\tau^*} - U_{t\wedge\tau^*} &= \mathbb{1}_{t+1\leq\tau^*}(U_{t+1} - U_t) \\ &= \mathbb{1}_{t+1\leq\tau^*}(U_{t+1} - E[U_{t+1}|\mathcal{F}_t]) \end{aligned}$$

Taking the expectations on both sides gives the martingale property of the process $(U_{t\wedge\tau^*})_{0\leq t\leq T}$. In particalar, we have

$$E[Y_{\tau^*}] = E[U_{\tau^*}] = U_0$$

(e) Since U is a supermartingale,

$$U_0 \geq E[U_\tau]$$

for any stopping time τ by the Optional Stopping Theorem. But since $U_t \geq Y_t$ by construction of the Snell envelope, we also have

$$U_0 \geq E[U_\tau] \geq E[Y_\tau]$$

Taking the supremum over stopping times $0 \leq \tau \leq T$ on both sides gives

$$U_0 \geq \sup\{E[Y_\tau] : \text{stopping times } 0 \leq \tau \leq T\}$$

For the other inequality we note that for $\tau^* = \min\{t \in \{0, ..., T\}$ such that $U_t = Y_t\}$, we have by the previous question

$$U_0 = E[Y_{\tau^*}]$$

and hence

$$U_0 \leq \sup\{E[Y_\tau] : \text{stopping times } 0 \leq \tau \leq T\}$$

(f) Follow directly from (d) and (e)

$$U_0 = E[Y_{\tau^*}] = \sup\{E[Y_\tau] : \text{stopping times } 0 \leq \tau \leq T\}$$

(g) Optimal exercise of American Options.

Exercise 9.2

(a) Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$ and $Y = (Y_k)_{k\in\mathbb{N}_0}$ a supermartingale with respect to P and $\mathbb{F}$. Show that Y can be uniquely decomposed as $Y = Y_0 + M - A$, where M is a martingale with $M_0 = 0$ and A is predictable and increasing (i.e., $A_k \leq A_{k+1}$ P-a.s. for all k) with $A_0 = 0$. This is the so-called *Doob decomposition* of Y.

(b) Now consider the trinomial market from Exercise 8.1 with $T = 1$ and define the process $U = (U_0, U_1)$ by $U_1 := H \in L^1_+(\mathcal{F}_1)$ and $U_0 :=$ superreplication price of H at time 0. Show that U is a Q-supermartingale for every $Q \in \mathbb{P}$, and find its Doob decomposition $U = U_0 + M^Q - A^Q$ for every $Q \in \mathbb{P}$.

Solution 9.2

(a) If Y has a Doob decomposition as in above, then, since M is a martingale and A is predictable, we have

$$E[Y_k - Y_{k-1}|\mathcal{F}_{k-1}] = E[M_k - M_{k-1}|\mathcal{F}_{k-1}] - E[A_k - A_{k-1}|\mathcal{F}_{k-1}] = A_{k-1} - A_k.$$

In particular, since $A_0 = 0$, we have

$$A_k = -\sum_{j=1}^{k} E[Y_j - Y_{j-1}|\mathcal{F}_{j-1}].$$

So defining A as such yields the Doob decomposition of Y. Uniqueness of the decomposition follow from the observation that predictable martingales are constant.

(b) Note that $U_0 = \sup_{Q\in\mathbb{P}} E_Q[H]$. Fix $Q \in \mathbb{P}$. Then

$$E_Q[U_1|\mathcal{F}_0] = E_Q[H] \leq U_0.$$

Therefore U is a Q-supermartingale.

As in part (a), we set

$$A_1 := -E[U_1 - U_0|\mathcal{F}_0] = \sup_{P\in\mathbb{P}} E_P[H] - E_Q[H]$$

and

$$M_1 := U_1 - U_0 + A_1 = H - E_Q[H].$$

So the Doob decomposition of U is

$$U_0 = \sup_{P\in\mathbb{P}} E_P[H],$$
$$U_1 = U_0 + M_1 + A_1 = \sup_{P\in\mathbb{P}} E_P[H] + (H - E_Q[H]) - (\sup_{P\in\mathbb{P}} E_P[H] - E_Q[H]).$$

Exercise 9.3 Let (S^0, S^1) be an *arbitrage-free* financial market with time horizon T and assume that the bank account process $S^0 = (S^0_k)_{k=0,1,\dots,T}$ is given by $S^0_k = (1+r)^k$ for a constant $r \geq 0$. As usual, denote the set of all EMMs for S^1 with numeraire S_0 by $\mathbb{P}(S^0)$. Fix a $K > 0$. The undiscounted payoff of a *European call option* on S^1 with strike K and maturity $k \in \{1, \dots, T\}$ is denoted by C^E_k and given by

$$C^E_k = \left(S^1_k - K\right)^+,$$

whereas the undiscounted payoff of an *Asian call option* on S^1 with strike K and maturity $k \in \{1, \dots, T\}$ is denoted by C^A_k and given by

$$C^A_k := \left(\frac{1}{k}\sum_{j=1}^{k} S^1_j - K\right)^+.$$

(a) Fix a $Q \in \mathbb{P}(S^0)$ and show that the function $\{1, \dots, T\} \to \mathbb{R}_+$, $k \mapsto E_Q\left[\frac{C^E_k}{S^0_k}\right]$ is increasing.
Hint: Use Jensen's inequality for conditional expectations.

(b) Fix a $Q \in \mathbb{P}(S^0)$ and show that for all $k = 1, \dots, T$, we have

$$E_Q\left[\frac{C^A_k}{S^0_k}\right] \leq \frac{1}{k}\sum_{j=1}^{k} E_Q\left[\frac{C^E_j}{S^0_j}\right]$$

(c) Fix a $Q \in \mathbb{P}(S^0)$ and deduce that for all $k = 1, \dots, T$, we have

$$E_Q\left[\frac{C^A_k}{S^0_k}\right] \leq E_Q\left[\frac{C^E_k}{S^0_k}\right].$$

Interpret this inequality.

Solution 9.3

(a) It clearly suffices to show that for all $k = 1, \dots, T-1$, we have

$$E_Q\left[\frac{C^E_{k+1}}{S^0_{k+1}}\right] \geq E_Q\left[\frac{C^E_k}{S^0_k}\right]$$

Fix a $k \in \{1, \dots, T-1\}$. Using the *tower property* of conditional expectation, *Jensen's inequality* for conditional expectations (for the convex function $x \mapsto x^+$), the fact that S^1 is a Q-martingale and that $S^0_k = (1+r)$ is deterministic with $r \geq 0$, we get

$$
\begin{aligned}
E_Q\left[\frac{C^E_{k+1}}{S^0_{k+1}}\right] &= E_Q\left[\frac{\left(S^1_{k+1}-K\right)^+}{S^0_{k+1}}\right] \\
&= E_Q\left[\Big(\frac{S^1_{k+1}}{S^0_{k+1}} - \frac{K}{S^0_{k+1}}\Big)^+\right] \\
&= E_Q\left[E_Q\left[\Big(\frac{S^1_{k+1}}{S^0_{k+1}} - \frac{K}{S^0_{k+1}}\Big)^+\middle|\mathcal{F}_k\right]\right] \\
&\geq E_Q\left[\left(E_Q\left[\frac{S^1_{k+1}}{S^0_{k+1}} - \frac{K}{S^0_{k+1}}\middle|\mathcal{F}_k\right]\right)^+\right] \\
&= E_Q\left[\Big(\frac{S^1_k}{S^0_k} - \frac{K}{S^0_{k+1}}\Big)^+\right] \\
&= E_Q\left[\Big(\frac{S^1_k}{S^0_k} - \frac{K}{(1+r)S^0_k}\Big)^+\right] \\
&\geq E_Q\left[\Big(\frac{S^1_k}{S^0_k} - \frac{K}{S^0_k}\Big)^+\right] \\
&= E_Q\left[\frac{C^E_k}{S^0_k}\right]
\end{aligned}
$$

(b) Since the function $x \mapsto x^+$ is convex, we have for $k = 1, \ldots, T$

$$
\begin{aligned}
C^A_k &= \left(\frac{1}{k}\sum_{j=1}^k S^1_j - K\right)^+ = \left(\sum_{j=1}^k \frac{1}{k}\Big(S^1_j - K\Big)\right)^+ \\
&\leq \sum_{j=1}^k \frac{1}{k}\left(S^1_j - K\right)^+ = \frac{1}{k}\sum_{j=1}^k C^E_j.
\end{aligned}
$$

By *linearity* and *monotonicity* of expectations and since $r \geq 0$, we get

$$
\begin{aligned}
E_Q\left[\frac{C^A_k}{S^0_k}\right] &= E_Q\left[\frac{C^A_k}{(1+r)^k}\right] \leq \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{C^E_j}{(1+r)^k}\right] \\
&\leq \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{C^E_j}{(1+r)^j}\right] = \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{C^E_j}{S^0_j}\right]
\end{aligned}
$$

(c) Putting the results of (a) and (b) together yields for $k = 1, \ldots, T$

$$
E_Q\left[\frac{C^A_k}{S^0_k}\right] \leq \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{C^E_j}{S^0_j}\right] \leq \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{C^E_k}{S^0_k}\right] = E_Q\left[\frac{C^E_k}{S^0_k}\right]
$$

This means nothing else than that for a fixed EMM, the price of an Asian call option on S^1 is smaller than the price of the European call option on the same asset with the same strike price K and same maturity $k \in \{1, \dots, T\}$.

This makes sense also intuitively since the price of a call option is increasing in the volatility of the underlying (because the probability of ending up in the money is higher), and averaging in the Asian call option amounts to reducing volatility of the underlying.