Introduction to Mathematical Finance

Exercise sheet 9

Exercise 9.1 Let $(Y_t)_{0\le t\le T}$ be a given integrable adapted discrete-time process. Define an adapted process $(U_t)_{0\le t\le T}$ by the recursion

$$\begin{aligned} U_T &= Y_T \\ U_t &= \max\left(Y_t, E[U_{t+1}|\mathcal{F}_t]\right) \quad \text{for } 0 \le t \le T-1 \end{aligned}$$

The process $(U_t)_{0\le t\le T}$ is called the Snell envelope of $(Y_t)_{0\le t\le T}$. For simplicity, we suppose in this exercise that $\mathcal{F}_0$ is the trivial σ-algebra.

(a) Show that the Snell envelope of a process is the smallest supermartingale dominating that process.

(b) Show that if Y is a supermartingale then $U_t = Y_t$ for all t, and if Y is submartingale, then $U_t = E[Y_T|\mathcal{F}_t]$.

(c) Let τ be any stopping time taking values in $\{0, ..., T\}$. Show that the process $(U_{t\wedge\tau})_{0\le t\le T}$ is a supermartingale.

Define the random time τ^* by

$$\tau^* = \min\{t \in \{0, ..., T\} \text{ such that } U_t = Y_t\}$$

(d) Show that τ^* is a stopping time. Furthermore, show that the process $(U_{t\wedge\tau^*})_{0\le t\le T}$ is a martingale and, in particular, $U_0 = E[Y_{\tau^*}]$

(e) Show that $U_0 = \sup\{E[Y_\tau] : \text{stopping times } 0 \le \tau \le T\}$

(f) Conclude that τ^* is an optimal stopping time, i.e. a solution to the problem of finding a stopping time $\tau \le T$ that achieves the supremum in $\sup_{\tau\le T} E[Y_\tau]$.

(g) Give a financial example where this result could be used.

Exercise 9.2

(a) Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$ and $Y = (Y_k)_{k\in\mathbb{N}_0}$ a supermartingale with respect to P and $\mathbb{F}$. Show that Y can be uniquely decomposed as $Y = Y_0 + M - A$, where M is a martingale with $M_0 = 0$ and A is predictable and increasing (i.e., $A_k \le A_{k+1}$ P-a.s. for all k) with $A_0 = 0$. This is the so-called *Doob decomposition* of Y.

(b) Now consider the trinomial market from Exercise 8.1 with $T = 1$ and define the process $U = (U_0, U_1)$ by $U_1 := H \in L^1_+(\mathcal{F}_1)$ and $U_0 :=$ superreplication price of H at time 0. Show that U is a Q-supermartingale for every $Q \in \mathbb{P}$, and find its Doob decomposition $U = U_0 + M^Q - A^Q$ for every $Q \in \mathbb{P}$.

Exercise 9.3 Let (S^0, S^1) be an *arbitrage-free* financial market with time horizon T and assume that the bank account process $S^0 = (S^0_k)_{k=0,1,\dots,T}$ is given by $S^0_k = (1+r)^k$ for a constant $r \geq 0$. As usual, denote the set of all EMMs for S^1 with numeraire S_0 by $\mathbb{P}(S^0)$. Fix a $K > 0$. The undiscounted payoff of a *European call option* on S^1 with strike K and maturity $k \in \{1, \dots, T\}$ is denoted by C^E_k and given by

$$C^E_k = \left(S^1_k - K\right)^+,$$

whereas the undiscounted payoff of an *Asian call option* on S^1 with strike K and maturity $k \in \{1, \dots, T\}$ is denoted by C^A_k and given by

$$C^A_k := \left(\frac{1}{k}\sum_{j=1}^{k} S^1_j - K\right)^+.$$

(a) Fix a $Q \in \mathbb{P}(S^0)$ and show that the function $\{1, \dots, T\} \to \mathbb{R}_+$, $k \mapsto E_Q\left[\frac{C^E_k}{S^0_k}\right]$ is increasing.
Hint: Use Jensen's inequality for conditional expectations.

(b) Fix a $Q \in \mathbb{P}(S^0)$ and show that for all $k = 1, \dots, T$, we have

$$E_Q\left[\frac{C^A_k}{S^0_k}\right] \leq \frac{1}{k}\sum_{j=1}^{k} E_Q\left[\frac{C^E_j}{S^0_j}\right]$$

(c) Fix a $Q \in \mathbb{P}(S^0)$ and deduce that for all $k = 1, \dots, T$, we have

$$E_Q\left[\frac{C^A_k}{S^0_k}\right] \leq E_Q\left[\frac{C^E_k}{S^0_k}\right].$$

Interpret this inequality.