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Introduction to Mathematical Finance

Solution sheet 9

Solution 9.1

(a) The integrability and adaptedness of U follow from the same properties of Y. Moreover, by definition, $U_t \geq Z_t$ and $U_t \geq E[U_{t+1}|\mathcal{F}_t]$ for all $0 \leq t \leq T$ hence the Snell envelope U of the process Y is a supermartingale dominating the process Y. Remains to show that U is the smallest such process. Let $V = (V_n)$ be any other supermartingale dominating Y, i.e. $V_n \geq Y_n$ for all n. We have to show that V dominates U as well. We do this by (backwards) induction. First, since $U_T = Y_T$ and V dominates Y, we have $V_T \geq U_T$. Assume inductively that $V_t \geq U_t$. Then

$$\begin{aligned} V_{t-1} &\geq E[V_t|\mathcal{F}_{t-1}] \quad \textit{as } V \textit{ is a supermartingale} \\ &\geq E[U_t|\mathcal{F}_{t-1}] \quad \textit{by the induction hypotheses} \end{aligned}$$

and also $V_{t-1} \geq Y_{t-1}$ as V dominates Y. Combining these two observations and using the definition of Snell enveleppe, we have

$$V_{t-1} \geq U_{t-1}$$

as required.

(b) In both cases we proceed by induction.

First suppose that Y is a supermartingale, and that we have proved $U_{t+1} = Y_{t+1}$ for some $t < T$. Then

$$U_t = \max\left(Y_t, E[U_{t+1}|\mathcal{F}_t]\right) = \max\left(Y_t, E[Y_{t+1}|\mathcal{F}_t]\right) = Y_t$$

where in the second equality we used the induction hypotheses and the last equality holds since Y is a supermartingale by assumption.

Now suppose that Y is a submartingale and that we have proved $U_{t+1} = E[Y_T|\mathcal{F}_{t+1}]$ for some $t < T$. Then

$$U_t = \max\left(Y_t, E[U_{t+1}|\mathcal{F}_t]\right) = \max\Big(Y_t, E\left[E[Y_T|\mathcal{F}_{t+1}]|\mathcal{F}_t\right]\Big) = E[Y_T|\mathcal{F}_t]$$

where in the second equality we used the induction hypotheses and the last equality holds by the tower property and the assumption that Y is a submartingale.

(c) Note that $U_{(t+1)\wedge\tau} - U_{t\wedge\tau} = \mathbb{1}_{t+1\leq\tau}(U_{t+1} - U_t)$. The supermartingale property of $(U_{t\wedge\tau})_{0\leq t\leq T}$ now immediately follows from the supermartingale property of U

$$\begin{aligned} E\left[U_{(t+1)\wedge\tau} - U_{t\wedge\tau}|\mathcal{F}_t\right] &= E\left[\mathbb{1}_{t+1\leq\tau}(U_{t+1} - U_t)|\mathcal{F}_t\right] \\ &= \mathbb{1}_{t+1\leq\tau}E\left[(U_{t+1} - U_t)|\mathcal{F}_t\right] \\ &\leq 0 \end{aligned}$$

where in the second line we used that $\mathbb{1}_{t+1\leq\tau} = 1 - \mathbb{1}_{\tau\leq t}$ is $\mathcal{F}_t$ measurable and the last inequality holds since U is a supermartingale.

(d) The event

$$\{\tau^* > t\} = \{Y_0 < U_0, ..., Y_t < U_t\}$$

is $\mathcal{F}_t$ measurable since both U and Y are adapted processes, hence τ^* is indeed a stopping time.

Now note that on the event $\{t+1 \leq \tau^*\}$, $U_t = E[U_{t+1}|\mathcal{F}_t]$ by definition of the Snell envelope U. Hence using the same observation as in (c), we have

$$\begin{aligned} U_{(t+1)\wedge\tau^*} - U_{t\wedge\tau^*} &= \mathbb{1}_{t+1\leq\tau^*}(U_{t+1} - U_t) \\ &= \mathbb{1}_{t+1\leq\tau^*}(U_{t+1} - E[U_{t+1}|\mathcal{F}_t]) \end{aligned}$$

Taking the expectations on both sides gives the martingale property of the process $(U_{t\wedge\tau^*})_{0\leq t\leq T}$. In particalar, we have

$$E[Y_{\tau^*}] = E[U_{\tau^*}] = U_0$$

(e) Since U is a supermartingale,

$$U_0 \geq E[U_\tau]$$

for any stopping time τ by the Optional Stopping Theorem. But since $U_t \geq Y_t$ by construction of the Snell envelope, we also have

$$U_0 \geq E[U_\tau] \geq E[Y_\tau]$$

Taking the supremum over stopping times $0 \leq \tau \leq T$ on both sides gives

$$U_0 \geq \sup\{E[Y_\tau] : \text{stopping times } 0 \leq \tau \leq T\}$$

For the other inequality we note that for $\tau^* = \min\{t \in \{0, ..., T\}$ such that $U_t = Y_t\}$, we have by the previous question

$$U_0 = E[Y_{\tau^*}]$$

and hence

$$U_0 \leq \sup\{E[Y_\tau] : \text{stopping times } 0 \leq \tau \leq T\}$$

(f) Follow directly from (d) and (e)

$$U_0 = E[Y_{\tau^*}] = \sup\{E[Y_\tau] : \text{stopping times } 0 \le \tau \le T\}$$

(g) Optimal exercise of American Options.

Solution 9.2

(a) If Y has a Doob decomposition as in above, then, since M is a martingale and A is predictable, we have

$$E[Y_k - Y_{k-1}|\mathcal{F}_{k-1}] = E[M_k - M_{k-1}|\mathcal{F}_{k-1}] - E[A_k - A_{k-1}|\mathcal{F}_{k-1}] = A_{k-1} - A_k.$$

In particular, since $A_0 = 0$, we have

$$A_k = -\sum_{j=1}^{k} E[Y_j - Y_{j-1}|\mathcal{F}_{j-1}].$$

So defining A as such yields the Doob decomposition of Y. Uniqueness of the decomposition follow from the observation that predictable martingales are constant.

(b) Note that $U_0 = \sup_{Q\in\mathbb{P}} E_Q[H]$. Fix $Q \in \mathbb{P}$. Then

$$E_Q[U_1|\mathcal{F}_0] = E_Q[H] \le U_0.$$

Therefore U is a Q-supermartingale.

As in part (a), we set

$$A_1 := -E[U_1 - U_0|\mathcal{F}_0] = \sup_{P\in\mathbb{P}} E_P[H] - E_Q[H]$$

and

$$M_1 := U_1 - U_0 + A_1 = H - E_Q[H].$$

So the Doob decomposition of U is

$$U_0 = \sup_{P\in\mathbb{P}} E_P[H],$$
$$U_1 = U_0 + M_1 + A_1 = \sup_{P\in\mathbb{P}} E_P[H] + (H - E_Q[H]) - (\sup_{P\in\mathbb{P}} E_P[H] - E_Q[H]).$$

Solution 9.3

(a) It clearly suffices to show that for all $k = 1, \dots, T-1$, we have

$$E_Q\left[\frac{C^E_{k+1}}{S^0_{k+1}}\right] \ge E_Q\left[\frac{C^E_k}{S^0_k}\right]$$

Fix a $k \in \{1, \dots, T-1\}$. Using the *tower property* of conditional expectation, *Jensen's inequality* for conditional expectations (for the convex function $x \mapsto x^+$), the fact that S^1 is a Q-martingale and that $S^0_k = (1+r)$ is deterministic with $r \geq 0$, we get

$$\begin{aligned}
E_Q\left[\frac{C^E_{k+1}}{S^0_{k+1}}\right] &= E_Q\left[\frac{\left(S^1_{k+1} - K\right)^+}{S^0_{k+1}}\right] \\
&= E_Q\left[\left(\frac{S^1_{k+1}}{S^0_{k+1}} - \frac{K}{S^0_{k+1}}\right)^+\right] \\
&= E_Q\left[E_Q\left[\left(\frac{S^1_{k+1}}{S^0_{k+1}} - \frac{K}{S^0_{k+1}}\right)^+ \middle| \mathcal{F}_k\right]\right] \\
&\geq E_Q\left[\left(E_Q\left[\frac{S^1_{k+1}}{S^0_{k+1}} - \frac{K}{S^0_{k+1}} \middle| \mathcal{F}_k\right]\right)^+\right] \\
&= E_Q\left[\left(\frac{S^1_k}{S^0_k} - \frac{K}{S^0_{k+1}}\right)^+\right] \\
&= E_Q\left[\left(\frac{S^1_k}{S^0_k} - \frac{K}{(1+r)S^0_k}\right)^+\right] \\
&\geq E_Q\left[\left(\frac{S^1_k}{S^0_k} - \frac{K}{S^0_k}\right)^+\right] \\
&= E_Q\left[\frac{C^E_k}{S^0_k}\right]
\end{aligned}$$

(b) Since the function $x \mapsto x^+$ is convex, we have for $k = 1, \dots, T$

$$\begin{aligned}
C^A_k = \left(\frac{1}{k}\sum_{j=1}^k S^1_j - K\right)^+ &= \left(\sum_{j=1}^k \frac{1}{k}\left(S^1_j - K\right)\right)^+ \\
&\leq \sum_{j=1}^k \frac{1}{k}\left(S^1_j - K\right)^+ = \frac{1}{k}\sum_{j=1}^k C^E_j.
\end{aligned}$$

By *linearity* and *monotonicity* of expectations and since $r \geq 0$, we get

$$\begin{aligned}
E_Q\left[\frac{C^A_k}{S^0_k}\right] = E_Q\left[\frac{C^A_k}{(1+r)^k}\right] &\leq \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{C^E_j}{(1+r)^k}\right] \\
&\leq \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{C^E_j}{(1+r)^j}\right] = \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{C^E_j}{S^0_j}\right]
\end{aligned}$$

(c) Putting the results of (a) and (b) together yields for $k = 1, \ldots, T$

$$E_Q\left[\frac{C_k^A}{S_k^0}\right] \leq \frac{1}{k}\sum_{j=1}^{k} E_Q\left[\frac{C_j^E}{S_j^0}\right] \leq \frac{1}{k}\sum_{j=1}^{k} E_Q\left[\frac{C_k^E}{S_k^0}\right] = E_Q\left[\frac{C_k^E}{S_k^0}\right]$$

This means nothing else than that for a fixed EMM, the price of an Asian call option on S^1 is smaller than the price of the European call option on the same asset with the same strike price K and same maturity $k \in \{1, \ldots, T\}$.

This makes sense also intuitively since the price of a call option is increasing in the volatility of the underlying (because the probability of ending up in the money is higher), and averaging in the Asian call option amounts to reducing volatility of the underlying.