Existence of Minimisers in the Plateau Problem

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The Plateau Problem

Given a boundary $\Gamma$ in $\mathbb{R}^n$, does there exist a surface that is bounded by $\Gamma$ such that the area of it is minimal.
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Sets of finite perimeter

We have the following Plateau type problem in some $A \subset \mathbb{R}^n$ with boundary data $E_0$,

$$\gamma = \inf\{ P(E) : E \setminus A = E_0 \setminus A, P(E) < \infty \}.$$
The existence of minimisers can be established using the direct method:

1. Construct a minimising sequence \( \{E_h\}_{h \in \mathbb{N}} \)
2. Extract a convergent subsequence, \( E_{h_k} \to E \)
3. Show that \( P(E) = \gamma \)
For a $k$-dimensional set, $M \subset \mathbb{R}^n$ we settle on the $k$-dimensional Hausdorff measure as our area functional. That is

$$A(M) = \int_M d\mathcal{H}^k.$$
"Bounded by"

We will take bounded by $\Gamma$ to mean that the surface $\Sigma$ satisfies $\partial \Sigma = \Gamma$. 
Surface
Outline

Submanifolds
Currents
Varifolds
Submanifolds

How can we tell if a given submanifold $M \subset \mathbb{R}^n$ has minimal area?
Monotonicity and Density

**Theorem (Monotonicity)**

Suppose $M$, $k$-dimensional, is stationary in $\mathbb{R}^n$ and fix $x \in \mathbb{R}^n$. Then

$$\omega_k^{-1} r^{-k} A(M \cap B_r(x))$$

is an increasing function of $r$ for $0 < r \leq \text{dist}(x, \partial M)$.

We define the density of $M$ at $p \in M \setminus \partial M$ to be

$$\Theta(M, p) = \lim_{r \to 0} \omega_k^{-1} r^{-k} A(M \cap B_r(x)).$$
The disk with Spines

**Figure:** A minimising sequence may not converge to a Minimal Surface
Fleming’s Example

**Figure:** A minimal surface with infinite genus
Currents

Let $U \subset \mathbb{R}^n$ and we let $D^k(U)$ denote the space of compactly supported $k$-forms on $U$.

**Definition (Currents)**

A $k$-current is a linear functional on $D^k(U)$. The space of $k$-currents, we will denote as $D_k(U)$.

Given $T \in D_k(U)$, the boundary is defined to be $\partial T \in D_{k-1}$ such that for any $\omega \in D^{k-1}(U)$

$$\partial T(\omega) = T(d\omega).$$
Mass of Currents

Definition (Mass)
Given a current $T$, we define its mass to be

$$
\mathbb{M}(T) = \sup_{\|\omega\| \leq 1} T(\omega).
$$
Theorem

The space of currents with finite mass is a Banach space with norm $\mathbb{M}$. 

Currents
Plateau Problem

Given a $k$-current $S$ with $\partial S = \emptyset$, is there a $k + 1$-current $T$ such that $\partial T = S$ and $\mathbb{M}(T)$ is minimal.
Rectifiable Integer Currents

Definition
Let $U \subset \mathbb{R}^n$. A rectifiable integer $k$-current (integral $k$-current) is a current $T$ such that for $\omega \in D^k(U)$

$$T(\omega) = \int_M <\omega(x), \xi(x)> \theta(x) d\mathcal{H}^k(x),$$

where $M \subset U$ is countably $k$-rectifiable, $\theta$ is a $\mathcal{H}^k$ integrable function that takes values positive integers and $\xi : M \rightarrow (\Lambda^k(\mathbb{R}^n))^*$ is $\mathcal{H}^k$ measurable which can be expressed at almost every point $x$, $\xi(x) = \tau_1 \wedge \cdots \wedge \tau_n$ where $\tau_1, \ldots, \tau_n$ is an orthonormal basis for the approximate tangent plane.

$\theta$ is called the multiplicity and $\xi$ is called the orientation.
Theorem (Federer and Flemming Compactness Theorem)

If \( \{ T_j \} \subset D_k(U) \) is a sequence of integral currents with

\[
\sup(\mathbb{M}_W(T_j) + \mathbb{M}_W(\partial T_j)) < \infty
\]

for all \( W \subset \subset U \), then \( \{ T_j \} \) is sequentially compact.
Plateau Problem

Given an integral $k$-current $S$ and $\partial S = \emptyset$, is there an integral $k + 1$-current $T$ such that $\partial T = S$ and $\mathbb{M}(T)$ is minimal.
Mobius band

Figure: Not every minimal surface is orientable
Rectifiable Varifolds

Let $M$ be a countably $k$-rectifiable subset of $\mathbb{R}^n$ and $\theta$ be a locally $\mathcal{H}^k$-integrable function on $M$. The rectifiable $n$-varifold $\nu(M, \theta)$ is the set of all equivalence classes of $(M, \theta)$, where $(M, \theta) \equiv (N, \phi)$ if $\mathcal{H}^k((M \setminus N) \cup (N \setminus M)) = 0$ and $\phi = \theta \mathcal{H}^k$-a.e on $M \cap N$. $\theta$ is called the multiplicity function. If it is integer valued we will call $V$ an integral varifold.
Rectifiable Varifolds

We associate to a varifold $V = v(M, \theta)$ the Radon measure $\mu_V$ such that for any $\mathcal{H}^k$ measurable set $A$,

$$\mu_V(A) = \int_{A \cap M} \theta d\mathcal{H}^k.$$ 

The mass of a varifold $V$ is

$$\mathcal{M}(V) = \mu_V(\mathbb{R}^n).$$

We say that $V_k \rightarrow V$ if $\mu_{V_k} \rightarrow \mu_V$. 

General Varifolds

We define the Grassmannian, $G(k, n)$ to be the collection of all $k$-dimensional subsets of $\mathbb{R}^n$. Given some $A \subset \mathbb{R}^n$, we define

$$G_k(A) = A \times G(n, k),$$

with the product metric.

**Definition (General Varifold)**
An $k$-varifold is a Radon measure on $G_k(\mathbb{R}^n)$. 
General Varifolds

Given a $k$-varifold $V$ on $G_k U$, there is an associated Radon measure $\mu_V$ on $U$ (weight of $V$) defined for $A \subset U$ as

$$\mu_V(A) = V(\pi^{-1}(A)).$$

The mass of the varifold is

$$\mathbb{M}(V) = \mu_V(U) = V(G_k(U)).$$
First Variation

\[ \delta V(X) = \frac{d}{dt} \mathbb{M}(\phi_t#(V \sqcup G_k(K))) \big|_{t=0} = \int_{G_k(U)} \text{div}_S X dV(x, S). \]  

(1)

\( V \) is stationary if \( \delta V(X) = 0 \) for all \( X : U \rightarrow \mathbb{R}^n \) continuous and compactly supported.
Theorem (Monotonicity)

Suppose \( V \) is stationary in \( U \), then

\[
r^{-n} \mu_V(B_r(x))
\]

is increasing for \( 0 < r < \text{dist}(x, \partial U) \).
Let $V$ be stationary in $U$. The density for $x \in U$ is defined to be

$$
\Theta^k(\mu_V, x) = \omega_k^{-1} r^{-k} \mu_V(B_r(x))
- \omega_k^{-1} \int_{G_k(B_r(x))} r^{-n-2} |p_{S^\perp}(y - x)|^2 dV(y, S),
$$

for $0 < r < \text{dist}(x, \partial U)$. 

**Definition (Density)**
Theorem (Rectifiability)

Suppose $V$ has locally bounded first variation in $U$ and that $\Theta^k(\mu_V, x) > 0$ for $\mu_V$-a.e. $x \in U$. Then $V$ is a $k$-rectifiable varifold.
Theorem (Compactness for Rectifiable Varifolds)

Suppose \( \{V_j\}_{j \in \mathbb{N}} \) is a sequence of rectifiable varifolds that have bounded first variation in \( U \), \( \Theta^k(\mu_{V_j}, x) \geq 1 \) in \( U \) and that

\[
\sup(\mu_{V_j}(U) + \|\delta V_j\|(U)) < \infty.
\]

Then \( \{V_j\}_{j \in \mathbb{N}} \) is sequentially compact and \( \Theta^k(\mu_V, x) \geq 1 \).