Existence of Minimisers in the Plateau Problem

Anthony Salib

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The Plateau Problem

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Sets of finite perimeter

We have the following Plateau type problem in some $A \subset \mathbb{R}^n$ with boundary data E_0,

$$\gamma = \inf\{P(E) : E \backslash A = E_0 \backslash A, P(E) < \infty\}.$$

Sets of finite perimeter

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The existence of minimisers can be established using the direct method:

Construct a minimising sequence $\{E_h\}_{h\in\mathbb{N}}$

Extract a convergent subsequence, $E_{h_k} \to E$

Show that $P(E) = \gamma$

"Area"

For a k-dimensional set, $M \subset \mathbb{R}^n$ we settle on the k-dimensional Hausdorff measure as our area functional. That is

$$\mathcal{A}(M) = \int_M d\mathcal{H}^k.$$

"Bounded by"

We will take bounded by Γ to mean that the surface Σ satisfies $\partial\Sigma = \Gamma$.

Surface

?

Outline

Submanifolds

How can we tell if a given submanifold $M \subset \mathbb{R}^n$ has minimal area?

Monotonicity and Density

Theorem (Monotonicity)

Suppose M, k-dimensional, is stationary in $\mathbb{R}^n$ and fix $x \in \mathbb{R}^n$. Then

$$\omega_k^{-1} r^{-k} \mathcal{A}(M \cap B_r(x))$$

is an increasing function of r for $0 < r \leq dist(x, \partial M)$.

We define the density of M at $p \in M \backslash \partial M$ to be

$$\Theta(M, p) = \lim_{r \to 0} \omega_k^{-1} r^{-k} \mathcal{A}(M \cap B_r(x)).$$

The disk with Spines

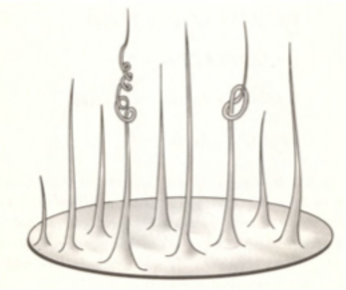

Figure: A minimising sequence may not converge to a Minimal Surface

Fleming's Example

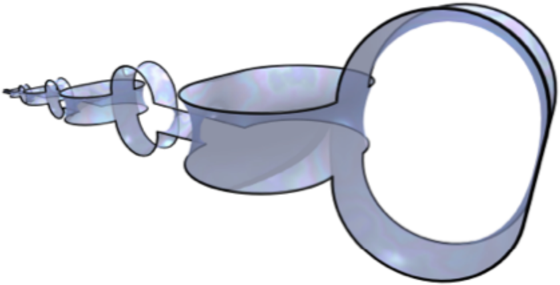

Figure: A minimal surface with infinite genus

Currents

Let $U \subset \mathbb{R}^n$ and we let $D^k(U)$ denote the space of compactly supported k-forms on U.

Definition (Currents)

A k-current is a linear functional on $D^k(U)$. The space of k-currents, we will denote as $D_k(U)$.

Given $T \in D_k(U)$, the boundary is defined to be $\partial T \in D_{k-1}$ such that for any $\omega \in D^{k-1}(U)$

$$\partial T(\omega) = T(d\omega).$$

Mass of Currents

Definition (Mass)

Given a current T, we define it's mass to be

$$\mathbb{M}(T) = \sup_{\|\omega\| \leq 1} T(\omega).$$

Currents

Theorem

The space of currents with finite mass is a Banach space with norm $\mathbb{M}$.

Plateau Problem

Given a k-current S with $\partial S = \emptyset$, is there a $k+1$-current T such that $\partial T = S$ and $\mathbb{M}(T)$ is minimal.

Rectifiable Integer Currents

Definition

Let $U \subset \mathbb{R}^n$. A rectifiable integer k-current (integral k-current)is a current T such that for $\omega \in D^k(U)$

$$T(\omega) = \int_M < \omega(x), \xi(x) > \theta(x) d\mathcal{H}^k(x),$$

where $M \subset U$ is countably k-rectifiable, θ is a $\mathcal{H}^k$ integrable function that takes values positive integers and $\xi : M \to (\Lambda_k(\mathbb{R}^n))^*$ is $\mathcal{H}^k$ measurable which can be expressed at almost every point x, $\xi(x) = \tau_1 \wedge \cdots \wedge \tau_n$ where $\tau_1, \ldots, \tau_n$ is an orthonormal basis for the approximate tangent plane.

θ is called the multiplicity and ξ is called the orientation.

Federer and Fleming Compactness Theorem

Theorem (Federer and Flemming Compactness Theorem)

If $\{T_j\} \subset D_k(U)$ *is a sequence of integral currents with*

$$\sup(\mathbb{M}_W(T_j) + \mathbb{M}_W(\partial T_j)) < \infty$$

for all $W \subset\subset U$, *then* $\{T_j\}$ *is sequentially compact.*

Plateau Problem

Given an integral k-current S and $\partial S = \emptyset$, is there an integral $k+1$-current T such that $\partial T = S$ and $\mathbb{M}(T)$ is minimal.

Mobius band

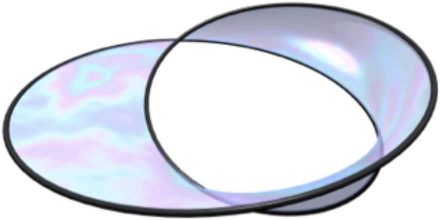

Figure: Not every minimal surface is orientable

Rectifiable Varifolds

Let M be a countably k-rectifiable subset of $\mathbb{R}^n$ and θ be a locally $\mathcal{H}^k$-integrable function on M. The rectifiable n-varifold $\underline{v}(M, \theta)$ is the set of all equivalence classes of (M, θ), where $(M, \theta) \equiv (N, \phi)$ if $\mathcal{H}^k((M \backslash N) \cup (N \backslash M)) = 0$ and $\phi = \theta$ $\mathcal{H}^k$-a.e on $M \cap N$.
θ is called the multiplicity function. If it is integer valued we will call V an integral varifold.

Rectifiable Varifolds

We associate to a varifold $V = \underline{v}(M, \theta)$ the Radon measure μ_V such that for any $\mathcal{H}^k$ measurable set A,

$$\mu_V(A) = \int_{A \cap M} \theta d\mathcal{H}^k.$$

The mass of a varifold V is

$$\mathbb{M}(V) = \mu_V(\mathbb{R}^n).$$

We say that $V_k \to V$ if $\mu_{V_k} \to \mu_V$.

General Varifolds

We define the Grassmannian, $G(k, n)$ to be the collection of all k-dimensional subsets of $\mathbb{R}^n$. Given some $A \subset \mathbb{R}^n$, we define

$$G_k(A) = A \times G(n, k),$$

with the product metric.

Definition (General Varifold)

An k-varifold is a Radon measure on $G_k(\mathbb{R}^n)$.

General Varifolds

Given a k-varifold V on $G_k U$, there is an associated Radon measure μ_V on U (weight of V) defined for $A \subset U$ as

$$\mu_V(A) = V(\pi^{-1}(A)).$$

The mass of the varifold is

$$\mathbb{M}(V) = \mu_V(U) = V(G_k(U)).$$

First Variation

$$\delta V(X) = \frac{d}{dt}\mathbb{M}(\phi_{t_\#}(V \llcorner G_k(K)))|_{t=0} = \int_{G_k(U)} \mathrm{div}_S X dV(x,S). \tag{1}$$

V is stationary if $\delta V(X) = 0$ for all $X : U \to \mathbb{R}^n$ continuous and compactly supported.

Monotonicity

Theorem (Monotonicity)

Suppose V is stationary in U, then

$$r^{-n}\mu_V(B_r(x))$$

is increasing for $0 < r < dist(x, \partial U)$.

Density

Definition (Density)

Let V be stationary in U. The density for $x \in U$ is defined to be

$$\Theta^k(\mu_V, x) = \omega_k^{-1} r^{-k} \mu_V(B_r(x)) - \omega_k^{-1} \int_{G_k(B_r(x))} r^{-n-2} |p_{S^\perp}(y-x)|^2 dV(y, S),$$

for $0 < r < dist(x, \partial U)$.

Rectifiability

Theorem (Rectifiability)

Suppose V has locally bounded first variation in U and that $\Theta^k(\mu_V, x) > 0$ for μ_V-a.e. $x \in U$. Then V is a k-rectifiable varifold.

Compactness

Theorem (Compactness for Rectifiable Varifolds)

Suppose $\{V_j\}_{j\in\mathbb{N}}$ *is a sequence of rectifiable varifolds that have bounded first variation in* U, $\Theta^k(\mu_{V_j}, x) \geq 1$ *in* U *and that*

$$\sup(\mu_{V_j}(U) + \|\delta V_j\|(U)) < \infty.$$

Then $\{V_j\}_{j\in\mathbb{N}}$ *is sequentially compact and* $\Theta^k(\mu_V, x) \geq 1$.