This section was presented by David Morselli with reference [2]. The notes are written by David Morselli, if you find any mistake please feel free to contact the author via dmorselli@student.ethz.ch.

7 Regularity of "multiplicity one" varifolds with small excess

The goal of this talk is to generalize the regularity results that we have studied for sets of finite perimeter in the settings of integral varifolds. The complications are due to the fact that we are no longer in codimension 1 and to the presence of the density: while the former is mainly a technical issue, the latter allows us to prove only a version of the regularity theorem that is weaker than in the case of constant density. The strategy of the proof is exactly the same and most of the intermediate results can be easily adapted; therefore, we will omit some steps and refer to Alessandro Pigati's talks[1] for that.

Before stating the main results, let us fix the notation and recall some basic results. Let $U \subset \mathbb{R}^N$ be an open set and $V = (\Gamma, g)$ be a k-dimensional integer-rectifiable varifold in U with associated measure $\mu_V(A) := \int_{\Gamma\cap A} g(x)\, d\mathcal{H}^k(x)$ for any Borel set $A \subset U$; we will always assume $\mu_V(U) < +\infty$. In the last talk we have defined the first variation of V with respect to a vector field $X \in C^1_c(U, \mathbb{R}^N)$, denoted by $\delta V(X)$, and we have proved that $\delta V(X) = \int_U \operatorname{div}_{T_x\Gamma} X\, d\mu_V(x)$. The last formula implies that the operator $X \mapsto \delta V(X)$ is linear; if it is also bounded, by the Riesz representation theorem there exists a Borel map $H\colon U \to \mathbb{R}^N$ called *generalized mean curvature* such that

$$\delta V(X) = -\int_U \langle X, H\rangle \; d\mu_V$$

If H is bounded in L^∞, it can be proven that for every $x \in U$ the map $r \mapsto e^{\|H\|_\infty r}\mu_V(B_r(x))/r^k$ is increasing. Therefore, it makes sense to define the *density*

$$\theta_V(x) := \lim_{r\to 0} \frac{\mu_V(B_r(x))}{\omega_k r^k}$$

Note that $\theta_V = g$ μ_V-a.e. by standard measure theoretic arguments and θ_V is upper semicontinuous (the proof is the same as in the case of sets of finite perimeter). Note also that $\theta_V \geq 1$ on $\operatorname{spt}(\mu_V) \cap U$: in fact, $\{\theta_V < 1\} \subseteq \{\theta_V \neq g\}$ as g is integer valued and so $\{\theta_V \geq 1\}$ is dense in $\operatorname{spt}(\mu_V)$ (otherwise, there would be an open set $A \subset \operatorname{spt}(\mu_V) \cap \{\theta_V < 1\}$; $\mu_V(A) > 0$ as $A \subset \operatorname{spt}(\mu_V)$ is open, but $\mu_V(A) = 0$ as $\mu_V(\{\theta_V < 1\}) = 0$); the conclusion follows by upper semicontinuity.

In many of the results of this notes we will make the following assumptions, that depends on ε:

$(\mathrm{HP})_\varepsilon$ V is a k-dimensional integral varifold with bounded mean curvature in $B_r(x_0) \subset \mathbb{R}^N$ such that

- $\theta_V(x_0) \geq 1$

[1] Whose notes are available here.

- $\mu_V(B_r(x_0)) < (\omega_k + \varepsilon) r^k$
- $\|H\|_\infty < \varepsilon / r$
- there exists a k-dimensional plane π such that

$$\mathrm{Exc}(\pi, x_0, r) := r^{-k} \int_{B_r(x_0)} \|T_x\Gamma - \pi\|^2 \, d\mu_V(x) < \varepsilon$$

Remark 7.1. In the case of sets of finite perimeter we only considered stationary sets, i.e. with $H = 0$, and in the rest of the seminar we will probably only need the results for the stationary case. The generalization for varifolds with bounded mean curvature is useful in the study of varifolds in compact Riemann manifolds: given a Riemann manifold $M \in \mathbb{R}^N$, we define an integral varifold in $U \cap M$ as an integral varifold in U such that $\mu_V(U \setminus M) = 0$. It is not true in general that $H = 0$ for a stationary varifold in $U \cap M$; however, if M is compact and isometrically embedded in $\mathbb{R}^N$, one can still prove that H is bounded. In any case, the presence of a nonzero mean curvature does not cause any major complication of the proofs.

For the rest of the notes we assume without loss of generality that $\Gamma = \bar{\Gamma} \cap U$ (i.e., Γ is closed in U) and $\Gamma = \mathrm{spt}(\mu_V) \cap U$. The letter C will denote a positive constant only depending on k, and N. By a tiny abuse of notation, we will use the same symbol for a vector subspace of $\mathbb{R}^N$ and the projection into that subspace.

Theorem 7.2. *There exist ε_1, γ positive constants only depending on k and N such that, if V satisfies* (HP) *with ε_1 in $B_r(x_0)$, then $\Gamma \cap B_{r/1000}(x_0)$ is a $C^{1,\gamma}$ submanifold in $B_{r/1000}(x_0)$ and $\theta_V \equiv 1$ on $\Gamma \cap B_{r/1000}(x_0)$.*

Corollary 7.3. *Let V be a k-dimensional integral varifold with bounded mean curvature in U such that $g \equiv h$ μ_V-a.e. for some constant h. Then, there exist an open set $W \subseteq U$ such that $\Gamma \cap W$ is a $C^{1,\gamma}$ submanifold of W, with γ the constant of Theorem 7.2, and $\mu_V(\Gamma \setminus W) = 0$.*

If we drop the assumption that g is constant a.e., we can only conclude that $W \cap \Gamma$ is dense in Γ.

Remark 7.4. This result is the analogous of Corollary 4.10. However, we can no longer conclude that the singular set is negligible without any assumption on the density; as a matter of fact, this is not true for varifolds with nonzero bounded mean curvature. It is still an open problem to understand whether it holds for stationary varifolds.

Proof. Assume without loss of generality that $h = 1$ (otherwise, we could just consider the varifold $(\Gamma, h^{-1}g)$). Fix $R > 0$ such that $R\|H\|_\infty < \varepsilon_1$. Since for $\mathcal{H}^k$-a.e. $x \in \Gamma$

$$1 = \theta_V(x) = \lim_{r \to 0} \frac{\mu_V(B_r(x))}{\omega_k r^k}$$

the first three assumptions of $(\mathrm{HP})_{\varepsilon_1}$ hold in $B_{r_x}(x)$ for some $r_x < R$. Now we observe that $\Gamma = N \cup \bigcup_{j \in \mathbb{N}} K_j$, where $\mathcal{H}^k(N) = 0$ and every K_j is a compact set included in the graph of a C^1 function $f_j \colon \pi_j \to \pi_j^\perp$ for some suitable k-dimensional plane π_j.[2] If $x \in K_j$, we can write

$$\operatorname{Exc}(T_x\Gamma, x, r) \le \Big[\frac{C}{r^k} \mathcal{H}^k(B_r(x) \cap (\Gamma \setminus K_j)) + \\ + \frac{1}{r^k} \int_{B_r(x)\cap(\Gamma\cap K_j)} |Df_j(z) - Df_j(x)|^2 \, d\mathcal{H}^k(z) \Big]$$

The second term is smaller than $C \sup_{z \in B_r(x) \cap K_j} |Df_j(z) - Df_j(x)|^2$, which vanishes as $r \to 0$ by continuity of Df. We now claim that for $\mathcal{H}^k$-a.e. $x \in K_j$ also the first term is infinitesimal. Define

$$F_n \coloneqq \left\{ x \in \Gamma \cap K_j \;\middle|\; \limsup_{r \to 0} \frac{\mathcal{H}^k(B_r(x) \cap (\Gamma \setminus K_j))}{r^k} \ge \frac{1}{n} \right\}$$

so that the union of all the F_n is the set of $x \in K_j$ such that the first term does not vanish. We apply the first conclusion of [1, Theorem 2.56] to the measure $\mu \coloneqq \mathcal{H}^k_{|\Gamma \setminus K_j}$ to obtain $\mathcal{H}^k((\Gamma \setminus K_j) \cap F_n) \ge n^{-1} \mathcal{H}^k(F_n)$; since $(\Gamma \setminus K_j) \cap F_n = \emptyset$ by definition, $\mathcal{H}^k(F_n) = 0$ for every n and so the claim is proved. Therefore, for $\mathcal{H}^k$-a.e. $x \in \Gamma$ we can apply Theorem 7.2 in $B_{r_x}(x)$ for some r_x and conclude that $\Gamma \cap B_{r_x/1000}(x)$ is a $C^{1,\gamma}$ submanifold. The set W is then the union of these balls.

In the general case, for every $n \in \mathbb{N} = \{1, 2, \dots\}$ define the sets $C_n \coloneqq \{ x \in \Gamma \mid \theta_V(x) \ge n \}$, which are closed by upper semicontinuity, $E_n \coloneqq C_n^\circ \setminus C_{n+1}$ and $E \coloneqq \bigcup_{n \in \mathbb{N}} E_n$. We claim that $\Gamma \setminus E \subseteq \bigcup_{n \in \mathbb{N}} C_n \setminus C_n^\circ$: take $x \in \Gamma \setminus E$, we know that $\theta_V(x) \ge 1$ and thus there exists $n \in \mathbb{N}$ such that $x \in C_n \setminus C_{n+1}$; if x were contained in C_n°, then we would have $x \in C_n^\circ \setminus C_{n+1} \subset E$; hence we conclude that $x \in C_n \setminus C_n^\circ$. $\bar\Gamma$ is a closed subset of a completed metric space and Γ is an open subset in the topology of $\bar\Gamma$, hence it is a Baire space. This implies that E is dense in Γ. Now for every n take an open set U_n such that $\Gamma \cap U_n = E_n$ and consider the varifold $V_n \coloneqq (E_n, g_{|U_n})$: by the first part of the proof there exists $W_n \subseteq U_n$ such that $\Gamma \cap W_n$ is a $C^{1,\gamma}$ submanifold and $\mu_{V_n}((\Gamma \cap U_n) \setminus W_n) = 0$. Then, $\Gamma \cap W_n$ is dense in $\Gamma \cap U_n$ (as any open set in $\Gamma \cap U_n$ has positive measure). Finally, the set $W \coloneqq \bigcup W_n$ satisfies the thesis, as $\Gamma \cap W$ is a $C^{1,\gamma}$ submanifold and $\Gamma \cap W$ is dense in E, which is dense in Γ. □

As in the case of sets of finite perimeter, the core of the proof of Theorem 7.2 is the following result.

Theorem 7.5. *There exist $\varepsilon_2 > 0$, $\eta \in (0, 1/2)$ only depending on k and N such that, if V satisfies* (HP) *with ε_2 in $B_r(x_0)$ and $\|H\|_\infty \le r^{-1} \operatorname{Exc}(\pi, x, r)$, then $\operatorname{Exc}(\tilde\pi, x_0, \eta r) \le \frac{1}{2} \operatorname{Exc}(\pi, x_0, r)$ for some k-dimensional plane $\tilde\pi$.*

Tilt-excess inequality

Proposition 7.6. *Let V be a k-dimensional integral varifold of with bounded mean curvature in $B_r(x_0)$, π a k-dimensional plane. Then,*

$$\operatorname{Exc}\Big(\pi, x_0, \frac{r}{2}\Big) \le \frac{C}{r^{k+2}} \int_{B_r(x_0)} \operatorname{dist}(x - x_0, \pi)^2 \, d\mu_V(x) + \frac{C}{r^{k-2}} \int_{B_r(x_0)} |H|^2 \, d\mu_V$$

[2] We mentioned this fact in the first class. A full proof can be obtained from the combination of [1, Theorem 2.76] and the Lusin type result for Lipschitz maps of [3, Theorem 6.11].

Proof. Assume without loss of generality that $x_0 = 0$ and $r = 1$. As in the situation of sets of finite perimeter, define $Y(x) := \pi^\perp x$ (so that $\mathrm{dist}(x,\pi) = |Y(x)|^2$) and $X := \varphi^2 Y$, where $\varphi \in C_c^\infty(B_1)$ is positive and equal to 1 in $B_{1/2}$. By definition of H,

$$\begin{aligned}\int_{B_1} \mathrm{div}_{T_x\Gamma} X \, d\mu_V(x) &= \int_{B_1} (\varphi^2 \,\mathrm{div}_{T_x\Gamma} Y + \underbrace{\varphi \langle Y, T_x\Gamma \nabla\varphi\rangle}_{(\bigstar)}) \, d\mu_V(x) = \\ &= -\int_{B_1} \langle \varphi^2 Y, H\rangle \, d\mu_V \le \frac{1}{2}\int_{B_1} (\varphi^4 |Y|^2 + |H|^2)\, d\mu_V\end{aligned} \tag{1}$$

Now choose $\{\nu_j\}_{j=1}^N$ orthonormal base of $\mathbb{R}^N$ such that $\pi = \mathrm{Span}\{\nu_j\}_{j=1}^k$ and $\{\xi_i\}_{i=1}^k$ orthonormal base of $T_x\Gamma$ for a given x. With this notation, $Y(w) = \sum_{j=k+1}^N \langle w, \nu_j\rangle \nu_j$ and thus

$$\mathrm{div}_{T_x\Gamma} Y = \sum_{i=1}^k \langle DY[\xi_i], \xi_i\rangle = \sum_{i=1}^k \langle \Big(\sum_{j=k+1}^N \langle \xi_i, \nu_j\rangle \nu_j\Big), \xi_i\rangle = \sum_{i=1}^k \sum_{j=k+1}^N \langle \nu_j, \xi_i\rangle^2$$

On the other hand, by definition of Hilbert-Schmidt norm,

$$\begin{aligned}\frac{1}{2}\|\pi - T_x\Gamma\|^2 &= \frac{1}{2}\left\|\sum_{j=1}^k \nu_j \otimes \nu_j - \sum_{i=1}^k \xi_i \otimes \xi_i\right\|^2 = k - \sum_{i,j=1}^k \langle \nu_j, \xi_i\rangle^2 = \\ &= \sum_{i=1}^k \Big(1 - \sum_{j=1}^k \langle \nu_j, \xi_i\rangle^2\Big) = \sum_{i=1}^k \sum_{j=k+1}^N \langle \nu_j, \xi_i\rangle^2\end{aligned}$$

and we conclude $2\,\mathrm{div}_{T_x\Gamma} Y = \|\pi - T_x\Gamma\|^2$. We now estimate $(\bigstar)$, recalling the inequality $ab \le a^2/4 + b^2$:

$$\begin{aligned}\varphi|\langle Y, T_x\Gamma\nabla\varphi\rangle| &= \varphi\left|\sum_{i=1}^k \langle Y, \xi_i\rangle \langle \nabla\varphi, \xi_i\rangle\right| = \varphi\left|\sum_{i=1}^k \sum_{j=k+1}^N \langle x, \nu_j\rangle \langle \nu_j, \xi_i\rangle \langle \nabla\varphi, \xi_i\rangle\right| \le \\ &\le \varphi |\nabla\varphi||Y| \sum_{i=1}^k \sum_{j=k+1}^N |\langle \nu_j, \xi_i\rangle| \le \\ &\le \frac{1}{4}\varphi^2 \sum_{i=1}^k \sum_{j=k+1}^N \langle \nu_j, \xi_i\rangle^2 + |\nabla\varphi|^2 |Y|^2 = \\ &= \frac{1}{4}\varphi^2 \,\mathrm{div}_{T_x\Gamma} Y + |\nabla\varphi|^2|Y|^2\end{aligned}$$

Therefore, (1) yields

$$\begin{gathered}\int_{B_1} \varphi^2 \,\mathrm{div}_{T_x\Gamma} Y \, d\mu_V(x) \le \frac{1}{2}\int_{B_1} (\varphi^4|Y|^2 + |H|^2)\, d\mu_V + \\ + \int_{B_1} \Big(\frac{1}{2}\varphi^2 \,\mathrm{div}_{T_x\Gamma} Y + 2|\nabla\varphi|^2|Y|^2\Big) d\mu_V(x) \\ \Downarrow \\ \frac{1}{2}\int_{B_1} \varphi^2 \,\mathrm{div}_{T_x\Gamma} Y \, d\mu_V(x) \le \frac{1}{2}\int_{B_1} (\varphi^4|Y|^2 + |H|^2 + 4|\nabla\varphi|^2|Y|^2)\, d\mu_V\end{gathered}$$

and recalling that $\varphi \equiv 1$ on $B_{1/2}$, we deduce

$$\mathrm{Exc}\Big(\pi, 0, \frac{1}{2}\Big) = 2^k \int_{B_{1/2}} \|T_x\Gamma - \pi\|^2 \, d\mu_V(x) \leq 2^{k+1} \int_{B_1} \varphi^2 \, \mathrm{div}_{T_x\Gamma} Y \, d\mu_V(x) \leq$$
$$\leq 2^{k+1} \int_{B_1} (\varphi^4 + 4|\nabla\varphi|^2)|Y|^2 \, d\mu_V + 2^{k+1} \int_{B_1} |H|^2 d\mu_V$$

Finally, $\varphi^4 + 4|\nabla\varphi|^2 < C$ (φ is fixed) and the inequality is proved. □

Remark 7.7. Using the notations of the previous Proposition (replacing $T_x\Gamma$ by T for simplicity), we can prove the inequality $|J_T\pi - 1| \leq C\|T - \pi\|^2$ that will be useful later, where $J_T\pi$ is the Jacobian of $\pi_{|T}$. First note that $J_T\pi = \sqrt{\det M}$, with M the $k \times k$ matrix with elements

$$M_{i,j} := \langle \pi\xi_i, \pi\xi_j \rangle = \langle \xi_i - \pi^\perp\xi_i, \xi_j - \pi^\perp\xi_j \rangle =$$
$$= \delta_{i,j} - \underbrace{\langle \xi_i, \pi^\perp\xi_j \rangle}_{=:A_{i,j}} - \langle \pi^\perp\xi_i, \xi_j \rangle + \underbrace{\langle \pi^\perp\xi_i, \pi^\perp\xi_j \rangle}_{=:B_{i,j}}$$

Now note that $|\pi^\perp\xi_i| = |\pi^\perp\xi_i - T^\perp\xi_i| \leq \|\pi - T\|$, hence $\|A\| \leq \|\pi - T\|$ and $\|B\| \leq \|\pi - T\|^2$. Then,

$$\det M = 1 - 2\,\mathrm{Tr}(A) + O(\|\pi - T\|^2) =$$
$$= 1 - 2\sum_{i=1}^{k} \langle \Big(\sum_{j=k+1}^{N} \langle \xi_i, \nu_j \rangle \, \nu_j \Big), \xi_i \rangle + O(\|\pi - T\|^2) =$$
$$= 1 - \|\pi - T\|^2 + O(\|\pi - T\|^2)$$

which implies $|\det M - 1| \leq C\|\pi - T\|^2$. Since $J_T\pi \geq 0$, we can conclude that

$$|J_T\pi - 1| \leq |\det M - 1|(J_T\pi + 1) = |\det M - 1| \leq C\|\pi - T\|^2$$

Lipschitz approximation

In the rest of the prove we will often use Lemma 5.2 (the "height lemma"), whose proof can be very easily adapted to our situation ($\int \mathrm{div}_{T_x\Gamma_i} X \, d\mu_V(x)$ is no longer zero, but it is bounded by $C\|H_i\|_\infty\|X\|_\infty$ and this is enough for the proof; for the details cf. [2, Lemma 5.2]).

Proposition 7.8. *For any $\ell, \beta \in (0,1)$ there exist $\lambda(\ell)$ and $\varepsilon_L(\ell,\beta)$ such that, if V satisfies* (HP) *with ε_L, then there exists $f\colon (\pi + x_0) \to \pi^\perp$ ℓ-Lipschitz map that satisfies the following properties:*

(i) $\theta_V \equiv 1$ $\mathcal{H}^k$*-a.e. on* $\Gamma \cap B_{r/100}(x_0)$*;*

(ii) $G := \left\{ x \in \Gamma \cap B_{r/100}(x_0) \;\middle|\; \theta_V(x) \geq 1, \mathrm{Exc}(\pi, x, s) \leq \lambda \quad \forall s \leq \frac{r}{10} \right\} \subseteq \Gamma_f$*;*

(iii) $\Gamma_f \cap B_{r/100}(x_0)$ *and* $\Gamma \cap B_{r/100}(x_0)$ *are included in the βr-neighborhood of π;*

(iv) the following estimate holds:

$$\mathcal{H}^k((\Gamma_f\setminus G)\cap B_{\frac{r}{100}}(x_0))+\mathcal{H}^k((\Gamma\setminus G)\cap B_{\frac{r}{100}}(x_0)) \le \frac{C}{\lambda}\operatorname{Exc}(\pi,x_0,r)r^k+C\|H\|_\infty r^{k+1}$$

Before the proof we present a small computation that will be useful for the estimate in (iv). In order to simplify notations, we assume without loss of generality that $\pi=\mathbb{R}^k\times\{0\}^{N-k}$.

Lemma 7.9. *$J(\mathrm{id}\times f)(y)-1=O(|Df(y)|^2)$, where $f\colon(\pi+x_0)\to\pi^\perp$ is the function given by Proposition 7.8.*

Proof. $J(\mathrm{id}\times f)(y)=\sqrt{\sum\det(k\times k \text{ minors of } (\mathrm{id}\times Df)(y))^2}$ by the Cauchy-Binet formula. Given $\alpha\subseteq\{k+1,\dots,N\}$ and $\beta\subseteq\{1,\dots,k\}$ we denote by $M^{\alpha,\beta}(y)$ the determinant of the $n\times n$ minor of $Df(y)$ containing rows α and columns β ($n\ge 2$). Hence,

$$J(\mathrm{id}\times f)(y)=\sqrt{1+|Df(y)|^2+\sum M^{\alpha,\beta}(y)^2}$$

and a Taylor expansion concludes the proof. □

Proof of Proposition 7.8. Assume without loss of generality $x_0=0$, $r=1$. As in the proof for sets of finite perimeter, we denoted by $\varepsilon_H(\delta)$ the constant that makes the estimates of Lemma 5.2 work with δ and set $\lambda:=\varepsilon_H(\ell/10)$ and $\varepsilon_L:=\varepsilon_H(\min\{\lambda,\beta\})$.

The choice of ε_L guarantees that $\mu_V(B_s(x))\le(\omega_k+\lambda)s^k$ for every $x\in\Gamma\cap B_{1/100}$ and s small enough; this means that $\theta_V(x)\le 1+\lambda/\omega_k<2$. Since $\theta_V=g\in\mathbb{N}$ $\mathcal{H}^k$-a.e., we can conclude that (i) holds. As a result, $\mu_V=\mathcal{H}^k_{|\Gamma}$.

The proof of (ii), (iii) and the bound of $\mathcal{H}^k((\Gamma\setminus G)\cap B_{1/100})$ is exactly the same as in the sets of finite perimeter (the extension of f to π is possible thanks to Kirszbraun theorem). In particular, note that the estimate of $\mathcal{H}^k((\Gamma\setminus G)\cap B_{1/100})$ is independent of H. We only need to prove the second part of (iv). By the area formula,

$$\begin{aligned}\mathcal{H}^k((\Gamma_f\setminus G)\cap B_{1/100})&\le\|J(\mathrm{id}\times f)\|_\infty\mathcal{H}^k(\pi(\Gamma_f\setminus G)\cap B_{1/100})\le\\&\le C[\mathcal{H}^k(B_{1/100}\cap\pi)-\mathcal{H}^k(\pi(G))]\end{aligned}\tag{2}$$

Now we estimate $\mathcal{H}^k(\pi(G))$ using the coarea formula, Remark 7.7 and the bound for $\mathcal{H}^k((\Gamma\setminus G)\cap B_{1/100})$:

$$\begin{aligned}\mathcal{H}^k(\pi(G))&=\int_G J_{T_xG}\,d\mathcal{H}^k(x)\ge\mathcal{H}^k(G)-C\int_G\|T_xG-\pi\|\,d\mathcal{H}^k(x)\overset{G\subseteq\Gamma}{\ge}\\&\ge\mathcal{H}^k(G)-C\operatorname{Exc}(\pi,0,1)\ge\\&\ge\mathcal{H}^k(\Gamma\cap B_{1/100})-\mathcal{H}^k((\Gamma\setminus G)\cap B_{1/100})-C\operatorname{Exc}(\pi,0,1)\overset{(2)}{\ge}\\&\ge\mathcal{H}^k(\Gamma\cap B_{1/100})-\frac{C}{\lambda}\operatorname{Exc}(\pi,0,1)\ge\\&\ge\theta(0)\,\frac{e^{-\|H\|_\infty/100}}{100^k}\omega_k-\frac{C}{\lambda}\operatorname{Exc}(\pi,0,1)\end{aligned}$$

Combining this with (2) we conclude

$$\begin{aligned}\mathcal{H}^k((\Gamma_f \setminus G) \cap B_{1/100}) &\le \frac{C}{\lambda}\operatorname{Exc}(\pi, 0, 1) \le +\frac{\omega_k}{100^k}(1 - e^{-\|H\|_\infty/100}) \\ &\le \frac{C}{\lambda}\operatorname{Exc}(\pi, 0, 1) + C\|H\|_\infty \end{aligned} \qquad \square$$

Harmonic approximation

For sets of finite perimeter, at this point of the proof we showed that, if the excess is small enough, we can normalize the Lipschitz function given by Proposition 5.1 so that the normalization is almost harmonic. Here we want to do the same for every component of the function f given by Proposition 7.8.

From now on we assume without loss of generality $x_0 = 0$ and $r = 1$. Choose $\{\nu_i\}_{i=1}^k$ orthonormal base of π, $\{e_j\}_{j=k+1}^N$ orthonormal base of $\pi^\perp$ and denote by $(y_1, \dots, y_k, z_{k+1}, z_N)$ the coordinates of $\mathbb{R}^N$ associated to this base. We will also use the notations $\mathcal{B}_r(x) \coloneqq B_r(x) \cap \pi$ and $\bar\nabla \coloneqq (\partial_1, \dots, \partial_k)$.

Proposition 7.10. *If ℓ and $E \coloneqq \operatorname{Exc}(\pi, 0, 1)$ are small enough and $\|H\|_\infty < E$, then for every $j \in \{k+1, \dots, N\}$*

$$\sup_{\varphi \in C_c^\infty(\mathcal{B}_{1/200})} \frac{|\int_{\mathcal{B}_{1/200}} \langle \bar\nabla \tilde f_j, \bar\nabla\varphi\rangle|}{\|\bar\nabla\varphi\|_\infty} \le \tilde C\sqrt{E}$$

where $\tilde f_j \coloneqq (c_0 E)^{-\frac{1}{2}} f_j$ and the constant c_0 is such that $\int_{\mathcal{B}_{1/200}} |\bar\nabla \tilde f_j|^2 \le 1$.

This result will allow to use Proposition 5.4 and find a harmonic approximation of $\tilde f_j$.

Proof. Fix $\varphi \in C_c^\infty(\mathcal{B}_{1/200})$ and define the vector field $X(y, z) \coloneqq \varphi(y)e_j$. Obviously X is not compactly supported in $B_{1/100}$, but we can easily fix that. We proved that $\Gamma \cap B_{1/100}$ is included in the β-neighborhood of π and we can assume $\beta < 1/200$; so we multiply X by a function $\zeta \in C_c^\infty(B_{1/100} \cap \pi^\perp)$ such that $\zeta(z) = 1$ for $|z| \le 1/200$.

$$\begin{aligned} &\left|\int_{\Gamma_f} \operatorname{div}_{T_x\Gamma_f} X \, d\mathcal{H}^k(x)\right| \le \\ &\quad\le \left|\int_{\Gamma_f} \operatorname{div}_{T_x\Gamma_f} X \, d\mathcal{H}^k(x) - \int_\Gamma \operatorname{div}_{T_x\Gamma} \zeta X \, d\mathcal{H}^k(x)\right| + |\delta V(\zeta X)| \le \\ &\le \|\bar\nabla\varphi\|_\infty[\mathcal{H}^k((\Gamma_f \setminus G) \cap B_{\frac{1}{100}}) + \mathcal{H}^k((\Gamma \setminus G) \cap B_{\frac{1}{100}})] + \|H\|_\infty\|\varphi\|_\infty\mathcal{H}^k(\Gamma \cap B_{\frac{1}{100}}) \\ &\qquad \le \frac{CE}{\lambda}\|\bar\nabla\varphi\|_\infty + CE\|\varphi\|_\infty \end{aligned}$$

Since φ has compact support, in $\mathcal{B}_{1/200}$, for every $y \in \mathcal{B}_{1/200}$ and $\tilde y \in \partial\mathcal{B}_{1/200}$ we have the inequality $|\varphi(y)| = |\varphi(y) - \varphi(\tilde y)| \le \|\bar\nabla\varphi\|_\infty|y - \tilde y| \le \frac{1}{100}\|\bar\nabla\varphi\|_\infty$ and so the previous estimate becomes

$$\left|\int_{\Gamma_f} \operatorname{div}_{T_x\Gamma_f} X \, d\mathcal{H}^k(x)\right| \le \frac{CE}{\lambda}\|\bar\nabla\varphi\|_\infty \tag{3}$$

Now for $i, h \in \{1, \ldots, k\}$ define $v_i := \nu_i + \sum_{j=k+1}^N e_j \partial_i f_j$, $g_{ih} := \langle v_i, v_h \rangle$ and observe that $|g_{ih} - \delta_{ih}| = \left| \sum_{j,j'} \partial_i f_j \partial_h f_{j'} \delta_{jj'} \right| \leq C|Df|^2$. Therefore, for ℓ is small enough the matrix g_{ih} is invertible and the inverse g^{ih} also satisfies $|g^{ih} - \delta_{ih}| \leq C|Df|^2$. Take $x = (y, f(y)) \in \Gamma_f$ and $\{\xi_i\}_{i=1}^k$ an orthonormal base of $T_x\Gamma_f$: the projection on $T_x\Gamma_f$ of a vector w is given by $\sum_{i,h} \langle w, v_i \rangle g^{ih} v_h$, so

$$
\begin{aligned}
\operatorname{div}_{T_x\Gamma_f} X &= \sum_{i=1}^k \langle DX(x)[\xi_i], \xi_i \rangle = \sum_{i=1}^k \langle e_j, \xi_i \rangle \langle \nabla\varphi(y), \xi_i \rangle = \langle T_x\Gamma_f(\nabla\varphi), e_j \rangle = \\
&= \sum_{i,h=1}^k \langle \nabla\varphi, v_i \rangle g^{ih} \langle v_h, e_j \rangle = \sum_{i,h=1}^k \partial_i \varphi \, g^{ih} \, \partial_h f_j = \\
&= \langle \bar\nabla\varphi(y), \bar\nabla f(y) \rangle + O(|\bar\nabla\varphi(y)||Df(y)|^3)
\end{aligned}
$$

We combine this with Lemma 7.9, (3) and the area formula to get

$$
\begin{aligned}
&\left| \int_{\mathcal{B}_{1/200}} \langle \bar\nabla\varphi(y), \bar\nabla f(y) \rangle \, dy \right| = \\
&\qquad = \left| \int_{\mathcal{B}_{1/200}} [\operatorname{div}_{T_x\Gamma_f} X + O(|\bar\nabla\varphi(y)||Df(y)|^3)] \, dy \right| = \\
&\qquad = \left| \int_{\mathcal{B}_{1/200}} [J(\mathrm{id} \times f) \operatorname{div}_{T_x\Gamma_f} X + O(|\bar\nabla\varphi(y)||Df(y)|^2)] \, dy \right| \leq \\
&\qquad \leq \left| \int_{\Gamma_f} \operatorname{div}_{T_x\Gamma_f} X \, d\mathcal{H}^k(x) \right| + O\Big(\|\bar\nabla\varphi(y)\|_\infty \int_{\mathcal{B}_{1/200}} |Df(y)|^2 \, dy \Big) \leq \\
&\qquad \leq \frac{CE}{\lambda} \|\bar\nabla\varphi\|_\infty + O\Big(\|\bar\nabla\varphi(y)\|_\infty \int_{\mathcal{B}_{1/200}} |Df(y)|^2 \, dy \Big)
\end{aligned}
\tag{4}
$$

Now we want to bound $\int_{\mathcal{B}_{1/200}} |Df(y)|^2 \, dy$ with the excess. By the area formula

$$
\begin{aligned}
\int_{\mathcal{B}_{1/200}} \|\pi - T_x\Gamma_f\|^2 J(\mathrm{id} \times f) \, dy &\leq \int_{B_{1/100} \cap \Gamma_f} \|\pi - T_x\Gamma_f\|^2 \, d\mathcal{H}^k(x) \leq \\
&\leq E + C\mathcal{H}^k((\Gamma_f \setminus G) \cap B_{1/100}) \leq \Big(1 + \frac{C}{\lambda} E\Big) E
\end{aligned}
\tag{5}
$$

On the other hand,

$$
\begin{aligned}
\|\pi - T_x\Gamma_f\|^2 &\geq \|\cancel{\pi(e_j)} - T_x\Gamma_f(e_j)\|^2 = \left| \sum_{i,h=1}^k \langle e_j, v_i \rangle g^{ih} v_h \right|^2 = \\
&= \left| \sum_{i,h=1}^k \partial_i f_j g^{ih} v_h \right|^2 = \sum_{i,h,i',h'} \partial_i f_j \partial_{i'} f_j g^{ih} g^{i'h'} \langle v_h, v_{h'} \rangle = \\
&= \sum_{i,i',h'} \partial_i f_j \partial_{i'} f_j g^{i'h'} \delta_{i,h'} = \sum_{i',h'} \partial_{h'} f_j \partial_{i'} f_j g^{i'h'} = \\
&= |\bar\nabla f_j|^2 (1 + O(|Df|^2))
\end{aligned}
$$

and we fix ℓ (and thus $\lambda(\ell)$) small enough so that $2\|\pi - T_x\Gamma_f\|^2 \geq |\bar{\nabla} f_j|^2$; this combined with (5) yields

$$\begin{aligned}\int_{\mathcal{B}_{1/200}} |\bar{\nabla} f_j(y)|^2 \leq \int_{\mathcal{B}_{1/200}} \|\pi - T_x\Gamma_f\|^2 \, dy \leq \\ \leq \int_{\mathcal{B}_{1/200}} \|\pi - T_x\Gamma_f\|^2 J(\mathrm{id} \times f) \, dy \leq \Big(1 + \frac{C}{\lambda} E\Big) E = CE\end{aligned} \tag{6}$$

(note that we do not need to keep track of ℓ and λ anymore). We also get $\|\pi - T_x\Gamma_f\|^2 \geq C|Df|^2$ by summing over every j and with the same computation as before

$$\int_{\mathcal{B}_{1/200}} |Df|^2 \, dy \leq CE$$

In conclusion, (4) becomes

$$\left| \int_{\mathcal{B}_{1/200}} \langle \bar{\nabla}\varphi(y), \bar{\nabla} f(y) \rangle \, dy \right| \leq \tilde{C}E \tag{7}$$

It is now trivial from (6) and (7) that $\tilde{f}_j$ has the desired properties. ☐

Conclusion

We are now ready to prove Theorem 7.5. As in the situation of sets of finite perimeter, the strategy is to bound the tilt of the tangent plane to the graph of a harmonic function close to f and then conclude using the tilt-excess inequality.

Proof of Theorem 7.5. Consider a constant $\rho > 0$ that will be fixed later and ε_A the constant that makes Proposition 5.4 work with ρ. Assume $\varepsilon_2 \leq (\varepsilon_A/\tilde{C})^2$; then we can combine Proposition 7.10 and Proposition 5.4 to deduce the existence of $\tilde{u}_j \colon \mathcal{B}_{1/200} \to \mathbb{R}$ harmonic such that $\int_{\mathcal{B}_{1/200}} |\bar{\nabla}\tilde{u}_j|^2 \leq 1$ and $\int_{\mathcal{B}_{1/200}} |\tilde{f}_j - \tilde{u}_j|^2 \leq \rho$. Define $u_j \coloneqq \tilde{u}_j\sqrt{c_0 E}$ and $u \coloneqq (u_{k+1}, \dots, u_N)$ so that

$$\int_{\mathcal{B}_{1/200}} |Du|^2 \leq CE, \qquad \int_{\mathcal{B}_{1/200}} |f - u|^2 \leq C\rho E \tag{8}$$

The plane $\tilde{\pi} \coloneqq \mathrm{Span}\left\{ \nu_i + \sum_{j=k+1}^N \langle \bar{\nabla} u_j(0), \nu_i \rangle \, e_j \right\}_{i=1}^k$ is our candidate for the excess improvement. Define $\tilde{x} \coloneqq (0, u(0))$. The harmonicity of u_j implies that $|u_j(0)|$ and $|\bar{\nabla} u_j(0)|$ are bounded by $C\|u_j\|_{L^1}$; therefore,

$$\mathrm{dist}(\tilde{x}, \pi) \leq |u(0)| \leq C\|u\|_{L^1} \leq C(\|u - f\|_{L^2} + \|f\|_{L^2}) \leq C(\sqrt{\rho E} + \beta)$$

$$\|\pi^\perp - \tilde{\pi}^\perp\| \leq C \sum_{j=k+1}^N |\bar{\nabla} u_j(0)| \leq C\|u\|_{L^1} \leq C(\sqrt{\rho E} + \beta)$$

The ball $B_{4\eta}(\tilde{x})$ is contained in $B_{1/100}$ if η, E, β are small enough. Now observe that

$$\int_{B_{4\eta}(\tilde{x})\setminus\Gamma_f} \operatorname{dist}(x-\tilde{x},\tilde{\pi})^2\, d\mu_V(x) \le \int_{B_{1/100}\setminus\Gamma_f} 2(\operatorname{dist}(\tilde{x},\tilde{\pi})^2 + |\tilde{\pi}^\perp(x)|^2)\, d\mu_V(x) \le$$
$$\le \int_{B_{1/100}\setminus\Gamma_f} (2\operatorname{dist}(\tilde{x},\tilde{\pi})^2 + 4|\pi^\perp(x)|^2 + 4\|\tilde{\pi}^\perp - \pi^\perp\|^2)\, d\mu_V(x) \le$$
$$\le C[(\rho E+\beta^2)+\beta^2+(\rho E+\beta^2)]\,\mathcal{H}^k((\Gamma\setminus\Gamma_f)B_{1/100}) \le$$
$$\le C(\rho E+\beta^2)E \quad (9)$$

(we have used the fact that $\Gamma\cap B_{1/100}$ is in the β-neighborhood of π). On the other hand, for $x=(y,f(y))$ we have $x-\tilde{x}=(y,f(y)-u(0))$ and thus, if ℓ is small enough so that $J(\mathrm{id}\times f)\le 2$,

$$\int_{B_{4\eta}(\tilde{x})\cap\Gamma_f} \operatorname{dist}(x-\tilde{x},\tilde{\pi})^2\, d\mu_V(x) \le 2\int_{\mathcal{B}_{4\eta}} |f(y)-u(0)-\overbrace{\sum_{j=k+1}^N \langle\bar{\nabla}u_j(0),y\rangle\, e_j}^{\in\tilde{\pi}}|^2\, dy \le$$
$$\le 4\int_{\mathcal{B}_{4\eta}} [|f(y)-u(y)|^2 - |u(y)-u(0)-\sum_{j=k+1}^N \langle\bar{\nabla}u_j(0),y\rangle\, e_j|^2]\, dy$$

Proposition 5.3 implies that for every j,

$$\sup_{y\in\mathcal{B}_{1/200}} \left|u_j(y)-u_j(0)-\langle\bar{\nabla}u_j(0),y\rangle\right| \le C\eta^2\|\bar{\nabla}u_j\|_{L^2}$$

and so we can conclude

$$\int_{B_{4\eta}(\tilde{x})\cap\Gamma_f} \operatorname{dist}(x-\tilde{x},\tilde{\pi})^2\, d\mu_V(x) \overset{(8)}{\le} C\rho E + C\eta^{k+4}E \qquad (10)$$

The combination of (9) and (10) yields

$$\frac{1}{(4\eta)^{k+2}}\int_{B_{4\eta}(\tilde{x})} \operatorname{dist}(x-\tilde{x},\tilde{\pi})^2\, d\mu_V(x) \le C(\eta^{-k-2}\rho E + \eta^{-k-2}\beta^2 + \eta^{-k-2}\rho + \eta^2)E$$

If β and ρ are small enough, then $|\tilde{x}|\le C(\sqrt{\rho E}+\beta) < \eta$ and $B_\eta\subset B_{2\eta}(\tilde{x})$, Therefore,

$$\operatorname{Exc}(\tilde{\pi},0,\eta) \le 2^k\operatorname{Exc}(\tilde{\pi},\tilde{x},2\eta) \overset{\text{tilt-excess}}{\le}$$
$$\le 2^k C\left(\frac{1}{(4\eta)^{k+2}}\int_{B_{4\eta}(\tilde{x})} \operatorname{dist}(x-\tilde{x},\tilde{\pi})^2\, d\mu_V(x) + \frac{1}{(4\eta)^{k-2}}\int_{B_{4\eta}(\tilde{x})} |H|^2\, d\mu_V\right) \le$$
$$\le 2^k C\left(\frac{1}{(4\eta)^{k+2}}\int_{B_{4\eta}(\tilde{x})} \operatorname{dist}(x-\tilde{x},\tilde{\pi})^2\, d\mu_V(x) + \frac{1}{(4\eta)^{\cancel{k}-2}}\|H\|^2_\infty \cancel{(4\eta)^k}\right) \le$$
$$\le C(\eta^{-k-2}\rho E + \eta^{-k-2}\beta^2 + \eta^{-k-2}\rho + \eta^2 + E^2\eta^2)E$$

It is now obvious that for $\eta\ll 1, \beta^2\ll\eta^{k+2}, \rho\ll\eta^{k+2}$ the last inequality implies that $\operatorname{Exc}(\tilde{\pi},0,\eta)\le E/2$. □

As in the situation of sets of finite perimeter, we now want to iterate the application of the previous result in order to show that every point close to 0 is in the set G of Proposition 7.8, and thus in Γ_f.

Proof of Theorem 7.2. If ε_1 is small enough, we can apply Lemma 5.2 to get that $\mu_V(B_s(x)) \leq (\omega_k+\varepsilon_2)s^k$ for all $x \in \Gamma \cap B_{1/2}$ and $s \leq 1/4$. It is immediate to check that $\mathrm{Exc}(\pi, x, 1/4) \leq 4^k\varepsilon_1$ (and also $< \varepsilon_2$ for ε_1 small), but this is not enough to apply Theorem 7.5, as we miss the condition $4\|H\|_\infty \leq \mathrm{Exc}(\pi, x, 1/2)$. Fix $x \in \Gamma \cap B_{1/2}$ and define

$$E(r) \coloneqq \min_\tau \mathrm{Exc}(\tau, x, r) \qquad F(r) \coloneqq E(r) + \frac{4}{\eta^k} r\|H\|_\infty$$

We claim that $F(r) \leq \varepsilon_2$ implies $F(\eta r) \leq \frac{3}{4}F(r)$. If $\|H\|_\infty r \leq E(r)$, we know by Theorem 7.5 that $E(\eta r) \leq E(r)/2$ and thus, recalling that $\eta < 1/2$,

$$F(\eta r) \leq \frac{1}{2}E(r) + \frac{4}{\eta^k} r\eta\|H\|_\infty \leq \frac{1}{2}\Big(E(r) + \frac{4}{\eta^k} r\|H\|_\infty\Big) = \frac{1}{2}F(r)$$

On the other hand, if $\|H\|_\infty r \geq E(r)$, then

$$F(\eta r) \leq \frac{1}{\eta^k}E(r) + \frac{4}{\eta^k} r\eta\|H\|_\infty \leq \Big(\frac{1}{4}+\eta\Big)\frac{4}{\eta^k} r\|H\|_\infty \leq \frac{3}{4}F(r)$$

and in both cases the claim is proved. Note also that

$$F\Big(\frac{1}{4}\Big) \leq 4^k E + \frac{1}{\eta^k}\|H\|_\infty \leq C\varepsilon_1$$

can be made smaller than ε_2; therefore, we can apply and iterate the previous result to get $F(\eta^n/4) \leq C(\frac{3}{4})^n\varepsilon_1$. For every r small enough set $n \coloneqq \lfloor \log_\eta 4r \rfloor$, so that $r \leq \eta^n/4 < r/\eta$ and

$$E(r) \leq \Big(\frac{\eta^n}{4r}\Big)^k E\Big(\frac{\eta^n}{4}\Big) \leq C\eta\Big(\frac{3}{4}\Big)^n\varepsilon_1 \leq C\Big(\frac{3}{4}\Big)^{\log_\eta 4r-1}\varepsilon_1 \leq Cr^\alpha\varepsilon_1 \tag{11}$$

for some constant α.

As in the case of sets of finite perimeter, one can prove that $\mathrm{Exc}(\pi, x, r) \leq C\varepsilon_1$ and conclude that $\Gamma \cap B_{1/100} \subseteq \Gamma_f$. Then the monotonicity formula allows to prove by contradiction that $\Gamma \cap B_{1/100} = \Gamma_f \cap B_{1/100}$. We omit this part of the proof, as there are no significant changes.[3]

We still need to prove the regularity of f. Fix $y \in \mathcal{B}_{1/400}$, $r < 1/400$ and define $x \coloneqq (y, f(y))$ and $\pi_{x,r}$ a k-dimensional plane that satisfies $\mathrm{Exc}(\pi_{x,r}, x, r) \leq Cr^\alpha\varepsilon_1$ (whose existence is guaranteed by (11)). Since $\|\pi - \pi_{x,r}\|^2 \leq C(\mathrm{Exc}(\pi, x, r) + \mathrm{Exc}(\pi_{x,r}, x, r)) \leq C\varepsilon_1$ (this can be easily proved by a straight forward computation), for ε_1 small there exists a linear map $A_{x,r}\colon \pi \to \pi^\perp$ with Hilbert-Schmidt norm smaller than 1 such that its graph coincides with $\pi_{x,r}$. Now consider two

[3] Cf. [2] for the details.

maps $A, B\colon \pi \to \pi^\perp$: if $|B| \le C\ell$ and ℓ is small, then $|\Gamma_B(v)| \le \frac{1}{2}|v|$ for every $v \in \pi^\perp$ (as $\Gamma_B \to \pi$ for $\ell \to 0$); therefore, for every $i \in \{1, \dots, k\}$

$$\begin{aligned} |A(\nu_i) - B(\nu_i)| = |(\nu_i + A(\nu_i)) - (\nu_i + B(\nu_i))| = \\ = |\Gamma_A(\nu_i + A(\nu_i)) - \Gamma_B(\nu_i + B(\nu_i))| \le \\ \le |\Gamma_A(\nu_i) - \Gamma_B(\nu_i)| + |\Gamma_A(A(\nu_i)) - \Gamma_B(A(\nu_i))| + |\Gamma_B(\underbrace{A(\nu_i) - B(\nu_i)}_{\in\pi^\perp})| \le \\ \le (1 + |A(\nu_i)|)\|\Gamma_A - \Gamma_B\| + \frac{1}{2}|A(\nu_i) - B(\nu_i)| \end{aligned}$$

and we conclude that $|A - B| \le C\|\Gamma_A - \Gamma_B\|$. We apply this for $A = A_{x,r}$, $B = Df(u)$ to get

$$\begin{aligned} \int_{\mathcal{B}_r(y)} |Df(u) - A_{x,r}|^2\, du \le C \int_{\mathcal{B}_r(y)} \|T_x\Gamma_f - \pi_{x,r}\|^2 J(\mathrm{id} \times f)\, du \le \\ \le C \int_{B_{4r}(x)\cap\Gamma_f} \|T_x\Gamma_f - \pi_{x,r}\|^2\, d\mathcal{H}(x) \le \\ \le Cr^k \operatorname{Exc}(\pi_{x,r}, x, 4r) \le Cr^{k+\alpha} \end{aligned}$$

The left hand side is minimized when we substitute $A_{x,r}$ by the average of Df on $\mathcal{B}_r(y)$, which we denote by $(Df)_{y,r}$. In conclusion, we proved that

$$\int_{\mathcal{B}_r(y)} |Df(u) - (Df)_{x,r}|^2\, du \le Cr^{k+\alpha} \qquad \forall y \in \mathcal{B}_{1/400}, \quad \forall r < 1/400$$

which implies that f is $C^{1,\gamma}$ in $\mathcal{B}_{1/1000}$ with $\gamma \coloneqq \alpha/2$.[4] $\square$

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[4] A sketch of the proof can be found at the end of [2].