3   De Giorgi's structure theorem

This section is presented by Riccardo Caniato on Mar 12, 2020 with the reference [3]. The notes are written by Riccardo Caniato, if you find any mistake please feel free to contact the writer via riccardo.caniato@math.ethz.ch.

Introduction

The main goal of this lecture will be to investigate the structure of the Gauss-Green measure associated to a set of locally finite perimeter $E \subset \mathbb{R}^n$.
We already know that, if $E \subset \mathbb{R}^n$ is an open set with C^1-boundary, it holds that

$$\mu_E = \nu_E \mathcal{H}^{n-1} \lfloor_{\partial E}$$

where $\nu_E : \partial E \to S^{n-1}$ is the outer unit normal to ∂E. Thus, in this particular case, it follows that

$$|\mu_E| = \mathcal{H}^{n-1} \lfloor_{\partial E}.$$

Of course, these facts cannot be generalized in this form to an arbitrary set of locally finite perimeter. Indeed, we know that a modification of a set of locally finite perimeter on (or with) a $\mathcal{L}^n$-negligible subset of $\mathbb{R}^n$ might modify widely its topological boundary but has no effect on its Gauss-Green measure.
On the other hand, we may wonder if another and apparently more reasonable generalization can be done. Namely, we may conjecture that

$$|\mu_E| = \mathcal{H}^{n-1} \lfloor_{\mathrm{spt}(\mu_E)}.$$

Although in principle the previous equation seems to make sense, the following example illustrates how it can't be true in general.

Example 3.1 (A "wild" set of finite perimeter)**.** Fix any $0 < \varepsilon < \omega_n$ and consider a dense and countable subset $\{x_k\}_{k \in \mathbb{N}}$ of the unit ball $B \subset \mathbb{R}^n$. Pick a sequence of radii $\{r_k\}_{k \in \mathbb{N}} \subset (0, \varepsilon)$ such that $B(x_k, r_k) \subset B$, for every $k \in \mathbb{N}$, and

$$n\omega_n \sum_{k \in \mathbb{N}} r_k^{n-1} \leq 1.$$

Then, the set

$$E := \bigcup_{k \in \mathbb{N}} B(x_k, r_k) \subset B$$

is a set of finite perimeter in $\mathbb{R}^n$ such that

$$P(E) = |\mu_E|(\mathbb{R}^n) \leq \varepsilon$$

and

$$|\mathrm{spt}(\mu_E)| \geq \omega_n - \varepsilon > 0.$$

In particular, $\mathcal{H}^{n-1} \lfloor_{\mathrm{spt}(\mu_E)}(\mathbb{R}^n) = \mathcal{H}^{n-1}\big(\mathrm{spt}(\mu_E)\big) = +\infty.$

Hence, the support of the Gauss-Green measure might still be too wild to represent a measure theoretic replacement for the topological boundary. What happens to be the correct measure theoretic analogue of the topological boundary is the so called reduced boundary, which is defined as follows.

Definition 3.2 (Reduced boundary)**.** Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter in $\mathbb{R}^n$.

We define the **reduced boundary** of E as the subset of $\mathbb{R}^n$ given by

$$\partial^* E := \left\{ x \in \mathbb{R}^n \text{ s.t. } \lim_{r \to 0^+} \frac{\mu_E\big(B(x,r)\big)}{|\mu_E|\big(B(x,r)\big)} \text{ exists and belongs to } S^{n-1} \right\}.$$

The Borel vector field $\nu_E : \partial^* E \to S^{n-1}$ defined as

$$\nu_E(x) := \lim_{r \to 0^+} \frac{\mu_E\big(B(x,r)\big)}{|\mu_E|\big(B(x,r)\big)}, \qquad \text{for every } x \in \partial^* E,$$

is called the **(measure theoretic) outer unit normal** to E.

Remark 3.3. Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter in $\mathbb{R}^n$.
As a direct application of the Lebesgue-Besicovitch differentiation theorem it follows immediately that

$$\mu_E = \nu_E |\mu_E|$$

and that $|\mu_E|$ is concentrated on $\partial^* E$, which implies $\mathrm{spt}(\mu_E) \subset \overline{\partial^* E}$. Moreover, since by definition $\partial^* E \subset \mathrm{spt}(\mu_E)$, we get that

$$\mathrm{spt}(\mu_E) = \overline{\partial^* E}.$$

The main tool that we will exploit to look at the local properties of the geometric objects that we are analyzing is the so called "blowing up" procedure, that is defined as follows.

Definition 3.4 (Blow-ups)**.** Let $E \subset \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$.
The **blow-ups** of E at x_0 are the elements of the family $\{E_{x_0,r}\}_{r>0}$ defined as

$$E_{x_0,r} := \frac{E - x_0}{r} = \Phi_{x_0,r}(E), \qquad \text{for every } r > 0,$$

where, for every fixed $r > 0$, the map $\Phi_{x_0,r} : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\Phi_{x_0,r}(x) := \frac{x - x_0}{r}, \qquad \text{for every } x \in \mathbb{R}^n.$$

Remark 3.5. By the Lebesgue points theorem, for every Lebesgue measurable $E \subset \mathbb{R}^n$ it holds that

$$x_0 \in E^{(0)} \text{ if and only if } E_{x_0,r} \xrightarrow{loc} \emptyset, \qquad \text{as } r \to 0^+,$$

and

$$x_0 \in E^{(1)} \text{ if and only if } E_{x_0,r} \xrightarrow{loc} \mathbb{R}^n, \qquad \text{as } r \to 0^+.$$

Remark 3.6. Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter. We will see during this lecture that we control the limit of the blow-ups family as $r \to 0^+$ for each point laying in the reduced boundary of E. Unfortunately, except for the points belonging to $E^{(0)} \cup E^{(1)} \cup \partial^* E$, the behaviour of the blow-ups family can be very wild. For example, it is possible to construct an open set of finite perimeter with a point $x_0 \in \partial E$ such that for every set of finite perimeter $F \subset \mathbb{R}^n$ there exists a sequence $r_k \to 0^+$ such that $E_{x_0,r_k} \xrightarrow{loc} F$ as $k \to +\infty$ (see e.g. [2]).

Tangential properties of the reduced boundary

Lemma 3.7. *If $E \subset \mathbb{R}^n$ is a set of locally finite perimeter in $\mathbb{R}^n$, $x_0 \in \mathbb{R}^n$ and $r > 0$, then $E_{x_0,r} \subset \mathbb{R}^n$ is a set of finite perimeter in $\mathbb{R}^n$ with*

$$\mu_{E_{x_0,r}} = \frac{(\Phi_{x_0,r})_{\#}\mu_E}{r^{n-1}}.$$

Proof. Pick any $\varphi \in C_c^1(\mathbb{R}^n)$ and set $\varphi_{x_0,r} := \varphi \circ \Phi_{x_0,r}$. Since $\nabla\varphi_{x_0,r} = r^{-1}(\nabla\varphi \circ \Phi_{x_0,r})$, it holds that

$$\int_{E_{x_0,r}} \nabla\varphi \, d\mathcal{L}^n = \frac{1}{r^n}\int_E (\nabla\varphi \circ \Phi_{x_0,r}) \, d\mathcal{L}^n = \frac{1}{r^{n-1}}\int \nabla\varphi_{x_0,r} \, d\mathcal{L}^n$$

$$= \frac{1}{r^{n-1}}\int_{\mathbb{R}^n} \varphi_{x_0,r} \, d\mu_E = \int_{\mathbb{R}^n} \varphi \, d\left(\frac{(\Phi_{x_0,r})_{\#}\mu_E}{r^{n-1}}\right).$$

As $r^{1-n}(\Phi_{x_0,r})_{\#}\mu_E$ is a Radon measure, the statement follows. $\qquad\square$

We have already seen that the intersection of two sets of locally finite perimeter has locally finite perimeter as well. During the present section though, we will need some more precise information about the structure of the Gauss-Green measure of the intersection $E \cap B(x,r)$ of a set of locally finite perimeter E with a ball.

Although the final result is pretty intuitive, we will be content to obtain it just for every $x \in \mathbb{R}^n$ and for a.e. $r > 0$, due to some measure theoretic details that need to be taken into account.

Lemma 3.8 (Intersection with a ball). *Let $E \subset \mathbb{R}^n$ is a set of locally finite perimeter in $\mathbb{R}^n$ and fix any $x \in \mathbb{R}^n$.*
Then, the intersection $E \cap B(x,r)$ is a set of locally finite perimeter in $\mathbb{R}^n$ for every $r > 0$. Moreover, for a.e. $r > 0$, it holds that:

$$\mu_{E \cap B(x,r)} = \nu_{B(x,r)}\mathcal{H}^{n-1}\llcorner_{E \cap \partial B(x,r)} + \mu_E\llcorner_{B(x,r)}, \tag{3.1}$$

$$|\mu_{E \cap B(x,r)}| = \mathcal{H}^{n-1}\llcorner_{E \cap \partial B(x,r)} + |\mu_E|\llcorner_{B(x,r)}, \tag{3.2}$$

$$P\big(E \cap B(x,r)\big) = \mathcal{H}^{n-1}\big(E \cap \partial B(x,r)\big) + P\big(E; B(x,r)\big). \tag{3.3}$$

Proof. First, we claim that if $E, F \subset \mathbb{R}^n$ are open sets with C^1-boundary such that

$$P(E; \partial F) = P(F; \partial E) = \mathcal{H}^{n-1}(\partial E \cap \partial F) = 0,$$

then $E \cap F$ (which is a set of locally finite perimeter) satisfies

$$\mu_{E \cap F} = \nu_E \mathcal{H}^{n-1}\llcorner_{F \cap \partial E} + \nu_F \mathcal{H}^{n-1}\llcorner_{E \cap \partial F}.$$

Indeed, as $\mu_E = \nu_E\mathcal{H}^{n-1}\llcorner_{\partial E}$ and $\mu_F = \nu_F\mathcal{H}^{n-1}\llcorner_{\partial F}$, exploiting Lemma 2.14 and the fact that $\mathcal{H}^{n-1}\big(\partial E \cap \partial F\big) = 0$ we get

$$|\mu_{E \cap F}| \leq |\mu_E| + |\mu_F| = \mathcal{H}^{n-1}\llcorner_{\partial E \cup \partial F}.$$

Thus, using again the fact that $\mathcal{H}^{n-1}\big(\partial E \cap \partial F\big) = 0$, we obtain that $|\mu_{E \cap F}|(\partial E \cap \partial F) = 0$. Hence, recalling that the Gauss-Green measure of a set of locally finite perimeter is

always concentrated on its boundary, it holds that $|\mu_{E\cap F}|$ is concentrated on the set $\partial(E \cap F) \smallsetminus (\partial E \cap \partial F) = (E \cap \partial F) \cup (F \cap \partial E)$, that is,

$$\mu_{E\cap F} = \mu_{E\cap F}\lfloor_{F\cap\partial E} + \mu_{E\cap F}\lfloor_{E\cap\partial F}.$$

Then, we get that

$$\int_{E\cap F} \operatorname{div} T \, d\mathcal{L}^n = \int_{\mathbb{R}^n} T \cdot d\mu_{E\cap F} = \int_{F\cap\partial E} T \cdot d\mu_{E\cap F} + \int_{E\cap\partial F} T \cdot d\mu_{E\cap F},$$

for every $T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$. In particular, picking $T \in C_c^1(F; \mathbb{R}^n)$, we obtain that

$$\int_{F\cap\partial E} T \cdot d\mu_{E\cap F} = \int_{E\cap F} \operatorname{div} T \, d\mathcal{L}^n = \int_{E} \operatorname{div} T \, d\mathcal{L}^n = \int_{\mathbb{R}^n} T \cdot d\mu_E = \int_F T \cdot d\mu_E.$$

By arbitrariness of $T \in C_c^1(F; \mathbb{R}^n)$ we conclude that $\mu_{E\cap F}\lfloor_{F\cap\partial E} = \mu_E\lfloor_F$. In the same way, we obtain that $\mu_{E\cap F}\lfloor_{E\cap\partial F} = \mu_F\lfloor_E$ and we complete the proof of our first claim. Next, we fix any $x \in \mathbb{R}^n$ and we prove the statement assuming that $E \subset \mathbb{R}^n$ is bounded. If E is bounded, in particular it holds that $|E| < +\infty$ and we can find a sequence $\{E_k\}_{k\in\mathbb{N}}$ of open sets with C^1-boundary such that

$$E_k \to E$$

and

$$|\mu_{E_k}| \overset{*}{\rightharpoonup} |\mu_E|$$

as $k \to +\infty$. In particular, by the coarea formula, it holds that

$$0 = \lim_{k\to+\infty} |E \triangle E_k| = \lim_{k\to+\infty} \int_0^{+\infty} \mathcal{H}^{n-1}\big((E\triangle E_k) \cap \partial B(x,r)\big) \, dr,$$

namely, the sequence $\big\{\mathcal{H}^{n-1}\big((E\triangle E_k) \cap \partial B(x,\cdot)\big)\big\}_{k\in\mathbb{N}}$ converges to 0 in $L^1_{loc}\big((0,+\infty)\big)$. Hence, there exists a subsequence $\{E_{k_h}\}_{h\in\mathbb{N}}$ such that

$$\lim_{h\to+\infty} \mathcal{H}^{n-1}\big((E\triangle E_{k_h}) \cap \partial B(x,r)\big) = 0, \qquad \text{for a.e. } r \in (0,+\infty). \tag{3.4}$$

In the following, we will omit the fact that we have taken a subsequence on order to simplify the notation. By well know properties of Radon measures, the sets

$$N := \big\{r \in (0,+\infty) \text{ s.t. } P\big(E; \partial B(x,r)\big) > 0\big\},$$

$$N_k := \big\{r \in (0,+\infty) \text{ s.t. } P\big(E_k; \partial B(x,r)\big) > 0\big\}, \qquad \text{for every } k \in \mathbb{N},$$

are at most countable. Thus, we conclude that for a.e. $r \in (0,+\infty)$ it holds that

$$P\big(E; \partial B(x,r)\big) = P\big(E_k; \partial B(x,r)\big) = 0, \qquad \text{for every } k \in \mathbb{N}. \tag{3.5}$$

Assume that $r \in (0,+\infty)$ is such that both (3.4) and (3.5) hold true. Then, by our first claim and by (3.5) it follows that

$$\mu_{E_k \cap B(x,r)} = \mu_{E_k}\lfloor_{B(x,r)} + \mu_{B(x,r)}\lfloor_{E_k}, \qquad \text{for every } h \in \mathbb{N}. \tag{3.6}$$

We show that:

1. $\mu_{B(x,r)} \lfloor_{E_k} \overset{*}{\rightharpoonup} \mu_{B(x,r)} \lfloor_E$ as $k \to +\infty$. Indeed, pick any $\varphi \in C_c^0(\mathbb{R}^n)$ and observe that

$$\left| \int_{\mathbb{R}^n} \varphi \, d\big(\mu_{B(x,r)} \lfloor_{E_k}\big) - \int_{\mathbb{R}^n} \varphi \, d\big(\mu_{B(x,r)} \lfloor_E\big) \right| \leq \int_{(E_k \triangle E) \cap \partial B(x,r)} |\varphi| \, d\mathcal{H}^{n-1}$$
$$\leq ||\varphi||_{L^\infty} \mathcal{H}^{n-1}\big((E_k \triangle E) \cap \partial B(x,r)\big)$$

and the statement then follows by (3.4).

2. $\mu_{E_k \cap B(x,r)} \overset{*}{\rightharpoonup} \mu_{E \cap B(x,r)}$ as $k \to +\infty$. Indeed, by (3.6), we get

$$\limsup_{k \to +\infty} P\big(E_k \cap B(x,r)\big) \leq \limsup_{k \to +\infty} P\big(E_k; B(x,r)\big) + P\big(B(x,r)\big)$$
$$\leq P\big(E; \overline{B(x,r)}\big) + P\big(B(x,r)\big) < +\infty.$$

Thus, the statement follows by $E_k \cap B(x,r) \to E \cap B(x,r)$ and by lower semicontinuity of perimeter.

3. $\mu_{E_k} \lfloor_{B(x,r)} \overset{*}{\rightharpoonup} \mu_E \lfloor_{B(x,r)}$ as $k \to +\infty$. For every $s > r > 0$, we consider a function $\psi_s \in C_c^0\big(B(x,s); [0,1]\big)$ such that $\psi_s \equiv 1$ on $B(x,r)$. Then, for every $\varphi \in C_c^\infty(\mathbb{R}^n)$, it holds that

$$\left| \int_{B(x,r)} \varphi \, d\mu_E - \int_{\mathbb{R}^n} \varphi \psi_s \, d\mu_E \right| \leq ||\varphi||_{L^\infty} |\mu_E|\big(\overline{B(x,s)} \smallsetminus B(x,r)\big)$$

and, by $\mu_{E_k} \overset{*}{\rightharpoonup} \mu_E$, we get also

$$\limsup_{k \to +\infty} \left| \int_{B(x,r)} \varphi \, d\mu_{E_k} - \int_{\mathbb{R}^n} \varphi \psi_s \, d\mu_{E_k} \right| \leq ||\varphi||_{L^\infty} |\mu_E|\big(\overline{B(x,s)} \smallsetminus B(x,r)\big).$$

Since

$$\int_{\mathbb{R}^n} \varphi \psi_s \, d\mu_{E_k} \to \int_{\mathbb{R}^n} \varphi \psi_s \, d\mu_E \qquad \text{as } k \to +\infty,$$

by letting first $k \to +\infty$ and then $s \to r^+$, we obtain

$$\limsup_{k \to +\infty} \left| \int_{B(x,r)} \varphi \, d\mu_{E_k} - \int_{B(x,r)} \varphi \, d\mu_E \right| \leq 2||\varphi||_{L^\infty} |\mu_E|\big(\partial B(x,r)\big) = 0,$$

where the last equality follows from (3.5).

By what we have proved so far, we can pass to the limit as $k \to +\infty$ is (3.6) in order to get (3.1). Again, by (3.5), the vector-valued Radon measures on the right-hand side of (3.1) are mutually singular, so that (3.2) and (3.3) follow immediately from (3.1).

Eventually, if E is unbounded, we simply fix $h \in \mathbb{N}$ and we apply the same procedure to the bounded set of locally finite perimeter $E \cap B(x,h)$ in order to prove the statement for a.e. $r \in (0,h)$. Since a countable union of negligible sets is negligible, the statement follows for a.e. $r \in (0, +\infty)$. $\qquad\square$

Lemma 3.9 (Characterization of half-spaces)**.** *If $E \subset \mathbb{R}^n$ is a set of locally finite perimeter in $\mathbb{R}^n$ and $\nu \in S^{n-1}$ is such that $\nu_E = \nu$ for $|\mu_E|$-a.e. $y \in \partial^* E$, then there exists $\alpha \in \mathbb{R}$ such that E is equivalent to the open half-space*

$$\{z \in \mathbb{R}^n \ \text{s.t.} \ z \cdot \nu < \alpha\}.$$

Proof. Fix an even regularizing kernel $\{\rho_\varepsilon\}_{\varepsilon>0} \subset C_c^\infty(\mathbb{R}^n)$ and let $u_\varepsilon := \chi_E \star \rho_\varepsilon \in C^\infty(\mathbb{R}^n)$, for every $\varepsilon > 0$. By Fubini's theorem, for very $T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$ it holds that

$$\int_{\mathbb{R}^n} u_\varepsilon \operatorname{div} T \, d\mathcal{L}^n = \int_E \operatorname{div}(T \star \rho_\varepsilon) \, d\mathcal{L}^n.$$

Moreover, since

$$\int_{\mathbb{R}^n} \operatorname{div}(u_\varepsilon T) \, d\mathcal{L}^n = \int_{\mathbb{R}^n} (\nabla u_\varepsilon \cdot T) \, d\mathcal{L}^n + \int_{\mathbb{R}^n} u_\varepsilon \operatorname{div} T \, d\mathcal{L}^n = 0,$$

we obtain that

$$-\int_{\mathbb{R}^n} (\nabla u_\varepsilon \cdot T) \, d\mathcal{L}^n = \int_E \operatorname{div}(T \star \rho_\varepsilon) \, d\mathcal{L}^n = \int_{\mathbb{R}^n} (T \star \rho_\varepsilon) \cdot \, d\mu_E = \int_{\mathbb{R}^n} (T \star \rho_\varepsilon) \cdot \nu \, d|\mu_E|. \tag{3.7}$$

Now fix any $\eta \in S^{n-1} \cap \nu^\perp$. For every $\varphi \in C_c^1(\mathbb{R}^n)$, by applying (3.7) with $T = \varphi \eta$ we obtain that

$$-\int_{\mathbb{R}^n} \frac{\partial u_\varepsilon}{\partial \eta} \varphi \, d\mathcal{L}^n = 0.$$

By arbitrariness of $\varphi \in C_c^1(\mathbb{R}^n)$ we conclude that

$$\frac{\partial u_\varepsilon}{\partial \eta} = 0 \qquad \text{on } \mathbb{R}^n.$$

On the other hand, for every fixed $\varphi \in C_c^1(\mathbb{R}^n)$ such that $\varphi \geq 0$, by applying (3.7) with $T = \varphi \nu$ we get

$$-\int_{\mathbb{R}^n} \frac{\partial u_\varepsilon}{\partial \nu} \varphi \, d\mathcal{L}^n = \int_{\mathbb{R}^n} \varphi_\varepsilon \, d\mu_E \geq 0.$$

Again, by arbitrariness of the non-negative test function φ, we get

$$\frac{\partial u_\varepsilon}{\partial \nu} \leq 0 \qquad \text{on } \mathbb{R}^n.$$

By what we have proved so far, we conclude that u_ε is a non-increasing function in the variable $x \cdot \nu$, i.e. there exists a non-increasing function $f_\varepsilon \in C^\infty(\mathbb{R}; [0,1])$ such that

$$\big(\chi_E \star \rho_\varepsilon\big)(x) = u_\varepsilon(x) = f_\varepsilon(x \cdot \nu), \qquad \text{for every } x \in \mathbb{R}^n.$$

By letting $\varepsilon \to 0^+$ in the previous equality, we get that

$$\lim_{\varepsilon \to 0^+} f_\varepsilon(t) \in \{0, 1\}, \qquad \text{for a.e. } t \in \mathbb{R}.$$

Hence, there exists $\alpha \in \mathbb{R}$ such that $f_\varepsilon(t) \to \chi_{(-\infty, \alpha)}(t)$, for a.e. $t \in \mathbb{R}$, that is

$$\big(\chi_E \star \rho_\varepsilon\big)(x) = u_\varepsilon(x) \to \chi_{(-\infty, \alpha)}(x \cdot \nu), \qquad \text{for a.e. } x \in \mathbb{R}^n.$$

Thus, the statement follows by $\chi_E \star \rho_\varepsilon \to \chi_E$ in $L^1_{loc}(\mathbb{R}^n)$. $\qquad\qquad \square$

Theorem 3.10 (Tangential properties of the reduced boundary). *Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter in $\mathbb{R}^n$ and $x_0 \in \partial^* E$.*
Then,

$$E_{x_0, r} \xrightarrow{loc} H_{x_0} = \{x \in \mathbb{R}^n \text{ s.t. } x \cdot \nu_E(x_0) \leq 0\} \qquad \text{as } r \to 0^+.$$

14

Moreover, if $\pi_{x_0} = \partial H_{x_0} = \nu_E(x_0)^{\perp}$, then

$$\mu_{E_{x_0,r}} \xrightarrow{*} \nu_E(x_0)\mathcal{H}^{n-1}\lfloor_{\pi_{x_0}}$$

and

$$\left|\mu_{E_{x_0,r}}\right| \xrightarrow{*} \mathcal{H}^{n-1}\lfloor_{\pi_{x_0}},$$

as $r \to 0^+$.

Proof. Fix $x_0 \in \partial^* E$. We divide the proof onto four steps.

Step one: we claim the existence of $r(x_0), C(n) > 0$ such that

$$P\big(E \cap B(x_0, r)\big) \leq 3\mathcal{H}^{n-1}\big(E \cap \partial B(x_0, r)\big), \qquad \text{for a.e. } 0 < r < r(x_0), \qquad (3.8)$$

$$P\big(E; B(x_0, r)\big) \leq C(n)r^{n-1}, \qquad \text{for every } 0 < r < r(x_0). \qquad (3.9)$$

Indeed, by Lemma 3.8-(3.1), for a.e. $r > 0$ we can pick $\varphi \in C^1_c(\mathbb{R}^n)$ such that $\varphi \equiv 1$ on $\overline{B(x_0, r)}$ in order to get that

$$0 = \int_{E \cap B(x_0,r)} \nabla\varphi \, d\mathcal{L}^n = \int_{\mathbb{R}^n} \varphi \, d\mu_{E \cap B(x_0,r)}$$

$$= \int_{E \cap \partial B(x_0,r)} \varphi \nu_{B(x_0,r)} \, d\mathcal{H}^{n-1} + \int_{B(x_0,r)} \varphi \, d\mu_E$$

$$= \int_{E \cap \partial B(x_0,r)} \nu_{B(x_0,r)} \, d\mathcal{H}^{n-1} + \mu_E\big(B(x_0,r)\big).$$

This implies that $\left|\mu_E\big(B(x_0,r)\big)\right| \leq \mathcal{H}^{n-1}\big(E \cap \partial B(x_0,r)\big)$, for a.e. $r > 0$. At the same time, since $x_0 \in \partial^* E$, there exists $r(x_0) > 0$ such that

$$\frac{1}{2} \leq \frac{\left|\mu_E\big(B(x_0,r)\big)\right|}{|\mu_E|\big(B(x_0,r)\big)} \Leftrightarrow P\big(E; B(x_0,r)\big) \leq 2\left|\mu_E\big(B(x_0,r)\big)\right|, \qquad \text{for every } 0 < r < r(x_0).$$

Combining the previous inequalities, we finally get that

$$P\big(E; B(x_0,r)\big) \leq 2\mathcal{H}^{n-1}\big(E \cap \partial B(x_0,r)\big), \qquad \text{for a.e. } 0 < r < r(x_0). \qquad (3.10)$$

By exploiting Lemma 3.8-(3.3), we eventually get (3.8). Moreover, as

$$\mathcal{H}^{n-1}\big(E \cap \partial B(x_0,r)\big) \leq n\omega_n r^{n-1},$$

by (3.10) it follows that (3.9) holds true for a.e. $0 < r < r(x_0)$. The validity of (3.9) *for every* $0 < r < r(x_0)$ follows by the fact that $P\big(E; B(x_0,r)\big)$ is an increasing function of $r > 0$.

Step two: we prove two lower bounds on the n-dimensional density ratios pf E and $\mathbb{R}^n \setminus E$ at x_0. More precisely, we want to show that

$$\frac{|E \cap B(x_0,r)|}{r^n} \geq \frac{1}{(3n)^n} \qquad \text{for every } 0 < r < r(x_0) \qquad (3.11)$$

and

$$\frac{\left|\big(\mathbb{R}^n \setminus E\big) \cap B(x_0,r)\right|}{r^n} \geq \frac{1}{(3n)^n} \qquad \text{for every } 0 < r < r(x_0). \qquad (3.12)$$

15

Since $\mu_E = -\mu_{\mathbb{R}^n \setminus E}$, it holds that $\partial^* E = \partial^*(\mathbb{R}^n \setminus E)$. Thus, it's enough to prove (3.11). To this end, we define the increasing function $m : (0, +\infty) \to [0, +\infty)$ by setting

$$m(r) := |E \cap B(x_0, r)|, \qquad \text{for every } r \in (0, +\infty).$$

By the coarea formula, it holds that

$$m(r) = |E \cap B(x_0, r)| = \int_0^r \mathcal{H}^{n-1}\big(E \cap \partial B(x_0, t)\big)\, dt, \qquad \text{for every } r \in (0 + \infty).$$

Thus, m is a absolutely continuous and

$$m'(r) = \mathcal{H}^{n-1}\big(E \cap \partial B(x_0, r)\big), \qquad \text{for a.e. } r \in (0, +\infty).$$

Moreover, $m(r) > 0$ for a.e. $r \in (0, +\infty)$. Indeed, since $x_0 \in \partial^* E \subset \mathrm{spt}(\mu_E)$, we get that $|\mu_E|\big(B(x_0, r)\big) > 0$, for every $r \in (0 + \infty)$. By Lemma 3.8-(3.3), we conclude that $P\big(E \cap B(x_0, r)\big) > 0$, for a.e. $r \in (0, +\infty)$. Hence $m(r) = |E \cap B(x_0, r)| \neq 0$ because otherwise it would be $P\big(E \cap B(x_0, r)\big) = 0$.

By a well known perimeter bound on the volume of a set of locally finite perimeter in $\mathbb{R}^n$ and by the first step, we get that

$$m(r)^{\frac{n-1}{n}} = |E \cap B(x_0, r)|^{\frac{n-1}{n}} \leq P\big(E \cap B(x_0, r)\big)$$
$$\leq 3\mathcal{H}^{n-1}\big(E \cap \partial B(x_0, r)\big) = 3m'(r), \qquad \text{for a.e. } 0 < r < r(x_0).$$

Dividing the previous inequality by $m(r)^{\frac{n-1}{n}} > 0$, we get

$$\frac{1}{3} \leq m'(r)m(r)^{\frac{1}{n}-1} = n\Big(m^{\frac{1}{n}}\Big)'(r),$$

which by integration over r gives $m(r)^{\frac{1}{n}} \geq r/(3n)$, for every $0 < r < r(x_0)$. *Step three*: we prove that $E_{x_0, r}$ locally converges to H_{x_0}, as $r \to 0^+$. To this end, it suffices to show that for every sequence $\{r_k\}_{k \in \mathbb{N}}$ such that $r_k \to 0^+$ as $k \to +\infty$ there exists a subsequence $\{r_{k_h}\}_{h \in \mathbb{N}}$ such that $E_{x_0, r_{k_h}}$ locally converges to H_{x_0} as $h \to +\infty$. By Lemma 3.7 and (3.9), for every $R > 0$ it holds that

$$P\big(E_{x_0, r}; B_R\big) = \frac{P\big(E; B(x_0, rR)\big)}{r^{n-1}} \leq C(n)R^{n-1} < +\infty, \qquad \text{for every } 0 < r < \frac{r(x_0)}{R}.$$

Hence, given a sequence $\{r_k\}_{k \in \mathbb{N}}$ such that $r_k \to 0^+$, by compactness of sets of locally finite perimeter we know that, up to extracting a subsequence, there exists a set of locally finite perimeter $F \subset \mathbb{R}^n$ such that

$$E_{x_0, r_k} \xrightarrow{loc} F$$

and

$$\mu_{E_{x_0, r_k}} \xrightarrow{*} \mu_F,$$

as $k \to +\infty$. We now claim that

$$\big|\mu_{E_{x_0, r_k}}\big| \xrightarrow{*} |\mu_F| \tag{3.13}$$

and

$$\nu_F(x) = \nu_E(x_0), \qquad \text{for } |\mu_F|\text{-a.e. } x \in \partial^* F. \tag{3.14}$$

16

Indeed, up to extracting a further subsequence, there exists a Radon measure λ such that $\left|\mu_{E_{x_0,r_k}}\right| \overset{*}{\rightharpoonup} \lambda$ as $k \to +\infty$. By well known properties of Radon measures, for a.e. $R > 0$ (more precisely, for every $R > 0$ such that $\lambda(\partial B_R) = 0$) it holds that

$$\lim_{k \to +\infty} \mu_{E_{x_0,r_k}}(B_R) = \mu_F(B_R). \tag{3.15}$$

At the same time, since $x_0 \in \partial^* E$, by Lemma 3.7 we get that

$$\lim_{r \to 0^+} \frac{\mu_{E_{x_0,r}}(B_R)}{\left|\mu_{E_{x_0,r}}\right|(B_R)} = \lim_{r \to 0^+} \frac{\mu_E\big(B(x_0, rR)\big)}{\left|\mu_E\right|\big(B(x_0, rR)\big)} = \nu_E(x_0).$$

In particular, we get that

$$\lim_{r \to 0^+} \frac{\mu_{E_{x_0,r}}(B_R) \cdot \nu_E(x_0)}{P\big(E_{x_0,r}; B_R\big)} = 1. \tag{3.16}$$

By lower semicontinuity of perimeter, (3.15) and (3.16), we conclude that

$$P(F; B_R) \leq \liminf_{k \to +\infty} P\big(E_{x_0,r_k}; B_R\big) = \lim_{k \to +\infty} \mu_{E_{x_0,r}}(B_R) \cdot \nu_E(x_0)$$
$$= \nu_E(x_0) \cdot \mu_F(B_R) \leq \left|\mu_F(B_R)\right| \leq \left|\mu_F\right|(B_R) = P(F; B_R),$$

for a.e. $R > 0$. In particular, we find that

$$\lim_{k \to +\infty} \left|\mu_{E_{x_0,r_k}}\right|(B_R) = \left|\mu_F\right|(B_R) \tag{3.17}$$

and

$$\left|\mu_F\right|(B_R) = \nu_E(x_0) \cdot \mu_F(B_R). \tag{3.18}$$

By well known properties of Radon measures, from $\mu_{E_{x_0,r_k}} \overset{*}{\rightharpoonup} \mu_F$ and (3.17) we deduce (3.13). Moreover, by (3.18), it follows that

$$0 = \int_{B_R} \big(1 - \nu_E(x_0) \cdot \nu_F(x)\chi_{\partial^* F}(x)\big)\, d|\mu_F|(x), \qquad \text{for a.e. } R > 0$$

and this implies (3.14), since $1 - \nu_E(x_0) \cdot \nu_F(x)\chi_{\partial^* F}(x) \geq 0$ for every $x \in \partial^* F$. By (3.14) and Lemma 3.9, we conclude that there exists $\alpha \in \mathbb{R}$ such that

$$\big|F \triangle \{x \in \mathbb{R}^n \text{ s.t. } \nu_E(x_0) \cdot x < \alpha\}\big| = 0.$$

If $\alpha < 0$, then $F \subset H_{x_0}$ and $|F \cap B_{-\alpha}| = 0$, so that

$$0 = \frac{|F \cap B_{-\alpha}|}{|B_{-\alpha}|} = \lim_{k \to +\infty} \frac{\left|E_{x_0,r_k} \cap B_{-\alpha}\right|}{|B_{-\alpha}|} = \lim_{k \to +\infty} \frac{|E \cap B(x_0, -r_k\alpha)|}{|B(x_0, -r_k\alpha)|},$$

in contradiction with (3.11). Similarly, $\alpha > 0$ would contradict (3.12). Hence it must be $\alpha = 0$ and the claim follows.

Step four: we have thus proved that $E_{x_0,r} \overset{loc}{\longrightarrow} H_{x_0}$, $\mu_{E_{x_0,r}} \overset{*}{\rightharpoonup} \mu_{H_{x_0}}$ and $\left|\mu_{E_{x_0,r}}\right| \overset{*}{\rightharpoonup} \left|\mu_{H_{x_0}}\right|$ as $r \to 0^+$. The remaining part of the statement then follows because, since H_{x_0} is an open set with C^1-boundary, it holds that $\mu_{H_{x_0}} = \nu_E(x_0)\mathcal{H}^{n-1}\lfloor_{\pi_{x_0}}$. $\qquad\square$

As an immediate application of Theorem 3.10, we can show that the n-dimensional density at each point of the reduced boundary is $1/2$.

Corollary 3.11 (Density at the points in the reduced boundary). *If $E \subset \mathbb{R}^n$ is a set of locally finite perimeter in $\mathbb{R}^n$ and $x_0 \in \partial^* E$, then :*

$$\lim_{r \to 0^+} \frac{|E \cap B(x_0, r)|}{\omega_n r^n} = \frac{1}{2} \tag{3.19}$$

and

$$\lim_{r \to 0^+} \frac{P\big(E; B(x_0, r)\big)}{\omega_{n-1} r^{n-1}} = 1. \tag{3.20}$$

In particular, $\partial^* E \subset E^{(1/2)}$.

Proof. Fix any $x_0 \in \partial^* E$ and let H_{x_0}, π_{x_0} as in Theorem 3.10. Since $|H_{x_0} \cap B| = \omega_n/2$, the local convergence of $E_{x_0,r}$ to H_{x_0} implies that

$$\frac{|E \cap B(x_0, r)|}{\omega_n r^n} = \frac{|E_{x_0,r} \cap B|}{\omega_n} \to \frac{|H_{x_0} \cap B|}{\omega_n} = \frac{1}{2},$$

as $r \to 0^+$. Thus, we have proved (3.19).
As $\pi_{x_0} \cap \partial B$ is a $(n-2)$-dimensional unit sphere, we have that

$$\mathcal{H}^{n-1} \lfloor_{\pi_{x_0}} (\partial B) = \mathcal{H}^{n-1}\big(\pi_{x_0} \cap \partial B\big) = 0.$$

Hence, as $\big|\mu_{E_{x_0,r}}\big| \overset{*}{\rightharpoonup} \mathcal{H}^{n-1} \lfloor_{\pi_{x_0}}$ as $r \to 0^+$, we get that

$$\lim_{r \to 0^+} \frac{P\big(E; B(x_0, r)\big)}{\omega_{n-1} r^{n-1}} = \lim_{r \to 0^+} \frac{|\mu_E|\big(B(x_0, r)\big)}{\omega_{n-1} r^{n-1}} = \lim_{r \to 0^+} \frac{\big|\mu_{E_{x_0,r}}\big|(B)}{\omega_{n-1}}$$

$$= \frac{\mathcal{H}^{n-1} \lfloor_{\pi_{x_0}} (B)}{\omega_{n-1}} = \frac{\mathcal{H}^{n-1}\big(\pi_{x_0} \cap B\big)}{\omega_{n-1}} = 1.$$

$\square$

The structure of the Gauss-Green measure and the essential boundary

We recall the following result about the characterization of the locally $\mathcal{H}^k$-rectifiable sets, which was presented by Prof. Rivière during the first lecture.

Theorem 3.12 (Rectifiability from convergence of blow-ups). *Let μ be a Radon measure on $\mathbb{R}^n$ and $M \subset \mathbb{R}^n$ be a Borel subset of $\mathbb{R}^n$ such that:*

1. *μ is concentrated on M;*

2. *for every $x_0 \in M$ there exists a k-dimensional plane π_{x_0} in $\mathbb{R}^n$ such that*

$$\frac{(\Phi_{x_0,r})_{\#}\mu}{r^k} \overset{*}{\rightharpoonup} \mathcal{H}^k \lfloor_{\pi_{x_0}}, \qquad as \ r \to 0^+.$$

Then, M is locally $\mathcal{H}^k$-rectifiable and $\mu = \mathcal{H}^k \lfloor_M$.

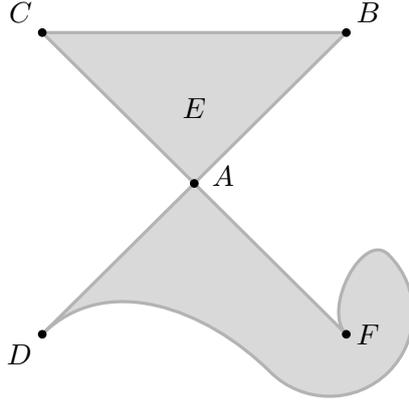By Theorem 3.10 and Theorem 3.12, the following celebrated result due to E. De Giorgi follows directly.

Figure 1: The following set of finite perimeter $E \subset \mathbb{R}^n$ is such that $\mathrm{spt}(\mu_E) = \partial E$, $\partial^e E = \partial E \setminus \{D, F\}$, $E^{(1/2)} = \partial E \setminus \{D, F, C, B\}$ and $\partial^* E = \partial E \setminus \{D, F, C, B, A\}$.

Theorem 3.13 (De Giorgi's structure theorem)**.** *Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter. Then, $\partial^* E$ is a locally $\mathcal{H}^{n-1}$-rectifiable subset of $\mathbb{R}^n$ and*

$$\mu_E = \nu_E \mathcal{H}^{n-1} \lfloor_{\partial^* E}.$$

Moreover, for every $x_0 \in \partial^* E$, the approximate tangent space of $\partial^* E$ at the point x_0 is $\pi_{x_0} = \nu_E(x_0)^\perp$.

Proof. By well know properties of Radon measures, we know that $\partial^* E$ is a Borel subset of $\mathbb{R}^n$ and we have already noticed that $|\mu_E|$ is concentrated on $\partial^* E$ by Remark 3.3. Moreover, by Theorem 3.10, for every $x_0 \in \partial^* E$ it holds that

$$\left| \mu_{E_{x_0,r}} \right| = \frac{(\Phi_{x_0,r})_{\#}\mu_E}{r^k} \overset{*}{\rightharpoonup} \mathcal{H}^k \lfloor_{\pi_{x_0}},$$

where $\pi_{x_0} = \nu_E(x_0)^\perp$. Hence, by applying Theorem 3.12 with $\mu = |\mu_E|$ and $M = \partial^* E$, we get that $\partial^* E$ is locally $\mathcal{H}^{n-1}$-rectifiable, $|\mu_E| = \mathcal{H}^{n-1} \lfloor_{\partial^* E}$ and, by definition, the approximate tangent space of $\partial^* E$ at every point $x_0 \in \partial^* E$ is given by $\nu_E(x_0)^\perp$. The remaining part of the statement follows directly from Remark 3.3. $\qquad\square$

Definition 3.14. Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable subset of $\mathbb{R}^n$.
We define the **essential boundary** of E as follows:

$$\partial^e E := \mathbb{R}^n \setminus \left(E^{(0)} \cup E^{(1)} \right).$$

Remark 3.15. Trivially, $E^{(1/2)} \subset \partial^e E$ for every Lebesgue measurable $E \subset \mathbb{R}^n$. Moreover, if $E \subset \mathbb{R}^n$ is a set of locally finite perimeter in $\mathbb{R}^n$, by Corollary 3.11 and by the characterization of $\mathrm{spt}(\mu_E)$, it follows that

$$\partial^* E \subset E^{(1/2)} \subset \partial^e E \subset \mathrm{spt}(\mu_E).$$

Eventually, notice that this chain of inclusions may very easily be strict at each level, as it is shown in Figure 1.

By Remark 3.5 and Theorem 3.10, we know what happens when we blow-up a set of locally finite perimeter E at every point in $\partial^* E \cup (\partial^e E)^c$. Moreover, by Remark 3.6, we know that we have no control on the behaviour of the blow-ups family at the remaining

points, namely, on $\partial^e E \setminus \partial^* E$. A deep result, due to H. Federer, states that the size of this set of bad points cannot be too large.

Before stating Federer's theorem, we recall the following proposition regarding a local perimeter bound on the n-dimensional volume of a set of locally finite perimeter (to find a proof of this statement, we refer the reader to [3, Proposition 12.37]).

Proposition 3.16 (Local perimeter bound on volume). *Let $n \in \mathbb{N}$ be such that $n \geq 2$, $t \in (0, 1)$, $x_0 \in \mathbb{R}^n$ and $r > 0$.*
Then, there exists a positive constant $c(n, t) > 0$ such that

$$P\big(E; B(x_0, r)\big) \geq c(n, t) \, |E \cap B(x_0, r)|^{\frac{n-1}{n}}, \tag{3.21}$$

for every set of locally finite perimeter $E \subset \mathbb{R}^n$ such that $|E \cap B(x_0, r)| \leq t |B(x_0, r)|$.

Remark 3.17. Fix any set of locally finite perimeter $E \subset \mathbb{R}^n$, $x_= \in \mathbb{R}^n$ and $r > 0$. Since $|B(x_0, r)| = |E \cap B(x_0, r)| + |B(x_0, r) \setminus E|$, then at least one between $E \cap B(x_0, r)$ and $B(x_0, r) \setminus E$ has volume smaller then $1/2|B(x_0, r)|$. Thus, since

$$
\begin{aligned}
P\big(E; B(x_0, r)\big) &= P\big(\mathbb{R}^n \setminus E; B(x_0, r)\big) \\
&\geq P\big(E \cap B(x_0, r); B(x_0, r)\big) = P\big(B(x_0, r) \setminus E; B(x_0, r)\big),
\end{aligned}
$$

by applying Proposition 3.16 with $t = 1/2$ to either $E \cap B(x_0, r)$ or $B(x_0, r) \setminus E$ we get that

$$P\big(E; B(x_0, r)\big) \geq C(n) \min\big\{ |E \cap B(x_0, r)|, |B(x_0, r) \setminus E| \big\}^{\frac{n-1}{n}},$$

where $C(n) := c(n; 1/2)$.

Theorem 3.18 (Federer's theorem). *Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter in $\mathbb{R}^n$. Then,*

$$\mathcal{H}^{n-1}\big(\partial^e E \setminus \partial^* E\big) = 0. \tag{3.22}$$

Proof. Trivially, for every $x_0 \in \mathbb{R}^n$ and for every $r > 0$ it holds that

$$|E \cap B(x_0, r)|^{\frac{n-1}{n}} \geq \frac{|E \cap B(x_0, r)|}{\omega_n^{1/n} r}$$

and

$$|B(x_0, r) \setminus E|^{\frac{n-1}{n}} \geq \frac{|B(x_0, r) \setminus E|}{\omega_n^{1/n} r}.$$

Hence, by Remark 3.17 and De Giorgi structure theorem, we conclude that there exists a constant $K(n) > 0$ such that

$$\frac{\mathcal{H}^{n-1}\lfloor_{\partial^* E}\big(B(x_0, r)\big)}{r^{n-1}} = \frac{P\big(E; B(x_0, r)\big)}{r^{n-1}} \geq K(n) \min\left\{ \frac{|E \cap B(x_0, r)|}{r^n}, \frac{|B(x_0, r) \setminus E|}{r^n} \right\},$$

for every $x_0 \in \mathbb{R}^n$ and for every $r > 0$. Hence,

$$\theta_{n-1}^*\big(\mathcal{H}^{n-1}\lfloor_{\partial^* E}\big)(x_0) = \limsup_{r \to 0^+} \frac{\mathcal{H}^{n-1}\lfloor_{\partial^* E}\big(B(x_0, r)\big)}{r^{n-1}} = 0$$

implies $x_0 \in E^{(0)} \cup E^{(1)}$. This means that

$$\partial^e E \subset \big\{ x \in \mathbb{R}^n \text{ s.t. } \theta_{n-1}^*\big(\mathcal{H}^{n-1}\lfloor_{\partial^* E}\big)(x) > 0 \big\}$$

20

and, thus,

$$\partial^e E \smallsetminus \partial^* E \subset \left\{ x \in \mathbb{R}^n \smallsetminus \partial^* E \text{ s.t. } \theta^*_{n-1}\left(\mathcal{H}^{n-1}\lfloor_{\partial^* E}\right)(x) > 0 \right\}.$$

The statement then follows since, by well know properties of the lower dimensional densities of Radon measures (see e.g. [3, Corollary 6.5]), the set appearing in the right-hand side of the last inclusion is $\mathcal{H}^{n-1}$-negligible. □

References

[1] Lawrence Evans and Ronald Gariepy. *Measure Theory and Fine Properties of Functions, Revised Edition.* Chapman and Hall/CRC, 2015. (Visited on 02/26/2020).

[2] Gian Paolo Leonardi. "Blow-up of oriented boundaries". In: *Rend. Sem. Mat. Univ. Padova* 103 (2000), pp. 211–232. ISSN: 0041-8994. URL: http://www.numdam.org/item?id=RSMUP_2000__103__211_0.

[3] Francesco Maggi. *Sets of Finite Perimeter and Geometric Variational Problems.* Cambridge University Press, 2012.