1 Rectifiable Sets and Preliminary Measure Theory

This section is presented by Prof. Dr. Tristan Rivière on Feb 20, 2020 with the main reference to [4]. The goal is to provide a common starting point and some reference to preliminaries. The notes here are written by Yujie Wu. The author tries to recover what the lecturer carried out, but it's possible that she introduced error in this process. If you find any mistake please feel free to contact the writer via yujwu@student.ethz.ch.

First of all, we assume familiarity with the notion of finite dimensional manifolds, immersions, embeddings and submanifiolds, especially in Euclidean spaces. A introductory reading can be [2]. For simplicity all the measures in this section are defined on subsets of $\mathbb{R}^n$.

We start with recalling the definition of Hausdorff measure.

Definition 1.1. The k-dimensional Hausdorff measure of step δ is defined through approximation by covering with sets with diameter less or equal to δ. To be precise,

$$\mathcal{H}^k_\delta(E) := \inf_{F\in\mathcal{F}_\delta(E)} \sum_j \omega_k(\frac{\text{diam}F_j}{2})^k$$

where $\mathcal{F}_\delta$ is the set of admissible covering, that is,

$$\mathcal{F}_\delta(E) := \{(F_j)_{j\in\mathbb{N}}, F_j \subset \mathbb{R}^n, \text{diam}F_j < \delta, E \subset \cup_j F_j\}.$$

If no such covering exists, then we say that E does not have finite H^k measure.

The k-dimensional Hausdorff measure of step δ increases as $\delta \to 0$, so we can define the k-dimensional Hausdorff measure as,

$$\mathcal{H}^k(E) := \sup_{\delta>0} \mathcal{H}^k_\delta(E) = \lim_{\delta\to 0} \mathcal{H}^k_\delta(E).$$

One can check that the k-dimensional Hausdorff measure is an outer measure.

Definition 1.2. We say that a set E is measurable with respect to an outer measure μ if for any set F,

$$\mu(F) = \mu(E \cap F) + \mu(F \setminus E).$$

The family of measurable sets form a σ-algebra, and μ is countably additive on measurable sets.

A Borel measure is an outer measure defined on the σ-algebra of Borel sets.

The following lemma is called the Carathéodory condition.

Lemma 1.3. *If μ is an outer measure on $\mathbb{R}^n$, then Borel sets are measurable with respect to μ if and only if for any two sets with positive distance, E_1, E_2, $d(E_1, E_2) = \inf_{x\in E_1, y\in E_2} d(x, y) > 0$, we have,*

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2).$$

Notice that Borel sets are not measurable with respect to $\mathcal{H}^k_\delta$, but with respect to $\mathcal{H}^k$.

Definition 1.4. An outer measure μ is Borel regular if Borel sets are μ-measurable, and for any set $E \subset \mathbb{R}^n$, there is a Borel set $F \supset E$, such that $\mu(F) = \mu(E)$.

The k-dimensional Hausdorff measure is Borel regular.

Definition 1.5. An outer measure is a Radon measure if it is locally finite (finite on compact sets), Borel regular.

We want to move on to the definition of rectifiability.

Definition 1.6. Assume an integer $0 \le k \le n$. A subset M in $\mathbb{R}^n$ is k-rectifiable, if $\mathcal{H}^k(M) < \infty$, and there is countably many Lipschitz maps $\varphi_j : \mathbb{R}^k \to \mathbb{R}^n$, such that,

$$\mathcal{H}^k(M \setminus \cup_j \varphi_j(\mathbb{R}^k)) = 0.$$

A set M is called locally k-rectifiable, if for any compact set K, $K \cap M$ is recitifiable.

With the following theorem we can see that the above definition can be slightly changed: the domain of the Lipschitz maps can be set as any subset of $\mathbb{R}^n$.

Theorem 1.7 (**Kirszbraun's Theorem**)**.** *A Lipschitz map $f : \mathbb{R}^{n_1} \supset E \to \mathbb{R}^{n_2}$ can be extended to $g : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ with the same Lipschitz constant.*

We mention that understanding of the Rademacher's theorem, Lusin's Theorem, Egorov's theorem and Area formula would also be useful for later use.

With these tools, one can show that k-rectifiable sets can be "almost" covered by C^1 submanifolds of dimension k of $\mathbb{R}^n$.

These discussions lead to the existence of approximate tangent planes.

Theorem 1.8. *If M is a k-rectifiable set, then for $\mathcal{H}^k$ almost everythere $x \in M$, there is a k-dimensional plane π_x in $\mathbb{R}^n$,*

$$\mathcal{H}^k \lfloor \left(\frac{M - x}{r} \right) \overset{*}{\rightharpoonup} \mathcal{H}^k \lfloor \pi_x, \quad \text{as } r \to 0.$$

We call this uniquely determined plane π_x as approximate tangent space to M at x.

Remark 1.9. The weak star convergence as measures is equivalent to

$$\lim_{r \to 0} \int_M \frac{1}{r^k} \varphi \left(\frac{y - x}{r} \right) d\mathcal{H}^k(y) = \int_{\pi_x} \varphi(z) d\mathcal{H}^k(z), \quad \forall \varphi \in C_c^0(\mathbb{R}^n).$$

In particular, the density of the Radon measure $\mu = \mathcal{H}^k \lfloor M$ (when compared with $\mathcal{H}^k$) exists μ almost everywhere and is equal to 1.

Lastly, we would prove the converse also holds true. The proof follows [4], and is a practice of the facts we have quickly mentioned. We also corrected several typos in the proof in [4].

Theorem 1.10 (Existence of Tangent Plane Implies Rectifiability)**.** *If μ is a Radon measure on $\mathbb{R}^n$, M is a Borel set in $\mathbb{R}^n$ where μ is concentrated on, and for every $x \in M$, there exists a k-dimensional plane π_x such that,*

$$\frac{(\Phi_{x,r})_{\#}\mu}{r^k} \overset{*}{\rightharpoonup} \mathcal{H}^k \lfloor \pi_x, \quad \text{as } r \to 0,$$

with $\Phi_{x,r}(z) = \frac{z-x}{r}$.

Furthermore, M is locally k-rectifiable and $\mu = \mathcal{H}^k \lfloor M$.

Remark 1.11. The pushforward of μ is defined to be,

$$\int_{\mathbb{R}^n} \varphi(y) d(\Phi_{x,r})_{\#}\mu = \int_{\mathbb{R}^n} \varphi(\frac{z - x}{r}) d\mu(z),$$

which holds for any $\varphi \in C_c^0(\mathbb{R}^n)$.

We use the following notation. Given π a k-dimensional plane, we denote $\mathbf{p}_\pi$ and $\mathbf{p}_\pi^\perp$ the orthogonal projections respectively onto π and $\pi^\perp$, and we define the following t-cone of π ($t \geq 0$),

$$K(\pi, t) = \{y \in \mathbb{R}^n, |\mathbf{p}_\pi^\perp(y)| \leq t|\mathbf{p}_\pi(y)|\}.$$

We start with a theorem for a sufficient condition for rectifiability.

Theorem 1.12. *If $M \subset \mathbb{R}^n$ is a bounded set, π is a k-dimensional plane, and there is $\delta, t > 0$ (independent of $x \in M$) such that,*

$$M \cap B(x, \delta) \subset x + K(\pi, t), \forall x \in M,$$

then one can find finitely many Lipshitz maps whose image in covers M, and thus M is k-rectifiable.

Proof. For any fixed $x_0 \in M$ and $x, y \in B(x_0, \frac{\delta}{2}) \cap M$, then $y \in B(x, \delta) \cap M \subset x + K(\pi, t)$, that is,

$$|\mathbf{p}_\pi^\perp(y - x)| < t|\mathbf{p}_\pi(y - x)|,$$

and we see that $\mathbf{p}_\pi$ is a bijection from $B(x_0, \frac{\delta}{2}) \cap M$ onto its image $G_{x_0} \subset \pi$. We have the following for the inverse $g_{x_0} : G_{x_0} \to M$,

$$g_{x_0} \circ \mathbf{p}_\pi(z) = z,$$
$$|g_{x_0}(z) - g_{x_0}(w)| = |g_{x_0}\mathbf{p}_\pi(\tilde{z}) - g_{x_0}\mathbf{p}_\pi(\tilde{w})| = |\tilde{z} - \tilde{w}| \leq (1 + t)|z - w|, \forall z, w \in G_{x_0}.$$

Applying a covering of the bounded set M we can find finitely many such Lipshitz maps in the form of g_{x_0}. We may compose with a transformation to allow the domain to be in $\mathbb{R}^k$ instead of π, and we have shown that M is rectifiable. □

Remark 1.13. Notice that in the above proof the k-plane π does not depend on $x \in M$, we shall construct such planes locally on M in the next proof.

Now we begin the proof of Theorem 1.10.

Proof. We split the proof into four steps.

Step 1

First we claim that for any $\lambda > 0$, there are finitely many k-dim planes $(\sigma_h)_{h=1,...,N(\lambda)}$, such that for any k-dim plane $\pi \subset \mathbb{R}^n$, there is a $h \in \{1, ..., N\}$, and π is contained in the λ-cone of σ_h. This can be achieved by first writing down the (not uniquely) k vectors of length one spanning π and approximate using unit-length vectors in S^n to form σ_h, then show that π lies in the λ-cone of σ_h.

Step 2

The second claim we will make is the following. We first assume $0 < \lambda < \lambda_1 < 1$, for λ_1 to be chosen soon. If σ and π are k-dim planes and lie in the λ-cone of each other, then for any $w \notin K(\sigma, 5\lambda)$, $B(w, \lambda|w|) \cap K(\pi, \lambda) = \emptyset$.

We prove this by estimating for $v \in B(w, \lambda|w|) \cap K(\pi, \lambda)$,

$$\begin{aligned}\frac{|\mathbf{p}_\sigma^\perp(w)|}{|w|} &\leq \frac{|\mathbf{p}_\pi^\perp(w)| + |\mathbf{p}_\sigma^\perp \mathbf{p}_\pi(w)|}{|w|} \\ &\leq \frac{|\mathbf{p}_\pi^\perp(v) - \mathbf{p}_\pi^\perp(v-w)|}{|w|} + \lambda \frac{|\mathbf{p}_\sigma \mathbf{p}_\pi(w)|}{|w|} \\ &\leq \lambda + \frac{|\mathbf{p}_\pi^\perp(v) - \mathbf{p}_\pi^\perp(v-w)|}{|w|} \\ &\leq \lambda + \frac{|v-w|}{|w|} + \lambda \frac{|\mathbf{p}_\pi(v)|}{|w|} \\ &\leq 2\lambda + \lambda(1+\lambda) \leq 4\lambda\end{aligned}$$

Now we may choose λ_1 small enough so that $\frac{4\lambda}{\sqrt{1-(4\lambda)^2}} < 5\lambda$ for all $\lambda < \lambda_1$, and the claim follows.

Step 3

Now we move on to the key claim in this proof. If for a bounded subset $M' \subset M$, the following limits hold uniformly for all $x \in M'$, then M' is $\mathcal{H}^k$-rectifiable,

$$\lim_{r \to 0} \frac{\mu(B(x,r))}{\omega_k r^k} = 1, \tag{1.1}$$

$$\lim_{r \to 0} \frac{\mu(B(x,r) \setminus (x + K(\pi, \lambda)))}{\omega_k r^k} = 0. \tag{1.2}$$

To show this we assume that for any $\varepsilon > 0$, there is $\delta > 0$ such that for every $x \in M'$ and $r < 2\delta$,

$$\frac{\mu(B(x,r))}{\omega_k r^k} \geq (1-\varepsilon)\omega_k r^k, \tag{1.3}$$

$$\mu(B(x,r) \setminus (x + K(\pi, \lambda))) \leq \varepsilon \omega_k r^k. \tag{1.4}$$

We set,

$$M_h' = \{x \in M' : \sigma_h \subset K(\pi_x, \lambda)\}, 1 \leq h \leq N.$$

If we manage to show that for small $\varepsilon > 0$,

$$B(x, \delta) \cap M_h' \subset x + K(\sigma_h, 5\lambda), \forall x \in M_h',$$

then by Theorem 1.12, we would have M_h' is k-rectifiable.

Indeed, assume not, then for some $x \in M_h'$, $y \in B(x,\delta) \cap M_h'$ but $y - x \notin K(\sigma_h, 5\lambda)$, then from Step 2 , we know,

$$B(y, \lambda|y-x|) \subset \mathbb{R}^n \setminus (x + K(\pi_x, \lambda)),$$

Since $\lambda < 1$, we have $B(y, \lambda|y-x|) \subset B(x, 2|y-x|)$, and we can apply equations (1.3) and (1.4)

$$(1-\varepsilon)\omega_k \lambda^k |y-x|^k \leq \mu(B(y, \lambda|y-x|)) \leq \varepsilon 2^k \omega_k |x-y|^k, \tag{1.5}$$

which is a contradiction to small ε.

Step 4

Now we are ready to prove that M is locally k-rectifiable. The limits in equation (1.1) and (1.2) exist for every $x \in M$, since

$$\omega_k = \mathcal{H}^k(\pi_x \cap B(0,1)) = \lim_{r \to 0} \frac{(\Phi_{x,r})_\# \mu(B(0,1))}{r^k} = \lim_{\to 0} \frac{\mu(B(x,r))}{r^k}. \tag{1.6}$$

This gives (1.1), and (1.2) is similar.

Thus one can apply Theorem 6.4 in [4], and conclude that, for any Borel set $E \subset \mathbb{R}^n$,

$$\mathcal{H}^k(E \cap M) \leq \mu(E \cap M) \leq 2^k \mathcal{H}^k(E \cap M). \tag{1.7}$$

Thus $\mathcal{H}^k \lfloor M$ is locally finite.

With this we can apply Egorov's theorem on compact subsets of M and follow Step 3, and since μ and $\mathcal{H}^k \lfloor M$ are comparable by (1.7), we can cover M with k-rectifiable subsets $\mathcal{H}^k$ almost everywhere. This gives that M is locally $\mathcal{H}^k$ rectifiable.

Finally, equation (1.6) gives us for any $x \in M$,

$$\lim_{r \to 0} \frac{\mu(B(x,r))}{\mathcal{H}^k \lfloor M(B(x,r))} \cdot \frac{\mathcal{H}^k \lfloor M(B(x,r))}{\omega_k r^k} = 1.$$

and Theorem 1.8 gives,

$$\lim_{r \to 0} \frac{\mathcal{H}^k \lfloor M(B(x,r))}{\omega_k r^k} = 1.$$

Together we may apply Theorem 5.8 in [4] (Lebesgue–Besicovitch differentiation theorem), and since μ is concentrated on M, we have that μ is absolute continuous with respect to $\mathcal{H}^k \lfloor M$ with density 1. Thus we have that $\mu = \mathcal{H}^k \lfloor M$. □

Remark 1.14. Another way of obtaining the estimates (1.3) and (1.4) without using Egorov's Theorem would be the following (thanks to Alessandro Pigati for pointing this out). Since we know that the limits in (1.1) and (1.2) exist by assumption, we may choose a sequence of $\delta_j \to 0$, and for the choice of ε in equation (1.5), we may cover M completely by the M_j's such that the estimates (1.3) and (1.4) holds with δ_j. Now we may apply Step 3 to these M_j's and continue our argument as in the latter part of Step 4.

References

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