2 Sets of Finite Perimeter

This section is presented by Yujie Wu on Feb 27, 2020 with the reference [2]. The notes are wriiten by Yujie Wu, if you find any mistake please feel free to contact the writer via yujwu@student.ethz.ch.

Motivation and the definition

If we take $E \subset \mathbb{R}^n$ to be an open set with C^1 boundary, then we have that ∂E is a C^1 hypersurface and the Gauss-Green formula holds true,

$$\int_E \text{div } T dx = \int_{\partial E} T \cdot \nu_E d\mathcal{H}^{n-1} = \int_{\mathbb{R}^n} T \cdot d\mu_E, \quad \forall T(x) \in C_c^1(\mathbb{R}^n) \tag{2.1}$$

with ν_E the outward-pointing unit normal vector on the boundary of E, and $\mu_E = \nu_E d\mathcal{H}^{n-1} \lfloor \partial E$ is a vector-valued Radon measure.
The perimeter of E would be $\mathcal{H}^{n-1}(\partial E) = |\mu_E|(\mathbb{R}^n)$ (could be infinite).
We would like to look at equation (2.1) for a general $\mathcal{H}^n$ measurable set. If there is such a Radon measure μ_E satisfying the right hand side and left hand side of equation (2.1), it would allow us to consider more general objects than open subset of $\mathbb{R}^n$ with smooth boundary in geometric variational problems, but does this generalization bring some meaningful geometric structure to the set E? We first propose the following definition.

Definition 2.1. Let E be a Lebesgue measurable set in $\mathbb{R}^n$. We say that E is a set of **locally finite perimeter** in $\mathbb{R}^n$ if for every compact set $K \subset \mathbb{R}^n$ we have,

$$\sup \left\{ \int_E \text{div } T(x), T(x) \in C_c^1(K, \mathbb{R}^n), \|T\|_\infty \leq 1, \right\} < \infty. \tag{2.2}$$

If instead one can take $T(x) \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ in equation (2.2) and the supremum is still finite, then we say that E is of finite perimeter in $\mathbb{R}^n$.

The following theorem justifies that the definition is a "generalization" of the Gauss-Green theorem above.

Theorem 2.2. *If E is a Lebesgue measurable set in $\mathbb{R}^n$, then E is a set of locally finite perimeter if and only if there exists a $\mathbb{R}^n$-valued Radon measure μ_E on $\mathbb{R}^n$ such that,*

$$\int_E \text{div } T(x) = \int_{\mathbb{R}^n} T(x) \cdot d\mu_E, \quad \forall T(x) \in C_c^1(\mathbb{R}^n, \mathbb{R}^n) \tag{2.3}$$

*We call the unique Radon measure μ_E the **Gauss-Green measure** of E, and define the **relative parimeter** of E in a subset $F \subset \mathbb{R}^n$ as $P(E; F) = |\mu_E|(F)$, and the **perimeter** of E as $P(E) = |\mu_E|(\mathbb{R}^n)$.*

Remark 2.3. We write the set difference of two sets as $E \triangle F = (E \setminus F) \cup (F \setminus E)$, and denote the Lebesgue measure $\mathcal{L}^n(\cdot)$ as $|\cdot|$. Then if $|E \triangle F| = 0$, the left hand side of equation (2.3) is invariant under modification of a set of Lebesgue measure zero, and thus by the uniqueness part we can get $\mu_E = \mu_F$ as Radon measures on $\mathbb{R}^n$. Thus one may modify the topological boundary of one set E corresponding to μ_E widely by adding for instance all the points in $\mathbb{R}^n$ with rational coordinates. Later we will try to "minimize" the topological boundary of a given set of locally finite perimeter.

Remark 2.4. Another way to introduce set of locally finite perimeter would be through the definition of functions of bounded variations (one may refer to [1]). A function $f(x) \in L^1_{loc}(\mathbb{R}^n)$ has locally bounded variation in $\mathbb{R}^n$ if for any compact $K \subset \mathbb{R}^n$,

$$\sup \left\{ \int_E f(x) \text{div } \phi(x), \phi(x) \in C^1_c(K, \mathbb{R}^n), \|\phi\|_\infty \leq 1, \right\} < \infty. \tag{2.4}$$

Thus, we can say that E has locally finite perimeter if and only if its characteristic function $\mathbb{1}_E$ has locally bounded variation, and the distributional derivative of $\mathbb{1}_E$ is represented by the Radon measure μ_E. Therefore sometimes equation (2.3) is referred to as distributional Gauss-Green and the perimeter of E as the distributional perimeter.

Proof. If such an Radon measure μ_E exists, then clearly E is of locally finite perimeter. On the other hand, We may define the functional $L(T)$ for $T \in C^1_c(K, \mathbb{R}^n)$ with K an arbitrary compact subset. Then $|\langle L, T\rangle| \leq C(K)\|T\|_\infty$. Thus we can extend L to be on $C^0_c(\mathbb{R}^n, \mathbb{R}^n)$, and apply the Riesz representation theorem to the bounded linear functional L to get a vector valued Radon measure μ_E. So that for any $T \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$

$$\int_{\mathbb{R}^n} T \cdot d\mu_E = \langle L, T\rangle = \int_E \text{div } T$$

□

Example 2.5. Thus an open set with C^1 (or Lipschitz) boundary is a set with locally finite perimeter by the Gauss-Green theorem. The Gauss-Green measure would be $\nu_E \mathcal{H}^{n-1} \lfloor \partial E$, with ν_E as the outward-pointing unit normal vector (defined Lebesgue almost everywhere if the boundary is Lipschitz).

Example 2.6. One can prove using the change of variable formula that if E is a set of finite perimeter then $E' = x + \lambda E$ is a set of finite perimeter and $\mu'_E = \lambda^{n-1}\Phi_{\#}\mu_E$.

Example 2.7. If E is a set of finite perimeter then the complement E^c is a set with finite perimeter and $\mu_{E^c} = -\mu_E$.

For the proof of the following lemma we refer to [2].

Lemma 2.8. *A Lebesgue measurable set $E \subset \mathbb{R}$ is of locally finite perimeter if and only if it is equivalent to a countable union of (possibly unbounded) open intervals lying at mutually positive distance.*

Lower semicontinuity of Perimeter

Given Lebesgue measurable sets $\{E_h\}_{h\in\mathbb{N}}$ and E in $\mathbb{R}^n$, we say that $\{E_h\}$ locally converges to E, and write $E_h \xrightarrow{loc} E$, if for any compact $K \subset \mathbb{R}^n$,

$$|K \cap (E_h \triangle E)| \to 0.$$

Theorem 2.9. *If $\{E_h\}_{h\in\mathbb{N}}$ is a sequence of sets with locally finite perimeter, and $E_h \xrightarrow{loc} E$ with the following bound for each compact $K \subset \mathbb{R}^n$,*

$$\liminf_h P(E_h; K) < \infty$$

, then E is also a set of locally finite perimeter, $\mu_{E_h} \xrightarrow{*} \mu_E$, and for any open set A,

$$P(E; A) \leq \liminf_h P(E_h; A). \tag{2.5}$$

Remark 2.10. The inequality (2.5) can be strict, one can check the example as of Figure 12.2 in [2].

Proof. For any $T \in C_c^1(K, \mathbb{R}^n)$ with K compact,

$$\int_E \text{div } T = \int 1_E \text{div } T = \lim_{h\to\infty} \int 1_{E_h} \text{div } T = \lim_{h\to\infty} \int T \cdot d\mu_{E_h} \le \liminf_{h\to\infty} |\mu_{E_h}|(K)\|T\|_\infty.$$

That is to say E has locally finite perimeter. Furthermore,

$$\int T \cdot d\mu_E = \int_E \text{div } T = \lim_{h\to\infty} \int_{E_h} \text{div } T = \lim_{h\to\infty} \int T \cdot d\mu_{E_h}.$$

For any open set A,

$$\begin{aligned}
|\mu_E|(A) &= \int 1_A \, d|\mu_E| \\
&= \sup\{\int T \cdot d\mu_E, T \in C_c^1(A, \mathbb{R}^n), \|T\|_\infty \le 1\} \\
&= \sup\{\int_E \text{div } T, T \in C_c^1(A, \mathbb{R}^n), \|T\|_\infty \le 1\} \\
&= \sup\{\lim_{h\to\infty} \int_{E_h} \text{div } T, T \in C_c^1(A, \mathbb{R}^n), \|T\|_\infty \le 1\} \\
&\le \liminf_{h\to\infty} \sup\{\int_{E_h} \text{div } T, T \in C_c^1(A, \mathbb{R}^n), \|T\|_\infty \le 1\} \\
&= \liminf_{h\to\infty} |\mu_{E_h}|(A)
\end{aligned}$$

□

Exercise 2.11. We prove a useful lemma using Theorem (2.9) that would be used in later proofs. If E is a Lebesgue measurable set in $\mathbb{R}^n$ and $C_c^1(\mathbb{R}^n) \ni \{u_h\}_{h\in\mathbb{N}} \to \mathbb{1}_E$ in $L^1_{loc}(\mathbb{R}^n)$, and for any compact $K \subset \mathbb{R}^n$,

$$\limsup_{h\to\infty} \int_K |\nabla u_h|^2 < \infty,$$

then E is a set with locally finite perimeter with $P(E; A) \le \liminf_{h\to\infty} \int_A |\nabla u_h|^2$, for $A \subset \mathbb{R}^n$ open.

The First Step to Minimize the Topological Boundary

Theorem 2.12. *If E is a set of locally finite perimeter, then*

$$\text{spt } \mu_E = \{x \in \mathbb{R}^n, 0 < |E \cap B_r(x)| < \omega_n r^n, \forall r > 0\} \subset \partial E$$

and there is a Borel set F in $\mathbb{R}^n$, such that,

$$|E \triangle F| = 0, \text{spt } \mu_F = \partial F.$$

Proof. Clearly the set $\{x \in \mathbb{R}^n, \forall r > 0, 0 < |E \cap B_r(x)| < \omega_n r^n\}$ is contained in $\overline{E} \setminus E^\circ = \partial E$. We first write out the characterization of the support,

$$\text{spt}\mu_E = \{x \in \mathbb{R}^n, \forall r > 0, |\mu_E|(B_r(x)) > 0\} = \{x \in \mathbb{R}^n, \exists r > 0, |\mu_E|(B_r(x)) = 0\}^c.$$

Now we show that,

$$(\mathrm{spt}\mu_E)^c = \{x \in \mathbb{R}^n, \exists r > 0, |E \cap B_r(x)| = 0, \text{or } |E \cap B_r(x)| = \omega_n r^n\}$$

First, if $\exists r > 0, |E \cap B_r(x)| = 0$, then for any $\varphi \in C_c^\infty(B_r(x))$,

$$0 = \int_E \nabla\varphi = \int \varphi d\mu_E$$

If we require $\varphi = 1$ on $B_{\frac{r}{2}}(x)$, then we get that $x \in (\mathrm{spt}\mu_E)^c$. Now we can apply the above argument to E^c with $\mu_{E^c} = -\mu_E$, to get that if $\exists r > 0, |E \cap B_r(x)| = w_n r^n$, then $x \in (\mathrm{spt}\mu_E)^c$.

For the other direction, that is, we assume $|\mu_E|(B_r(x)) = 0$ for some $r > 0$, then for any $\varphi \in C_c^\infty(B_r(x))$,

$$0 = \int \varphi\mu_E = \int_E \nabla\varphi.$$

Thus the weak derivative of 1_E in $B_r(x)$ is zero, and we have that either $(1_E)\lfloor_{B_r(x)}= 0$ or $(1_E)\lfloor_{B_r(x)}= 1$. This gives either $|E \cap B_r(x)| = 0$ or $|E \cap B_r(x)| = w_n r^n$.

Now we show the second part of the conclusion, without loss of generality we may assume E is Borel (using the outer Borel regularity mentioned last time). We build the set F as,

$$F = A_1 \cup E \setminus A_0,$$

where A_1, A_0 are two disjoint open set defined as,

$$A_1 = \{x \in \mathbb{R}^n, \exists r > 0, |E \cap B_r(x)| = \omega_n r^n\}$$
$$A_0 = \{x \in \mathbb{R}^n, \exists r > 0, |E \cap B_r(x)| = 0, \}$$

Notice that by a covering argument $|A_0 \cap E| = 0$ and $|A_1 \setminus E| = |A_1 \cap E^c| = 0$. Thus $|F \triangle E| = 0$, F is Borel, and by the first part of the proof, $\mathrm{spt}\mu_F \subset \partial F$. We only need to show that $\partial F \subset (A_1 \cup A_0)^c \subset \mathrm{spt}\mu_F$. Notice that A_1 is an open set contained in F and A_0 is an open set contained in F^c. □

Regularization and Convergence Properties

We assume thar ρ_ε is a mollifier (approximation of identity). For instance we may take a nonnegative function $\rho \in C_c^\infty(B_2(0), \mathbb{R})$, $\rho = 1$ on $B_1(0)$ and $\rho_\varepsilon = \varepsilon^{-n}\rho(\frac{x}{\varepsilon})$. Then $\rho_\varepsilon \in C_c^\infty(B_{2\varepsilon}(0), \mathbb{R})$, $\rho_\varepsilon = 1$ on $B_\varepsilon(0)$, and $\int_{\mathbb{R}^n} \rho_\varepsilon = 1$. For the process below we also assume that ρ is symmetric, that is $\rho(x) = \rho(-x)$.

Then we have the smooth function, $0 \leq 1_E \star \rho_\varepsilon \leq 1$, and for any $\varphi \in C_c^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \nabla(1_E \star \rho_\varepsilon)\varphi = -\int_{\mathbb{R}^n} 1_E \star \rho_\varepsilon \nabla\varphi = -\int_E \nabla(\varphi \star \rho_\varepsilon) \tag{2.6}$$

where we used that ρ is symmetric.

Theorem 2.13. *Assume E is a set of locally finite perimeter, then*

$$(\mu_E)_\varepsilon := -\nabla(1_E \star \rho_\varepsilon)dx \overset{*}{\rightharpoonup} \mu_E$$

On the other hand if E is a measurable set with, for any compact set K,

$$\liminf_{\varepsilon\to 0} |(\mu_E)_\varepsilon|(K) = \int_K |\nabla(1_E \star \rho_\varepsilon)|dx < \infty,$$

then E is a set with locally finite perimeter bounded by $\limsup_{\varepsilon\to 0} |(\mu_E)_\varepsilon|$.

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Proof. Firstly, we take equation (2.6) and apply it to E to get

$$-\int_{\mathbb{R}^n} \nabla(1_E \star \rho_\varepsilon)\varphi = \int_{\mathbb{R}^n} \varphi \star \rho_\varepsilon d\mu_E$$

Let $\varepsilon \to 0$ we get the first part. For the second part, we can work backwards and we get that for any $\varphi \in C_c^\infty(K)$ with K compact, $\|\varphi\|_\infty \leq 1$, then

$$\begin{aligned}\int_E \nabla\varphi &= \lim_{\varepsilon\to 0}\int_E (\nabla\varphi) \star \rho_\varepsilon \\ &= \lim_{\varepsilon\to 0}\int (\nabla\varphi) 1_E \star \rho_\varepsilon \\ &= \lim_{\varepsilon\to 0}\int -\varphi\nabla(1_E \star \rho_\varepsilon) \\ &\leq \liminf_{\varepsilon\to 0} |\nabla(1_E \star \rho_\varepsilon)dx|(K) < \infty\end{aligned}$$

and we obtained the second part of the result. □

One may use the above theorem to prove the following useful lemma whose proof we refer to [2].

Lemma 2.14. *Given E, F two sets with (locally) finite perimeter, then $E \cap F$ and $E \cup F$ are also of (locally) finite perimeter, furthermore, for any open set $A \subset \mathbb{R}^n$*

$$P(E \cap F; A) + P(E \cup F; A) \leq P(E; A) + P(F; A)$$

Compactness Theorem for Sets with Bounded Perimeter

Theorem 2.15. *Let $(E_h)_{h\in\mathbb{N}}$ be a sequence of sets of finite perimeter all contained in B_R for some $R > 0$, and*

$$\sup_h P(E_h) < \infty,$$

then there is a set $E \subset B_R$ of finite perimeter and a subsequence $h(k)$ such that,

$$E_{h(k)} \to E, \quad \mu_{E(h)} \overset{*}{\rightharpoonup} \mu_E$$

Before coming to the proof we first remark that the requirement $E_h \subset B_R, \forall h \in \mathbb{N}$ is necessary. For instance the sequence of disjoint open balls $E_h = B_1(x_h)$ with $|x_h| \to \infty$ would not have a convergence subsequence with respect to $|(\cdot)\triangle(\cdot)|$. But we can have the following version of local convergence.

Lemma 2.16. *Let $(E_h)_{h\in\mathbb{N}}$ be a sequence of sets of locally finite perimeter, and for any $R > 0$,*

$$\sup_h P(E_h; B_R) < \infty,$$

then there is a set E of locally finite perimeter and a subsequence $h(k)$ such that,

$$E_{h(k)} \xrightarrow{\text{loc}} E, \quad \mu_{E(h)} \overset{*}{\rightharpoonup} \mu_E$$

Proof. We take $m \in \mathbb{N}$, and apply the above theorem to get (for a subsequence) $E_{h,m} = E_h \cap B_m \to \tilde{E}_m$. Since the limit is unique up to a set a measure zero, we may assume that $1_{\tilde{E}_m} = 1_{\tilde{E}_{m+1}}$ on B_m, thus we may take $E = \cup \tilde{E}_m$, and then by a diagonal sequence argument we would have

$$E_{h(k)} \xrightarrow{loc} E$$

and the weak-star convergence of measure follows. □

Proof. We now begin to proof theorem 2.15. We first recall the following variation of the Poincaré inequality for cubes (for the proof one can refer to [2]). For any $u \in C^1(\mathbb{R}^n)$, and any cube of length $r > 0$, $Q = x_0 + (0, r)^n$,

$$\int_Q |u - \bar{u}| \leq \sqrt{n} r \int_Q |\nabla u|.$$

with $\bar{u} = \frac{1}{r^n} \int_Q u$.

Secondly, we prove that any E of finite perimeter and finite n-dimensional measure can be approximated by a set as a union of cubes. That is, there is a finite union of disjoint cubes $T = \cup_{i=1}^N Q_i$ of length $r > 0$, such that

$$|E \triangle T| \leq \sqrt{n} r P(E)$$

We first divide $\mathbb{R}^n$ as cubes of length $r > 0$, that is each cube Q can be represented as $(k_1 r, (k_1 + 1)r) \times ... \times (k_n r, (k_n + 1)r)$ for $(k_1, ..., k_n) \in \mathbb{N}^n$. We write such a division as $\mathbb{R}^n = \cup_{j \in \mathbb{N}} Q_j$. If a cube Q has the following property,

$$|Q \cap E| \geq \frac{r^n}{2},$$

then we say that Q is a cube with large portions in E, and we select T as the union of all cubes with large portions in E. Since E is bounded, there are finitely many such cubes, without loss of generality, we call these cubes $Q_1, ..., Q_N$, and we want to prove that T is a good approximation of E.

We first have $u_\varepsilon = 1_E \star \rho_\varepsilon$, and by the Poincaré inequality,

$$\sqrt{n} r \int |\nabla u_\varepsilon| = \sqrt{n} r \sum_{\cup j \in \mathbb{N}} \int_{Q_j} |\nabla u_\varepsilon| \geq \sum_{\cup j \in \mathbb{N}} \int_{Q_j} |u_\varepsilon - \bar{u}_\varepsilon|.$$

Now we let $\epsilon \to 0$, and by the convergence properties of the regularization, and since E is bounded, the sum on both sides of the above inequality is actually finite,

$$\begin{aligned}
\sqrt{n} r P(E) &\geq \sum_{j \in \mathbb{N}} \int_{Q_j} \left| 1_E - \frac{|E \cap Q_j|}{r^n} \right| \\
&= \sum_{j \in \mathbb{N}} |E \cap Q_j| \left| 1 - \frac{|E \cap Q_j|}{r^n} \right| + \sum_{j \in \mathbb{N}} |Q_j \setminus E| \left| \frac{|E \cap Q_j|}{r^n} \right| \\
&= 2 \sum_{j \in \mathbb{N}} \frac{|E \cap Q_j| |Q_j \setminus E|}{r^n}.
\end{aligned}$$

Now we can make the approximation,

$$|T \triangle E| = |T \setminus E| + |E \setminus T| = \sum_{j \leq N} |Q_j \setminus E| + \sum_{j > N} |E \cap Q_j| \leq \sqrt{n} r P(E).$$

where in the last inequality we used if $j \leq N$,

$$\frac{|E \cap Q_j|}{r^n} \geq \frac{1}{2},$$

and if $j > N$,

$$\frac{|Q_j \setminus E|}{r^n} \geq \frac{1}{2}.$$

Now we move on to show the compactness of the following set as a subset of L^1 functions on $\mathbb{R}^n$,

$$\begin{aligned} \mathcal{C}_{R,p} = \{ f = 1_E \in L^1(\mathbb{R}^n), & \text{ for some measurable subset} \\ & E \subset B_R, \text{ with finite perimeter } P(E) \leq p \}. \end{aligned}$$

The above set is closed under the L^1 norm with the help of theorem 2.9. We are left to show that is totally bounded. For this we use the above approximation. For any arbitrary small $r > 0$, we can pick finitely many cubes $(Q_m)_{m=1,\dots,M}$ of length $\frac{r}{2\sqrt{n}p}$ covering B_{R+1}, we write $\mathcal{Q} = \{Q_1, \dots Q_M\}$. Then let

$$\mathcal{T} = \mathcal{P}(\mathcal{Q}),$$

with $\mathcal{P}(\cdot)$ denoting the power set of a subset. For any $E \in \mathcal{C}_{R,p}$, there is an element T in $\mathcal{T}$, such that

$$\|1_E - 1_T\|_{L^1} = |T \triangle E| \leq r\frac{r}{2\sqrt{n}p} < r.$$

Thus we get compactness of $\mathcal{C}_{R,p}$.

Now we get back to the conclusions, given a sequence $E_h \in \mathcal{C}_{R,p}$ for some $R > 0, p \geq 0$, there is a convergence subsequence to an element $E \in \mathcal{C}_{R,p}$. The weak-star convergence of measure follows. □

An Example of the Direct Method: A Simple Plateau Type Problem

For more examples for instance on relative isoperimetric problems and prescribed mean curvature problems please refer to [2].

We solve existence of minimizer to the following Plateau type problem.

Theorem 2.17. *Let A to be an open bounded set in $\mathbb{R}^n$ and E_0 a set of finite perimeter. Consider,*

$$\gamma(A, E_0) = inf\{P(E), E is\ a\ set\ of\ finite\ perimeter, E \setminus A = E_0 \setminus A\}$$

Then a minimizer of $\gamma(A, E_0)$ exists.

Proof. The minimum is nontrivial since E_0 is a candidate with finite perimeter. We take a minimizing sequence E_h, and define $M_h = E_h \triangle E_0$, then by Theorem 2.15, we can find a convergent subsequence (without relabelling) $M_h \to M$. Now we know that $E_h = E_0 \cup M_h \setminus (E_0 \cap M_h) \to E_0 \cup M \setminus (E_0 \cap M) = E$, and by theorem 2.9, $P(E) \leq \gamma(A, E_0)$, and since

$$|(E \triangle E_0) \setminus A| = \lim_{h \to \infty} |(E_h \triangle E_0) \setminus A| = 0.$$

We get that E is a minimizer.

Another way to tackle this problem does not require A to be bounded. One still can take a minimizing sequence and apply the local convergence results. Then for any compact set K,

$$|K \cap (E \triangle E_0) \setminus A| = \lim_{h \to \infty} |K \cap (E_h \triangle E_0) \setminus A| = 0.$$

Now we can use,

$$|(E \triangle E_0) \setminus A| = \lim_{R \to \infty} |B_R \cap (E \triangle E_0) \setminus A| = 0$$

Similarly one can show that $P(E) \leq \liminf_{h \to \infty} P(E_h) = \gamma(A, E_0)$. $\square$

References

[1] Lawrence Evans and Ronald Gariepy. *Measure Theory and Fine Properties of Functions, Revised Edition.* Chapman and Hall/CRC, 2015. (Visited on 02/26/2020).

[2] Francesco Maggi. *Sets of Finite Perimeter and Geometric Variational Problems.* Cambridge University Press, 2012.