Percolation Theory - Exercise Sheet 2

NCCR SwissMAP - Master Class in Mathematical Physics
Spring term 2020 (401-4604-20L)

Exercise 2.1. Consider percolation on $(\mathbb{Z}^d, E)$.

(a) Let A, B be two decreasing events. Prove that

$$\mathbb{P}_p[A \cap B] \geq \mathbb{P}_p[A] \cdot \mathbb{P}_p[B].$$

(b) Let A be an increasing event, B a decreasing event. Prove that

$$\mathbb{P}_p[A \cap B] \leq \mathbb{P}_p[A] \cdot \mathbb{P}_p[B].$$

Exercise 2.2. [Square Root Trick]

(a) Let $k \geq 2$, $\epsilon > 0$. Let $A_1, \ldots, A_k$ be k increasing events. Assume that

$$\mathbb{P}_p\Big[\bigcup_{1\leq i\leq k} A_i\Big] \geq 1 - \epsilon.$$

Prove that

$$\max_{1\leq i\leq k} \mathbb{P}_p[A_i] \geq 1 - \epsilon^{\frac{1}{k}}.$$

(b) Assume that

$$\mathbb{P}_p\big[\exists \text{ an open path in } \Lambda_n \text{ from left to right}\big] \xrightarrow{n\to\infty} 1.$$

Use (a) to prove that

$$\mathbb{P}_p\big[\exists \text{ an open path in } \Lambda_n \text{ from left to } \{n\} \times \{0, \ldots n\}\big] \xrightarrow{n\to\infty} 1.$$

Exercise 2.3. [Proof of Harris-FKG inequality]

(a) Prove by induction on $n \geq 1$ that for any increasing functions $f, g : \{0,1\}^n \to \mathbb{R}$,

$$\mathbb{E}_p[f(\omega_1, \ldots, \omega_n)g(\omega_1, \ldots, \omega_n)] \geq \mathbb{E}_p[f(\omega_1, \ldots, \omega_n)] \cdot \mathbb{E}_p[g(\omega_1, \ldots, \omega_n)].$$

(b) On $(\mathbb{Z}^d, E)$, let $X, Y : \{0,1\}^E \to \mathbb{R}$ be two bounded, increasing random variables. Use (a) and the martingale convergence theorem to prove that

$$\mathbb{E}_p[XY] \geq \mathbb{E}_p[X] \cdot \mathbb{E}_p[Y].$$

Exercise 2.4. Let $G = (V, E)$ be a finite graph, and let $A \subseteq \{0,1\}^E$ be an increasing event, $B \subseteq \{0,1\}^E$ a decreasing event.

(a) Let $\omega \in A$. Prove that there exists a witness I for A in ω such that for all $e \in I$, $\omega(e) = 1$.

(b) Prove that $A \circ B = A \cap B$. Deduce that $\mathbb{P}_p[A \circ B] \leq \mathbb{P}_p[A] \cdot \mathbb{P}_p[B]$ in this case.

Exercise 2.5. Let $x, y \in \mathbb{Z}^d$ such that $x \neq y$. The goal of this exercise is to prove that $f(p) := \mathbb{P}_p[x \longleftrightarrow y]$ is *strictly* increasing in p.

(a) Use the monotone coupling to prove this.

(b) Use Russo's formula to prove this.