Percolation Theory - Exercise Sheet 3

NCCR SwissMAP - Master Class in Mathematical Physics
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Exercise 3.1. Let $p < p_c$. Prove that $\mathbb{E}_p[|C_0|] < \infty$.

Exercise 3.2. [Fekete's lemma]
Let $(u_n)_{n\geq 1}$ be a sequence of numbers in $[-\infty, \infty)$ satisfying

$$u_{m+n} \leq u_m + u_n \qquad \text{(subadditivity)}$$

for all $m, n \geq 1$. Prove that the limit of $\left(\frac{u_n}{n}\right)$ exists in $[-\infty, \infty)$ and that

$$\lim_{n\to\infty} \frac{u_n}{n} = \inf_{n\geq 1} \frac{u_n}{n}.$$

Exercise 3.3. [Correlation length]

(a) Let $e_1 = (1, 0, \ldots, 0)$ and $p \in [0, 1]$. Show that the correlation length, defined by

$$\xi(p) = \Big(\lim_{n\to\infty} -\frac{1}{n} \log\left(\mathbb{P}_p[0 \longleftrightarrow ne_1]\right)\Big)^{-1},$$

is well-defined, and finite for $p < p_c$.

Hint: Use Fekete's lemma.

(b) Let $p < p_c$. Prove that there exists $C > 0$ (not depending on p) such that for all $n \geq 1$,

$$\frac{1}{Cn^{d-1}} e^{-\frac{n}{\xi(p)}} \leq \theta_n(p) \leq Cn^{d-1} e^{-\frac{n}{\xi(p)}}.$$

Hint: To obtain the upper bound, show first that

$$\max_{x\in\partial\Lambda_n} \mathbb{P}_p[0 \longleftrightarrow x] \leq \mathbb{P}_p[0 \longleftrightarrow 2ne_1]^{\frac{1}{2}}.$$

Exercise 3.4. [Percolation with long-range interactions]
Let $(J_{x,y})_{x,y\in\mathbb{Z}^d}$ be a family of non-negative, translation-invariant numbers, i.e. $J_{x,y} \geq 0$ and $J_{x,y} = J_{x+z,y+z}$ for all $x, y, z \in \mathbb{Z}^d$.
Let $\mathbb{P}_\beta, \beta \geq 0$, be the bond percolation measure on $\mathbb{Z}^d$ defined as follows:

$$\mathbb{P}_\beta[\{x, y\} \text{ is open}] = 1 - e^{-\beta J_{x,y}},\ \mathbb{P}_\beta[\{x, y\} \text{ is closed}] = e^{-\beta J_{x,y}}$$

for $x, y \in \mathbb{Z}^d$ (not necessarily neighbors).

(a) Assume that $\sum_{x\in\mathbb{Z}^d} J_{0,x} = +\infty$. Prove that $\mathbb{P}_\beta[0 \longleftrightarrow \infty] = 1$ for all $\beta > 0$, where $\{0 \longleftrightarrow \infty\}$ denotes the event that 0 is connected to Λ_n^{c} for all $n \geq 1$.

From now on, we assume that $\sum_{x\in\mathbb{Z}^d} J_{0,x} < \infty$.

(b) Define the analogues β_c, $\tilde{\beta}_c$, $\phi_\beta(S)$ of p_c, $\tilde{p}_c$, $\phi_p(S)$ in this context.

(c) Show that there exists $c > 0$ such that for all $\beta \geq \tilde{\beta}_c$,

$$\mathbb{P}_\beta[0 \longleftrightarrow \infty] \geq c\,(\beta - \tilde{\beta}_c).$$

(d) Assume that the interactions are finite-range (i.e. $\exists R$ s.t. $J_{x,y} = 0$ if $|x - y| \geq R$). Show that for all $\beta < \tilde{\beta}_c$, there exists $c > 0$ such that

$$\mathbb{P}_\beta[0 \longleftrightarrow \Lambda_n^{\mathsf{c}}] \leq e^{-cn}.$$

(e) In the general case (i.e. no finite-range assumption), show that for all $\beta < \tilde{\beta}_c$,

$$\sum_{x \in \mathbb{Z}^d} \mathbb{P}_\beta[0 \longleftrightarrow x] < \infty.$$

Deduce that $\tilde{\beta}_c = \beta_c$.

Hint: Consider S with $\phi_\beta(S) < 1$ and show that for all $n \geq 1$,

$$\sum_{x \in \Lambda_n} \mathbb{P}_\beta[0 \overset{\Lambda_n}{\longleftrightarrow} x] \leq \frac{|S|}{1 - \phi_\beta(S)}.$$