Percolation Theory - Exercise Sheet 4

NCCR SwissMAP - Master Class in Mathematical Physics

Spring term 2020 (401-4604-20L)

Exercise 4.1. [Properties of the correlation length]
Show that the correlation length as a function $\xi : [0, p_c] \to [0, \infty]$ is non-decreasing, continuous, and satisfies $\xi(0) = 0$ and $\xi(p_c) = \infty$.

Exercise 4.2. [Volume correlation length]

(a) Let $p \in (0, 1)$. Show that for all $m, n \geq 1$,

$$\frac{1}{m+n}\,\mathbb{P}_p\big[|C_0| = m+n\big] \geq \frac{p}{(1-p)^2}\,\frac{1}{n}\,\mathbb{P}_p\big[|C_0| = n\big]\,\frac{1}{m}\,\mathbb{P}_p\big[|C_0| = m\big].$$

(b) Let $p \in [0, 1]$. Prove that the volume correlation length, defined by

$$\zeta(p) = \Big(\lim_{n\to\infty} -\frac{1}{n}\log\big(\mathbb{P}_p\big[|C_0| = n\big]\big)\Big)^{-1},$$

is well-defined, and finite for $p < p_c$. Also prove that

$$\mathbb{P}_p\big[|C_0| = n\big] \leq \frac{(1-p)^2}{p}\,n\,e^{-\frac{n}{\zeta(p)}}.$$

(c) Let $p < p_c$. Show that

$$\mathbb{P}_p\big[|C_0| \geq n\big] = e^{-\frac{n}{\zeta(p)} + o(n)}.$$

Deduce that $\zeta(p) \geq \xi(p)$. What about the other direction?

Exercise 4.3. [Exponential decay in volume]
The goal of this exercise is to give an alternative proof of exponential decay in volume.

(a) Let $X \geq 0$ be a random variable. Assume that $\mathbb{E}[e^{\epsilon X}] =: C < \infty$ for some $\epsilon > 0$. Show that for all $x \geq 0$,

$$\mathbb{P}[X \geq x] \leq C\,e^{-\epsilon x}.$$

Let $p < p_c$. To prove that there exists $c > 0$ such that for all $n \geq 1$

$$\mathbb{P}_p\big[|C_0| \geq n\big] \leq e^{-cn},$$

it therefore suffices to prove $\mathbb{E}[e^{\epsilon |C_0|}] < \infty$ for some $\epsilon > 0$.

(b) Using BK-Reimer inequality, show that

$$\mathbb{E}_p\big[|C_0|^2\big] \leq \mathbb{E}_p\big[|C_0|\big]^3.$$

(c) More generally, show that

$$\mathbb{E}_p\big[|C_0|^n\big] \leq 2^n\,n!\,\mathbb{E}_p\big[|C_0|\big]^{2n-1},$$

and conclude from there.