Percolation Theory - Exercise Sheet 5

NCCR SwissMAP - Master Class in Mathematical Physics
Spring term 2020 (401-4604-20L)

Throughout this exercise sheet, we consider percolation on $(\mathbb{Z}^d, E)$ and define $\theta(p) := \mathbb{P}_p[0 \longleftrightarrow \infty]$.

Exercise 5.1. Let $N(\omega)$ be the number of infinite clusters in the percolation configuration $\omega \in \{0,1\}^E$. Prove that

$$N = \begin{cases} 0 \text{ a.s.} & \text{if } \theta(p) = 0, \\ 1 \text{ a.s.} & \text{if } \theta(p) > 0. \end{cases}$$

Exercise 5.2. Define the random variable

$$X_n = \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \mathbb{1}_{x \longleftrightarrow \infty}.$$

Prove that

$$\lim_{n\to\infty} X_n = \theta(p). \qquad \text{(in probability)}$$

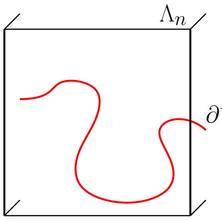
Hint: Look at the expectation and the variance of X_n.

Exercise 5.3. Let $x, y \in \mathbb{Z}^d$. Prove that $p \mapsto \mathbb{P}_p[x \longleftrightarrow y]$ is continuous on $[0,1]$.

Exercise 5.4. Let $p \in [0,1]$ such that $\theta(p) > 0$. Define

$$\partial^- \Lambda_n = \{x \in \Lambda_n : x_1 = -n\},\ \partial^+ \Lambda_n = \{x \in \Lambda_n : x_1 = n\},$$

which are two opposite sides of the boundary of $\Lambda_n = \{-n, \dots, n\}^d$.
Prove that

$$\lim_{n\to\infty} \mathbb{P}_p\Big[\partial^- \Lambda_n$$


$$\partial^+ \Lambda_n\Big] = 1,$$

where the drawing represents an open path in Λ_n from $\partial^- \Lambda_n$ to $\partial^+ \Lambda_n$.