

Percolation Theory - Exercise Sheet 5  
 NCCR SwissMAP - Master Class in Mathematical Physics  
 Spring term 2020 (401-4604-20L)

Throughout this exercise sheet, we consider percolation on  $(\mathbb{Z}^d, E)$  and define  $\theta(p) := \mathbb{P}_p[0 \longleftrightarrow \infty]$ .

**Exercise 5.1.** Let  $N(\omega)$  be the number of infinite clusters in the percolation configuration  $\omega \in \{0, 1\}^E$ . Prove that

$$N = \begin{cases} 0 \text{ a.s.} & \text{if } \theta(p) = 0, \\ 1 \text{ a.s.} & \text{if } \theta(p) > 0. \end{cases}$$

**Exercise 5.2.** Define the random variable

$$X_n = \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \mathbb{1}_{x \longleftrightarrow \infty}.$$

Prove that

$$\lim_{n \rightarrow \infty} X_n = \theta(p). \quad (\text{in probability})$$

*Hint:* Look at the expectation and the variance of  $X_n$ .

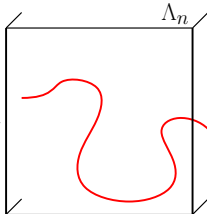
**Exercise 5.3.** Let  $x, y \in \mathbb{Z}^d$ . Prove that  $p \mapsto \mathbb{P}_p[x \longleftrightarrow y]$  is continuous on  $[0, 1]$ .

**Exercise 5.4.** Let  $p \in [0, 1]$  such that  $\theta(p) > 0$ . Define

$$\partial^- \Lambda_n = \{x \in \Lambda_n : x_1 = -n\}, \quad \partial^+ \Lambda_n = \{x \in \Lambda_n : x_1 = n\},$$

which are two opposite sides of the boundary of  $\Lambda_n = \{-n, \dots, n\}^d$ .

Prove that

$$\lim_{n \rightarrow \infty} \mathbb{P}_p \left[ \partial^- \Lambda_n \overset{\Lambda_n}{\curvearrowright} \partial^+ \Lambda_n \right] = 1,$$


The diagram shows a square domain labeled  $\Lambda_n$  at the top right corner. The left boundary is labeled  $\partial^- \Lambda_n$  and the right boundary is labeled  $\partial^+ \Lambda_n$ . A red, wavy path starts from the left boundary and ends at the right boundary, representing an open path within the domain.

where the drawing represents an open path in  $\Lambda_n$  from  $\partial^- \Lambda_n$  to  $\partial^+ \Lambda_n$ .