Multiple Choice 2.1 True or false? Motivate your answers.

Consider the ODE $y''(x) - \sigma^2 y(x) = 0$, where $\sigma > 0$. Then:

		True	False
(a)	It has at least one bounded solution.	☐	☐
(b)	If y is a nonzero solution, then y is unbounded.	☐	☐
(c)	For any two values $a, b \in \mathbb{R}$, there is only one solution with $y(0) = a$ and $y(1) = b$.	☐	☐
(d)	There is no solution which is surjective from $\mathbb{R}$ to $\mathbb{R}$.	☐	☐

Multiple Choice 2.2 Choose the correct statement(s). Motivate your answers.

A particular solution of the ODE $y'' + 2y' + y = \mathrm{e}^{-x}$ is:

(a) $k\mathrm{e}^{-x}$ for some constant k. ☐

(b) $kx\mathrm{e}^{-x}$ for some constant k. ☐

(c) $kx^2\mathrm{e}^{-x}$ for some constant k. ☐

(d) $k\mathrm{e}^{-x^2}$ for some constant k. ☐

Exercise 2.1 A piece of mass (of mass m) connected to a coil spring that can stretch along its length. If $k > 0$ denotes the spring constant, the equation of motion of such system is given, according to Hooke and Newton's laws, by

$$m\ddot{x}(t) = -kx(t), \tag{†}$$

where $x = x(t)$ denotes the position in time of the piece of mass along the vertical direction. Call $\omega = \sqrt{\frac{k}{m}}$. Find the solution of (†):

(a) with initial position $x(0) = 1$ and initial velocity $\dot{x}(0) = 2\omega$.

(b) with initial position $x(0) = 1$ and final position $x(\frac{\pi}{2\omega}) = 1$.

(c) Is is possible to find a solution so that $x(t) \to -\infty$ as $t \to +\infty$?

Exercise 2.2 It is observed that the populations of certain species of bacteria grow, when there is plenty of food and space, with a rate proportional to the number of present individuals. So, if $y(t)$ represents the size of the bacteria population with respect to time $t \geq 0$, it satisfies

$$\begin{cases} y'(t) = \kappa y(t) & \text{for } t > 0, \\ y(0) = y_0, \end{cases} \tag{$\circ$}$$

where y_0 represents the initial size of the population and $\kappa > 0$ is a constant determined by the biology of the bacteria in consideration.

(a) Find the solution of the problem ($\circ$). Looking at the solution, can you guess what is this kind of growth called?

(b) Suppose you are in a lab where the technology to observe the population delivers one picture per ε seconds, for some small $\varepsilon > 0$. Explain why ($\circ$) is replaced by

$$\begin{cases} \dfrac{y(t+\varepsilon) - y(t)}{\varepsilon} = \kappa y(t) & \text{for } t = 0, \varepsilon, 2\varepsilon, \ldots \\ y(0) = y_0, \end{cases} \tag{$\diamond$}$$

and find the solution of the problem ($\diamond$).

(c) How does the solution to ($\diamond$) behave as $\varepsilon \to 0$?

Exercise 2.3 Consider the differential equation

$$xy' = 2y - 3xy^2 \qquad \text{for } x > 0,$$

in the unknown $y = y(x)$.

(a) Rewrite the equation using the change of variable $y = \frac{1}{u}$.

(b) Solve the equation in the new variable u and then write the solution in the original variable y.

(c) There is one solution that cannot be obtained with the procedure (a) & (b). What is it, and why?