Multiple Choice 5.1 True or false? Motivate your answer.

Let $m, n \in \mathbb{N}$ and $L : \mathbb{R}^m \to \mathbb{R}^n$ be a *linear* function. Then if L is continuous at one point $x_0 \in \mathbb{R}^m$, it is continuous at every point in $\mathbb{R}^m$.

☐ True ☐ False

Multiple Choice 5.2 Choose the correct statement. Motivate your answer.

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(0,0) = 0$. For f to be continuous at $(0,0)$, the fact that

$$\lim_{x\to 0} f(x,0) = 0 \quad \text{and} \quad \lim_{y\to 0} f(0,y) = 0$$

(a) is *necessary*, but in general not sufficient. ☐

(b) is *sufficient*, but in general not necessary. ☐

(c) is *necessary and sufficient.* ☐

Note: Looking at Exercise 5.1 may be useful!

Exercise 5.1 Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Prove that f is continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$.

(b) Prove that, for every fixed $y \in \mathbb{R}$, there holds

$$\lim_{x\to 0} f(x,y) = 0,$$

and that or every fixed $x \in \mathbb{R}$, there holds

$$\lim_{y\to 0} f(x,y) = 0.$$

(c) Consider now the parabola $P = \{(x, x^2) : x \in \mathbb{R}\} \subset \mathbb{R}^2$. Prove that

$$\lim_{\substack{(x,y)\to 0 \\ (x,y)\in P}} f(x,y) = \lim_{x\to 0} f(x,x^2) = \frac{1}{2}.$$

Without any further computation, what can you say about

$$\lim_{(x,y)\to 0} f(x,y) \ldots ?$$

Is f continuous at $(0,0)$?

Exercise 5.2 Compute the limits

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$,

(b) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{y^2 \log x}{(x-1)^2+y^2}$,

(c) $\displaystyle\lim_{(x,y,z)\to\infty} f(x,y,z)$,
where $f(x,y,z) = x^4 + y^2 + z^2 - x^3 + xyz - x + 4$.

Note: similarly as in one variable, "$\lim_{(x,y,z)\to\infty} f(x,y,z) = \alpha$" means that for every $\varepsilon > 0$ there exists $M > 0$, so that if $\|(x,y,z)\| \geq M$, then $|f(x,y,z) - \alpha| \leq \varepsilon$.

Exercise 5.3 Which of the following subsets of the Euclidean space are compact?

(a) $A = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 2020\}$;

(b) $B = \{(a,b,c) \in \mathbb{R}^3 \,|\, a, b, c \text{ are integers and } a^2 + b^2 + c^2 < 2020\}$;

(c) $C = \{(x, f(x)) \in \mathbb{R}^2 \,|\, x \in (0,1], f(x) = \sin(\frac{1}{x})\}$;

(d) $D = \{(x,y) \in \mathbb{R}^2 \,|\, x \geq 0 \text{ and } y \geq 0\}$;

(e) $E = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 \leq 2\}$.