Multiple Choice 10.1 Choose the correct statement. Motivate your answer.

Let $V : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field and consider its *Jacobian matrix*, namely the 3×3 matrix of its 1st derivatives:

$$DV = \begin{pmatrix} \partial_{x_1} V_1 & \partial_{x_2} V_1 & \partial_{x_3} V_1 \\ \partial_{x_1} V_2 & \partial_{x_2} V_2 & \partial_{x_3} V_2 \\ \partial_{x_1} V_3 & \partial_{x_2} V_3 & \partial_{x_3} V_3 \end{pmatrix}.$$

Knowing that, in this matrix, there are three *distinct* coinciding pairs of elements, is, in general,

(a) *necessary*, but not sufficient, ☐

(b) *sufficient*, but not necessary, ☐

(c) *necessary and sufficient*, ☐

(d) neither necessary nor sufficient, ☐

for V to be conservative.

Multiple Choice 10.2 Choose the correct statement. Motivate your answer.

Let $V : \mathbb{R}^3 \to \mathbb{R}^3$ and V be as in the previous question.

Knowing that the matrix DV is *symmetric*, is

(a) *necessary*, but not sufficient, ☐

(b) *sufficient*, but not necessary, ☐

(c) *necessary and sufficient*, ☐

(d) neither necessary nor sufficient, ☐

for V to be conservative.

Exercise 10.1 In each of the following, find a parametrization of the curve γ and compute the line integral $\int_\gamma F \cdot d\vec{s}$.

(a) $F(x, y) = (x + y, x - y)$ and γ runs through the parabola $\{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$ from the point $(-1, 1)$ to the point $(1, 1)$.

(b) $F(x, y) = (0, xy^2)$ and γ runs through the half-circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 4, y \geq 0\}$ in counter-clockwise direction.

(c) $F(x, y) = (x^2 + y^2, x^2 - y^2)$ and γ runs trough the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ in counter-clockwise direction.

Exercise 10.2 In each of the following, determine whether the vector field $V : \mathbb{R}^3 \to \mathbb{R}^3$ admits a potential and compute the line integral $\int_\gamma V \cdot d\vec{s}$ along the curve $\gamma \colon [0, 1] \to \mathbb{R}^3$, $\gamma(t) = (t^3, t^2 + t, t)$.

$$\text{(a)} \quad V(x, y, z) = \begin{pmatrix} 2xy^3 \\ 3x^2y^2 + 2yz \\ y^2 \end{pmatrix}, \qquad \text{(b)} \quad V(x, y, z) = \begin{pmatrix} x + z \\ x + y + z \\ x + z \end{pmatrix}.$$

Exercise 10.3 The following vector field describes, according to the *Biot-Savart law*, the magnetic field generated by an infinitely long, constant-current electric wire displaced along the z-axis:

$$B(x, y, z) = \frac{\mu_0 I}{2\pi} \frac{1}{x^2 + y^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \quad \text{defined for } (x, y) \neq 0,$$

where μ_0 and I are, respectively, the magnetic constant and I the (also constant) current.

(a) Prove that it is

$$\frac{\partial}{\partial x_i} B_j = \frac{\partial}{\partial x_j} B_i \quad \forall\, i, j \in \{1, 2, 3\},$$

where we denoted $(x_1, x_2, x_3) = (x, y, z)$.

(b) Consider the curves $\gamma_m : [0, 2\pi m] \to \mathbb{R}^3$, $\gamma_m(t) = (\cos(t), \sin(t), 0)$ for $m \in \mathbb{Z}$, and compute the line integrals $\int_{\gamma_m} B \cdot d\vec{s}$.

(c) Does B admit a potential in $\mathbb{R}^3 \setminus \{z\text{-axis}\}$?