Christmas Serie

Differential Equations

Exercise 14.1

(a) Determine the solutions to the differential equation

$$y''' - 4y'' + 6y' = 0$$

satisfying the conditions

$$\lim_{t\to-\infty} y(t) = 0 \quad \text{and} \quad y'(0) = 0.$$

(b) Determine a solution of the equation

$$y''' - 4y'' + 6y' = e^{2t} + 9t^2.$$

Hint: since the equation is linear, you may divide the right-hand side in 2 parts and then sum the solutions (superposition principle).

(c) Determine the solution of the problem

$$\begin{cases} y = 2t^2 y' & \text{for } t \geq 1, \\ y(1) = 1. \end{cases}$$

Exercise 14.2

(a) Solve the following differential equation

$$\ddot{y} - 3\dot{y} - 4y = t + t\,e^{-2t}.$$

(b) Solve the following differential equation

$$\ddot{y} - y = \cosh t,$$

and determine the solution satisfying the conditions

$$y(0) = 1 \quad \text{and} \quad y'(0) = -1.$$

Exercise 14.3 Solve the following differential equations. For each of the solutions, determine the subset of $\mathbb{R}$ where it is defined.

$(a)\quad y' = y^2 - 1,$ $\qquad (b)\quad y' = e^{x+y},$

$(c)\quad y' = \dfrac{e^{x+y} - e^{x-y}}{\cosh(y)},$ $\qquad (d)\quad y' = \cos(2x)\, y \log(y).$

Note: If you use the method of separation of variables, be careful not to divide by zero.

Multivariable Calculus

Exercise 14.4 Determine the critical points of the following functions and whether they are local maximum, minimum, or saddle points.

(a) $f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = x^3 + y^3 + 3xy,$

(b) $f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = x^2 + y^2 - 2xy,$

(c) $f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = y(x-1)e^{-(x^2+y^2)},$

(d) $f : \mathbb{R}^3 \to \mathbb{R}, f(x,y,z) = x^2 + y^2 + z^2 + 2xyz.$

Exercise 14.5 For which $a \in \mathbb{R}$ there exists a function $f : \mathbb{R}^3 \to \mathbb{R}$ whose gradient is the vector field

$$(x,y,z) \mapsto \begin{pmatrix} \log(1+x^2) + ay^2 \\ xy + y^2 \\ z^3 \end{pmatrix} ? \tag{1}$$

For such a's, find one function f. How do any such two such solutions (for the same a) differ?

Exercise 14.6

(a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \cos(x^2 + y^2)$$

Compute the Taylor polynomial of f of 3rd order at the origin.

(b) Let $f : \{(x,y) \in \mathbb{R}^2 \mid y > 0\} \to \mathbb{R}$ be given by

$$f(x,y) = e^x \log(y)$$

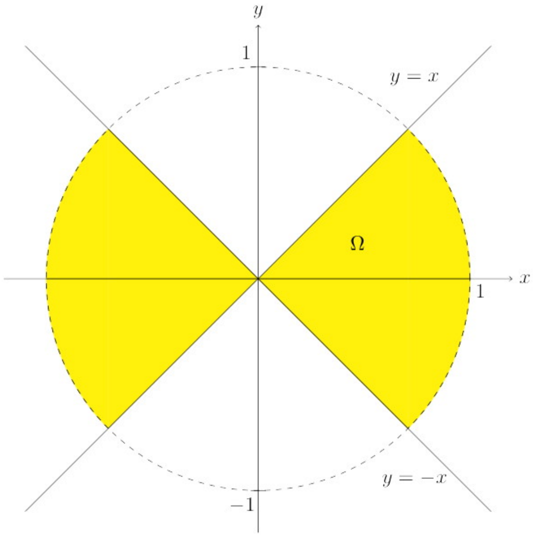
Approximate the value of f at the point $(0.102\,,\, 1.121)$ with the help of the Taylor polynomial of 1st order.

Multiple Integrals

Exercise 14.7 Integrate the function

$$f(x, y) = |x|\sqrt{x^2 + y^2}$$

over the set Ω indicated below.



Exercise 14.8 The Piriform curve is the planar curve given by

$$C := \{(x, y) \in \mathbb{R}^2 \mid y^2 = x^3(2 - x)\}.$$

A parametrization of C is given by $\gamma : [-\frac{\pi}{2}, \frac{3\pi}{2}] \to \mathbb{R}^2$,

$$\gamma(t) = \begin{pmatrix} 1 + \sin(t) \\ \cos(t)(1 + \sin(t)) \end{pmatrix}$$

Determine the area of the set Ω enclosed by C by means of Green's theorem.

Exercise 14.9 Consider the following curve in the plane:

$$\Gamma = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = \sqrt{x^2 + y^2} + x\right\}$$

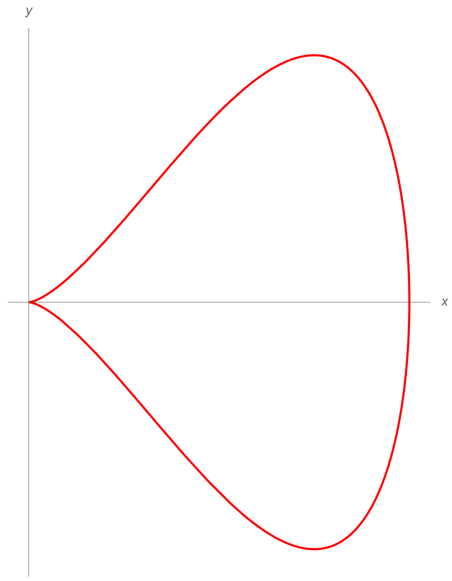


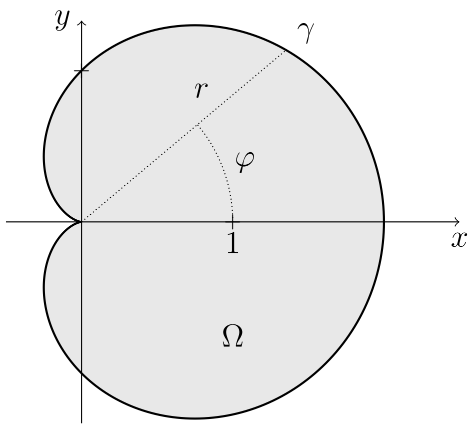
Figure 1: The Piriform Curve

(a) Express Γ in polar coordinates $(x, y) = (r\cos\varphi, r\sin\varphi)$ and find a parametrization $\gamma : [0, 2\pi) \to \mathbb{R}^2$ for Γ.

With the help of (a) compute now the area of the domain Ω enclosed by γ in both of the following ways:

(b) Using Green's formula with $\operatorname{curl}(v) = 1$ for the vector field $v(x, y) = (-y, 0)$,

(c) Expressing the double integral in polar coordinates.



Exercise 14.10 Let $Q = (0,1) \times (0,1) \subset \mathbb{R}^2$ be the unit square. Find those $\alpha, \beta \in \mathbb{R}$ for which the integral

$$\int_Q x^\alpha y^\beta \, dxdy$$

is convergent, and for such values compute it.

Exercise 14.11 Determine the values $\alpha, \beta \in \mathbb{R}$ for which integral in $\mathbb{R}^3$

$$\int_{\mathbb{R}^3} |x|^\alpha \mathrm{e}^{-|x|^\beta} d\mu$$

is convergent (no need to compute it).