Multiple Choice 8.1 True or False? Motivate your answers.

The function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\Big(\frac{1}{\sqrt{x^2+y^2}}\Big), & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

at the point $(0,0)$ is:

		True	False
(a)	discontinuous	☐	☐
(b)	continuous	☐	☐
(c)	differentiable	☐	☐
(d)	C^1 (i.e. continuously differentiable).	☐	☐

Solution. It is

		True	False
(a)	discontinuous	☐	☒
(b)	continuous	☒	☐
(c)	differentiable	☒	☐
(d)	C^1 (i.e. continuously differentiable).	☐	☒

We are going to show that f is differentiable, and hence continuous, at $(0,0)$ but not of class C^1. For $(x,y) \neq (0,0)$ the partial derivatives are

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= 2x\sin\Big(\frac{1}{\sqrt{x^2+y^2}}\Big) + (x^2+y^2)\cos\Big(\frac{1}{\sqrt{x^2+y^2}}\Big)\cdot\frac{2x}{-2(x^2+y^2)^{\frac{3}{2}}} \\ &= 2x\sin\Big(\frac{1}{\sqrt{x^2+y^2}}\Big) - \frac{x}{\sqrt{x^2+y^2}}\cos\Big(\frac{1}{\sqrt{x^2+y^2}}\Big), \\ \frac{\partial f}{\partial y}(x,y) &= 2y\sin\Big(\frac{1}{\sqrt{x^2+y^2}}\Big) - \frac{y}{\sqrt{x^2+y^2}}\cos\Big(\frac{1}{\sqrt{x^2+y^2}}\Big),\end{aligned}$$

that are clearly continuous away from $(0,0)$. To see that f is differentiable at $(0,0)$ we see that

$$\left|\frac{f(x,y)-f(0,0)}{\|(x,y)\|}\right| = \left|\frac{(x^2+y^2)\sin\Big(\frac{1}{\sqrt{x^2+y^2}}\Big) - 0}{\sqrt{x^2+y^2}}\right| \leq \sqrt{x^2+y^2} \xrightarrow{(x,y)\to(0,0)} 0.$$

which implies f is differentiable with $df(0,0) = (0,0)$.

However, the partial derivatives of f are not continuous at $(0,0)$, since for $x > 0$ we have

$$\frac{\partial f}{\partial x}(x,0) = 2x\sin\Big(\frac{1}{\sqrt{x^2}}\Big) - \frac{x}{\sqrt{x^2}}\cos\Big(\frac{1}{\sqrt{x^2}}\Big) = 2x\sin\Big(\frac{1}{x}\Big) - \cos\Big(\frac{1}{x}\Big).$$

The term $-\cos(\frac{1}{x})$ is divergent $(x,0) \to (0,0)$. Consequently, $\frac{\partial f}{\partial x}$ is not continuous at $(0,0)$ and f is not of class C^1.

Multiple Choice 8.2 Choose the correct statement. Motivate your answer.

Recall that a *critical point* of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ is an $x_0 \in \mathbb{R}^n$ so that $df(x_0) = 0$. At such point, the tangent plane to the graph of f is:

(a) not defined ☐

(b) horizontal (looking at $\mathbb{R}^3$ in the usual way with upward-pointing z-axis) ☐

(c) vertical (looking at $\mathbb{R}^3$ in the usual way with upward-pointing z-axis) ☐

(d) none of the above, in general. ☐

Solution. The correct answer is

(b) horizontal (looking at $\mathbb{R}^3$ in the usual way with upward-pointing z-axis) ☒

Indeed, (see also Exercise 8.1 below) the equation of the plane is just

$$z = f(x_0) + df(x_0)\cdot(x,y) = f(x_0)$$

which means that it is parallel to the x-y plane, and therefore horizontal.

Exercise 8.1 Let $\mathcal{G} = \{(x,y,f(x,y)) : (x,y) \in \mathbb{R}^2\} \subset \mathbb{R}^3$ be the graph of the function

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = e^{-(x^2+y^2-2x+3y+2)}.$$

(a) Find the equation of the tangent plane E to $\mathcal{G}$ at the point $(0,0,e^{-2})$, both in *Cartesian form*, i.e. with an equation:

$$E = \{(x,y,z) \in \mathbb{R}^3 : \text{“equation in } x,y,z\text{”}\},$$

and in *parametric form* i.e. with a function:

$$\varphi\colon \mathbb{R}^2 \to E \subset \mathbb{R}^3, \quad \varphi(s,t) = \big(x(s,t), y(s,t), z(s,t)\big).$$

(b) Use a plotting software of your choice to verify that φ actually plots a plane that is tangent to $\mathcal{G}$ as above.

(c) Find all the points in $\mathcal{G}$ where the tangent plane is parallel to the x-y plane $\Pi = \{(x, y, 0) : (x, y) \in \mathbb{R}^2\}$.

Solution. Recall that, in general, the tangent plane for $\mathcal{G}$ at (x_0, y_0) is given by

$$E = \left\{(x, y, A(x, y)) : (x, y) \in \mathbb{R}^2\right\},$$

where A is the affine approximation of f:

$$\begin{aligned} A(x, y) &= f(x_0, y_0) + df(x_0) \cdot (x, y) \\ &= f(x_0, y_0) + \frac{\partial}{\partial x} f(x_0, y_0)\,(x - x_0) + \frac{\partial}{\partial y} f(x_0, y_0)(y - y_0). \end{aligned}$$

consequently, a paremetrization and a Cartesian equation E are

$$\varphi(s, t) = (s, t, A(s, t)), \quad \text{and} \quad z = A(x, y).$$

(a) The partial derivatives of f are

$$\frac{\partial f}{\partial x}(x, y) = (-2x + 2)\, f(x, y), \qquad \frac{\partial f}{\partial y}(x, y) = (-2y - 3)\, f(x, y),$$

consequently, we have $df(0, 0) = (2e^{-2}, -3e^{-2})$ and the affine approximation of f is

$$A(x, y) = f(0, 0) + df(0, 0) \cdot (x, y) = e^{-2} + 2e^{-2}x - 3e^{-2}y.$$

Thus, it is

$$\varphi(s, t) = (s, t, e^{-2} + 2e^{-2}s - 3e^{-2}t)$$

and

$$E = \{(x, y, z) : z = e^{-2} + 2e^{-2}x - 3e^{-2}y\}.$$

(b) This is left to the student.

(c) The tangent plane in (x_0, y_0) is parallel to Π if and only if A is constant, hence if and only if $df(x_0, y_0) \equiv 0$. From the computation in (a), since $f > 0$ this means that $-2x_0 + 2 = 0$ and $2y_0 - 3 = 0$, i.e. $(x_0, y_0) = (1, -\frac{3}{2})$. Consequently, E is parallel to Π at the point

$$\left(1, -\tfrac{3}{2}, f(1, -\tfrac{3}{2})\right) = \left(1, -\tfrac{3}{2}, e^{\frac{5}{4}}\right) \in \mathcal{G}.$$

Exercise 8.2 Consider the function

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = e^x \sin(y).$$

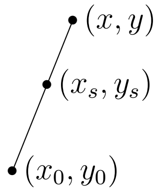
(a) Compute the Taylor polynomials of 1st and 2nd order of f at $(x_0, y_0) = (0, \frac{\pi}{2})$; approximate with each of them the value of f at $(x_1, y_1) = (0, \frac{\pi}{2} + \frac{1}{4})$. Compare the results approximating numerically the value of $f(x_1, y_1)$ with a software of your choice.

(b) Similarly as for the one variable case, one can prove that if a function is C^2, one can write

$$f(x) = f(x_0) + df(x_0) \cdot (x - x_0) + R_1 f(x,y),$$

where $R_1 f$ is the *rest*, given by

$$\begin{aligned} R_1 f(x,y) = {} & \frac{1}{2}\frac{\partial^2 f}{\partial x \partial x}(x_s, y_s) \cdot (x - x_0)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y \partial y}(x_s, y_s) \cdot (y - y_0)^2 \\ & + \frac{\partial^2 f}{\partial x \partial y}(x_s, y_s) \cdot (x - x_0)(y - y_0) \end{aligned}$$



where $(x_s, y_s) = \Big(x_0 + s(x - x_0), y_0 + s(y - y_0)\Big)$ for some $s \in [0,1]$ (see e.g. Satz 7.5.2 of Struwe's script).

With this information, quantify how precise in the linear approximation in the ball $B_{\frac{1}{4}}(0, \frac{\pi}{2})$ by giving an upper bound for the corresponding error.

Solution. (a) The partial derivatives of f are $f(x,y) = e^x \sin(y)$ sind

$$\begin{aligned} &\frac{\partial f}{\partial x}(x,y) = e^x \sin(y), && \frac{\partial}{\partial x}\frac{\partial f}{\partial x}(x,y) = e^x \sin(y), && \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x,y) = e^x \cos(y), \\ &\frac{\partial f}{\partial y}(x,y) = e^x \cos(y), && \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x,y) = e^x \cos(y), && \frac{\partial}{\partial y}\frac{\partial f}{\partial y}(x,y) = -e^x \sin(y). \end{aligned}$$

Consequently, at $(x_0, y_0) = (0, \frac{\pi}{2}) \in \mathbb{R}^2$ we have

$$\begin{aligned} &\frac{\partial f}{\partial x}(0, \tfrac{\pi}{2}) = 1, && \frac{\partial}{\partial x}\frac{\partial f}{\partial x}(0, \tfrac{\pi}{2}) = 1, && \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(0, \tfrac{\pi}{2}) = 0, \\ &\frac{\partial f}{\partial y}(0, \tfrac{\pi}{2}) = 0, && \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(0, \tfrac{\pi}{2}) = 0, && \frac{\partial}{\partial y}\frac{\partial f}{\partial y}(0, \tfrac{\pi}{2}) = -1. \end{aligned}$$

Since moreover $f(0, \frac{\pi}{2}) = 1$ sowie, the taylor polynomials are

$$\begin{aligned}
T_1 f(x,y) &= f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\cdot(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\cdot(y-y_0) \\
&= 1 + x, \\
T_2 f(x,y) &= T_1 f(x,y) + \frac{1}{2}\frac{\partial^2 f}{\partial x \partial x}(x_0,y_0)\cdot(x-x_0)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y \partial y}(x_0,y_0)\cdot(y-y_0)^2 \\
&\qquad + \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)\cdot(x-x_0)(y-y_0) \\
&= 1 + x + \frac{1}{2}x^2 - \frac{1}{2}\Big(y - \frac{\pi}{2}\Big)^2.
\end{aligned}$$

Evaluating them at $(0, \frac{\pi}{2} + \frac{1}{4})$ gives

$$T_1 f(0, \tfrac{\pi}{2} + \tfrac{1}{4}) = 1, \qquad T_2 f(0, \tfrac{\pi}{2} + \tfrac{1}{4}) = 1 - \tfrac{1}{2}(\tfrac{1}{4})^2 = \tfrac{31}{32} = 0.96875.$$

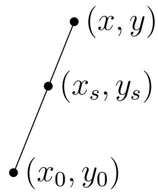
The value approximated numerically by a program is

$$f(0, \tfrac{\pi}{2} + \tfrac{1}{4}) = \sin(\tfrac{\pi}{2} + \tfrac{1}{4}) \approx 0.96891,$$

so the approximation of T_2 is relatively good.

(b) Since $f(x,y) = T_1 f(x,y) + R_1 f(x,y)$, where the rest $R_1 f$ is given by

$$\begin{aligned}
R_1 f(x,y) = &\frac{1}{2}\frac{\partial^2 f}{\partial x \partial x}(x_s,y_s)\cdot(x-x_0)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y \partial y}(x_s,y_s)\cdot(y-y_0)^2 \\
&+ \frac{\partial^2 f}{\partial x \partial y}(x_s,y_s)\cdot(x-x_0)(y-y_0)
\end{aligned}$$



where $(x_s, y_s) := \big(x_0 + s(x-x_0), y_0 + s(y-y_0)\big)$ for some $s \in [0,1]$. To estimate this rest, we see that

$$|\partial^2_{xx} f(x,y)|, |\partial^2_{xy}(x,y)|, |\partial^2 f_{yy}(x,y)| \le e^x,$$

since $|\sin x| \le 1$ und $|\cos x| \le 1$. So for $(x_0,y_0) = (0, \frac{\pi}{2})$ and $(x,y) \in B_{\frac{1}{4}}(x_0,y_0)$ we have $|x - x_0| \le \frac{1}{4}$, $|y - y_0| \le \frac{1}{4}$ and $x_0 + s(x-x_0) \le \frac{1}{4}$, so

$$|R_1(x,y)| \le 4 \cdot \left(\frac{1}{2}e^{\frac{1}{4}} \cdot \left(\frac{1}{4}\right)^2\right) = \frac{1}{8}e^{\frac{1}{4}} \approx 0.1605.$$

Exercise 8.3 Compute the Taylor polynomials of the following functions at the given point and of the given order.

(a) $f(x,y) = \frac{1}{1-xy}$, at $(0,0)$, $2n-$th order with $n \geq 1$.

(b) $f(x,y) = \arctan(x^2 y)$, at $(0,0)$, 2nd order.

(c) $f(z) = \log(|z|^2+1)$ ($z \in \mathbb{C} \simeq \mathbb{R}^2$), at $z=0$, $2n-$th order with $n \geq 1$.

(d) $f(x_1,\dots,x_n) = \prod_{i=1}^n x_i$, at $x_0 = (2,\dots,2)$ 2nd order.

Solution. All the polynomials can be computed directly by working out each of the partial derivatives; we try to give below some alternative methods. We denote with $T_n f$ the required polynomial.

(a) Recall that for $|b| < 1$ one has the geometric series formula: $\frac{1}{1-b} = \sum_{n=0}^\infty b^n$. Consequently, we can write, as $(x,y) \to (0,0)$,

$$\frac{1}{1-xy} = \sum_{n\in\mathbb{N}} (xy)^n = \sum_{n=0}^N (xy)^n + \sum_{n=N+1}^\infty (xy)^n = \sum_{n=0}^N (xy)^n + o(|(x,y)|^{2N}),$$

consequently, it has to be, for every $n \in \mathbb{N}$,

$$T_{2n} f(x,y) = 1 + \sum_{k=1}^n (xy)^k.$$

(b) Recall that $\arctan(t) = t + O(t^3)$, so $\arctan(x^2y) = x^2y + O(x^6y^3)$ and in particular

$$T_2 f(x,y) = 0.$$

(c) Recall that for any $|t| < 1$ we have

$$\log(1+t) = \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} t^k,$$

consequently $|z| < 1$

$$\log(1+|z|^2) = \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} |z|^{2k},$$

and thus for $z = x+iy \in \mathbb{C} \simeq \mathbb{R}^2$ we have

$$T_{2n} f(z) = \sum_{k=1}^n \frac{(-1)^{k-1}}{k} |z|^{2k} = \sum_{k=1}^n \frac{(-1)^{k-1}}{k} (x^2+y^2)^k.$$

(d) Note that for any $i \neq j$ we have

$$\partial_{x_i} f(x) = \prod_{k \neq i} x_k, \quad \partial^2_{x_i} f(x) = 0, \quad \partial^2_{x_i,x_j} f(x) = \prod_{k \neq i,j} x_k,$$

hence, for $1 \leq i \neq j \leq n$,

$$f(2) = 2^n, \quad \partial_{x_i} f(2) = 2^{n-1}, \quad \partial^2_{x_i,x_j} f(2) = 2^{n-2},$$

Thus writing $x_0 = (2, \ldots, 2)$ and $y = x - x_0$ we have

$$\begin{aligned} T_2 f(x) &= f(x_0) + df(x_0) \cdot y + \frac{1}{2} y \cdot \mathrm{Hess} f(x_0) \cdot y \\ &= 2^n + 2^{n-1} \sum_{i=1}^n y_i + 2^{n-2} \sum_{1 \leq i < j \leq n} y_i y_j. \end{aligned}$$

(In the terms of order 2, we have a coefficient $\frac{1}{2}$ in Taylor formula, but we have two symmetric terms $\partial_i \partial_j$ and $\partial_j \partial_i$ for $1 \leq i < j \leq n$. We combine this 2 terms and end up with $(\frac{1}{2} + \frac{1}{2}) \partial^2_{x_i,x_j} f(2)$.)