Multiple Choice 10.1 Choose the correct statement. Motivate your answer.

Let $V : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field and consider its *Jacobian matrix*, namely the 3×3 matrix of its 1st derivatives:

$$DV = \begin{pmatrix} \partial_{x_1} V_1 & \partial_{x_2} V_1 & \partial_{x_3} V_1 \\ \partial_{x_1} V_2 & \partial_{x_2} V_2 & \partial_{x_3} V_2 \\ \partial_{x_1} V_3 & \partial_{x_2} V_3 & \partial_{x_3} V_3 \end{pmatrix}.$$

Knowing that, in this matrix, there are three *distinct* coinciding pairs of elements, is, in general,

(a) *necessary*, but not sufficient, ☐

(b) *sufficient*, but not necessary, ☐

(c) *necessary and sufficient*, ☐

(d) neither necessary nor sufficient, ☐

for V to be conservative.

Solution. It is

(a) *necessary*, but not sufficient, ☒

(b) *sufficient*, but not necessary, ☐

(c) *necessary and sufficient*, ☐

(d) neither necessary nor sufficient, ☐

Indeed since $\mathbb{R}^3$ is star-shaped V is conservative if any only if $\partial_i V_j = \partial_j V_i$ for $i, j \in \{1, 2, 3\}$, which means that it has to be

$$\begin{pmatrix} \partial_{x_1} V_2 \\ \partial_{x_1} V_3 \\ \partial_{x_2} V_3 \end{pmatrix} = \begin{pmatrix} \partial_{x_2} V_1 \\ \partial_{x_3} V_1 \\ \partial_{x_3} V_2 \end{pmatrix}$$

So if this holds, V is conservative. But consider for instance $V(x, y, z) = (x + y, y, z)$: since $\partial_2 V_1 \neq \partial_1 V_2$, V is not conservative however its Jacobian matrix is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We see for instance that the couples of elements with entries $(1, 1) - (2, 2)$, $(2, 2) - (3, 3)$ and $(2, 3) - (3, 2)$ have the same values.

Multiple Choice 10.2 Choose the correct statement. Motivate your answer.

Let $V : \mathbb{R}^3 \to \mathbb{R}^3$ and V be as in the previous question.

Knowing that the matrix DV is *symmetric*, is

(a) *necessary*, but not sufficient, ☐

(b) *sufficient*, but not necessary, ☐

(c) *necessary and sufficient*, ☐

(d) neither necessary nor sufficient, ☐

for V to be conservative.

Solution. It is

(a) *necessary*, but not sufficient, ☐

(b) *sufficient*, but not necessary, ☐

(c) *necessary and sufficient*, ☒

(d) neither necessary nor sufficient, ☐

Indeed since $\mathbb{R}^3$ is star-shaped V is conservative if any only if $\partial_i V_j = \partial_j V_i$ for $i, j \in \{1, 2, 3\}$, and this precisely means that DV is symmetric.

Exercise 10.1 In each of the following, find a parametrization of the curve γ and compute the line integral $\int_\gamma F \cdot d\vec{s}$.

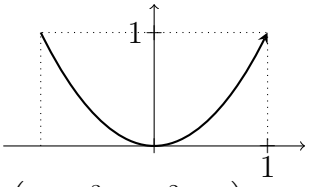
(a) $F(x, y) = (x + y, x - y)$ and γ runs through the parabola $\{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$ from the point $(-1, 1)$ to the point $(1, 1)$.

(b) $F(x, y) = (0, xy^2)$ and γ runs through the half-circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 4, y \geq 0\}$ in counter-clockwise direction.

(c) $F(x, y) = (x^2 + y^2, x^2 - y^2)$ and γ runs trough the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ in counter-clockwise direction.

Solution. (a) We choose $\gamma\colon [-1,1] \to \mathbb{R}^2$ with $\gamma(t) = (t, t^2)$. Then

$$\begin{aligned}\int_\gamma F \cdot d\vec{s} &= \int_{-1}^{1} F(\gamma(t)) \cdot \gamma'(t)\, dt \\ &= \int_{-1}^{1} \begin{pmatrix} t + t^2 \\ t - t^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} dt \\ &= \int_{-1}^{1} \Big((t + t^2) + 2t(t - t^2) \Big)\, dt = \int_{-1}^{1} \Big(-2t^3 + 3t^2 + t \Big)\, dt = 2.\end{aligned}$$

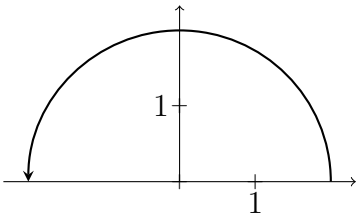


(b) We choose $\gamma\colon [0,\pi] \to \mathbb{R}^2$ with $\gamma(t) = (2\cos t, 2\sin t)$. Then

$$\begin{aligned}\int_\gamma F \cdot d\vec{s} &= \int_0^\pi F(\gamma(t)) \cdot \gamma'(t)\, dt = \int_0^\pi \begin{pmatrix} 0 \\ (2\cos t)(2\sin t)^2 \end{pmatrix} \cdot \begin{pmatrix} -2\sin t \\ 2\cos t \end{pmatrix} dt \\ &= \int_0^\pi (2\cos t)^2 (2\sin t)^2\, dt = 2\int_0^\pi 1 - \cos(4t)\, dt = 2\Big[t - \tfrac{1}{4}\sin(4t)\Big]_0^\pi = 2\pi,\end{aligned}$$

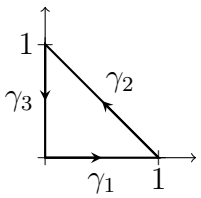
where we used the trigonometric identities

$$\begin{aligned}\sin(2\alpha) &= 2(\sin\alpha)(\cos\alpha), \\ \cos(2\alpha) &= (\cos\alpha)^2 - (\sin\alpha)^2 \\ &= 1 - 2(\sin\alpha)^2, \\ \Rightarrow\ 8(\sin t)^2(\cos t)^2 &= 2(\sin(2t))^2 = 1 - \cos(4t).\end{aligned}$$



(c) We parametrize the triangle piecewise-linearly by

$$\begin{aligned}&\gamma_1\colon [0,1] \to \mathbb{R}^2, \quad \gamma_2\colon [0,1] \to \mathbb{R}^2, \quad \gamma_3\colon [0,1] \to \mathbb{R}^2 \\ &t \mapsto \begin{pmatrix} t \\ 0 \end{pmatrix} \qquad t \mapsto \begin{pmatrix} 1-t \\ t \end{pmatrix} \qquad t \mapsto \begin{pmatrix} 0 \\ 1-t \end{pmatrix}\end{aligned}$$



The line integral over each piece of the curve is

$$\begin{aligned}\int_{\gamma_1} F \cdot d\vec{s} &= \int_0^1 \begin{pmatrix} t^2 \\ t^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt = \int_0^1 t^2\, dt = \frac{1}{3}, \\ \int_{\gamma_2} F \cdot d\vec{s} &= \int_0^1 \begin{pmatrix} (1-t)^2 + t^2 \\ (1-t)^2 - t^2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} dt = \int_0^1 -2t^2\, dt = -\frac{2}{3}, \\ \int_{\gamma_3} F \cdot d\vec{s} &= \int_0^1 \begin{pmatrix} (1-t)^2 \\ -(1-t)^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} dt = \int_0^1 (1-t)^2\, dt = \frac{1}{3},\end{aligned}$$

so, by additivity of the lintegral, we conclude

$$\int_\gamma F \cdot d\vec{s} = \int_{\gamma_1} F \cdot d\vec{s} + \int_{\gamma_2} F \cdot d\vec{s} + \int_{\gamma_3} F \cdot d\vec{s} = \frac{1}{3} - \frac{2}{3} + \frac{1}{3} = 0.$$

Exercise 10.2 In each of the following, determine whether the vector field $V\colon \mathbb{R}^3 \to \mathbb{R}^3$ admits a potential and compute the line integral $\int_\gamma V \cdot d\vec{s}$ along the curve $\gamma\colon [0,1] \to \mathbb{R}^3$, $\gamma(t) = (t^3, t^2+t, t)$.

$$\text{(a)} \quad V(x,y,z) = \begin{pmatrix} 2xy^3 \\ 3x^2y^2 + 2yz \\ y^2 \end{pmatrix}, \qquad \text{(b)} \quad V(x,y,z) = \begin{pmatrix} x+z \\ x+y+z \\ x+z \end{pmatrix}.$$

Solution. Recall that, to determine whether there exists a potential, namely a function $f\colon \mathbb{R}^3 \to \mathbb{R}$ so that $\nabla f = V$, the necessary condition is

$$\frac{\partial}{\partial x_i} V_j = \frac{\partial}{\partial x_j} V_i \quad \forall\, i,j \in \{1,2,3\}, \tag{$\star$}$$

where we denoted $(x_1, x_2, x_3) = (x, y, z)$. Since $\mathbb{R}^3$ is star-shaped the condition is also sufficient.

(a) It is a simple computation to see that $(\star)$ is satisfied, so V admits a potential f. To find, one, we see that

$$\begin{aligned} \frac{\partial f}{\partial x} \overset{!}{=} 2xy^3 \;&\Rightarrow f(x,y,z) = x^2y^3 + g(y,z) \\ &\Rightarrow \frac{\partial f}{\partial y} = 3x^2y^2 + \frac{\partial g}{\partial y} \overset{!}{=} 3x^2y^2 + 2yz \\ &\Rightarrow g(y,z) = y^2 z + h(z) \\ &\Rightarrow \frac{\partial f}{\partial z} = y^2 + h'(z) \overset{!}{=} y^2. \end{aligned}$$

So h is constant. Hence $f(x,y,z) = x^2y^3 + y^2z$ is a potential for V. For the integral along γ we then have

$$\int_\gamma v \cdot d\vec{s} = f(\gamma(1)) - f(\gamma(0)) = f(1,2,1) - f(0,0,0) = 12.$$

(b) Since $\partial_2 V_1 = 0 \neq 1 = \partial_1 V_2$, V does not fulfill $(\star)$ and so it does not admit a

potential. We then compute the line integral with its definition:

$$\begin{aligned}\int_\gamma V \cdot d\vec{s} &= \int_0^1 v(\gamma(t)) \cdot \gamma'(t)\, dt = \int_0^1 \begin{pmatrix} t^3+t \\ t^3+t^2+2t \\ t^3+t \end{pmatrix} \cdot \begin{pmatrix} 3t^2 \\ 2t+1 \\ 1 \end{pmatrix} dt \\ &= \int_0^1 (t^3+t)(3t^2+1)\, dt + \int_0^1 2t^4+3t^3+5t^2+2t\, dt \\ &= \left[\tfrac{1}{2}(t^3+t)^2\right]_0^1 + \left[\tfrac{2}{5}t^5+\tfrac{3}{4}t^4+\tfrac{5}{3}t^3+t^2\right]_0^1 \\ &= 2+\tfrac{2}{5}+\tfrac{3}{4}+\tfrac{5}{3}+1 = \tfrac{349}{60}.\end{aligned}$$

Exercise 10.3 The following vector field describes, according to the *Biot-Savart law*, the magnetic field generated by an infinitely long, constant-current electric wire displaced along the z-axis:

$$B(x,y,z) = \frac{\mu_0 I}{2\pi}\frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \quad \text{defined for } (x,y) \neq 0,$$

where μ_0 and I are, respectively, the magnetic constant and I the (also constant) current.

(a) Prove that it is

$$\frac{\partial}{\partial x_i}B_j = \frac{\partial}{\partial x_j}B_i \quad \forall\, i,j \in \{1,2,3\},$$

where we denoted $(x_1,x_2,x_3) = (x,y,z)$.

(b) Consider the curves $\gamma_m : [0, 2\pi m] \to \mathbb{R}^3$, $\gamma_m(t) = (\cos(t), \sin(t), 0)$ for $m \in \mathbb{Z}$, and compute the line integrals $\int_{\gamma_m} B \cdot d\vec{s}$.

(c) Does B admit a potential in $\mathbb{R}^3 \setminus \{z\text{-axis}\}$?

Solution. (a) This is a direct calculation.

(b) We see that

$$\begin{aligned}\int_{\gamma_m} B \cdot d\vec{s} &= \frac{\mu_0 I}{2\pi}\int_0^{2\pi m} \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 0 \end{pmatrix} dt \\ &= \frac{\mu_0 I}{2\pi}\int_0^{2\pi m} dt = \mu_0 I m.\end{aligned}$$

(c) No, since the γ_m's are loops but the line integral are not zero. Note that this is not in conflict with (a) since $\mathbb{R}^3 \setminus \{z\text{-axis}\}$ is not star-shaped.