Class 1

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Mose Müller
mosmuell@student.ethz.ch

Exercises:
- Uploded on Friday (I'll give some comments / hints the Monday after)
- No manatory hand-in, no bonus!
- Please do work on them, it will benefit you!
- You do not have to hand in all exercises / questions!
- Hand-in via SAM UpTool (link on course website)
- You can hand-in exercises from all weeks, I'm happy to correct them & give you feedback!
- I will try to correct the exercises before mondays, I'd appreciate it if you could hand-in on Friday if you want immediate feedback
- Give some explanation for the MC

Concept of this class

- give a structure if possible
- provide with concepts / definition needed for the exercises
  → give examples
- go through theory if needed
- answer any questions you may have (as good as I can :-))
  → feel free to mail any questions that I should discuss

Unit outline

1. Ordinary differential equations (ODE)

   - linear ODE's (homogeneous, inhom.)

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- linear ODE's of 1^{st} order $\quad y' + ay = b$
- linear ODE's with constant coefficients
- Example: harmonic oscillator

2. Differential calculus in $\mathbb{R}^n$

(some sort of structure will be given when the time comes)

3. Integration in $\mathbb{R}^n$

(")

<u>Exercise sheet</u>

MC (again, please provide explanation to your answer so that I can give feedback)

If a statement is <u>true</u>, there needs to be a <u>proof</u> (proof by contradiction),
↳ $A \Rightarrow B$
$\neg B \Rightarrow \neg A$

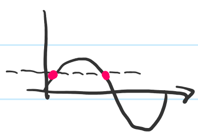
a <u>false</u> statement has at least one <u>counter example</u>!

1.1. <u>Def</u>: $f: X \to Y$, X, Y are sets. (domain: X, codomain: Y)

- f is injective $\Leftrightarrow$ $f(x_0) = f(x_1) \Rightarrow x_0 = x_1$, $x_0, x_1 \in X$

$\Leftrightarrow \quad x_0 \neq x_1 \Rightarrow f(x_0) \neq f(x_1)$

Intuitively: f should not cross a line parallel to the x-axis more than once!



Example: $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x$ is injective

$g: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ is not injective

$g: \mathbb{R} \to \mathbb{R}, \ x \mapsto x^2$ is not injective

but $\tilde{g}: \mathbb{R}_{\geq 0} \to \mathbb{R}, \ x \mapsto x^2$ is injective

$h: [0, 2\pi] \to [-1, 1], \ x \mapsto \sin(x), \cos(x)$ are not inj.

- f is surjective $\Leftrightarrow$ $f(X) = Y$ $(\operatorname{Im}(f) = Y)$

  $\forall y \in Y \ \exists x \in X: \ f(x) = y$

Ex: $f: \mathbb{R} \to \mathbb{R}, \ x \mapsto x$ is surjective

$g: \mathbb{R} \to \mathbb{R}, \ x \mapsto x^2$ is not surjective

$\tilde{g}: \mathbb{R} \to \mathbb{R}_{\geq 0}, \ x \mapsto x^2$ is surjective

1.2. Beware that $n \geq 1$!

Ex 1

Look at definitions given in the lecture!

<u>Def.</u>: • ODE

- function <u>f is unknown</u>
- f has only <u>one argument</u> ($\to$ ordinary)
- f & its derivatives are evaluated at the <u>same point</u>

• Order of an ODE

- highest derivative that appears in equation

• (in-) homogeneous linear ODE

$$y^{(k)} + a_{k-1}(x) y^{(k-1)} + \ldots + a_1(x) y' + a_0(x) y = b(x) \qquad (*)$$

If $b(x) = 0$, the linear ODE is homogeneous, otherwise its inhomog.

part b) There are case distinctions ($a_i = 0$?)

Examples:
- $y'(\frac{1}{2}x) + y(x) = 0$
- $y'' + 2y - 5 = 0$
- $(y')^2 + 2y = 2x$
- $y'y - y = 0$

Ex 2

What to know: $\frac{d}{dx} e^{a(x)} = a'(x) \cdot e^{a(x)}$ $\qquad \frac{d}{dx} e^{-x} = -e^{-x}$

How do you verify whether a given $y(x)$ solves an ODE? → plug in!

seen already?

Example:
- $y(x) = c \cdot e^{x^2}$ solves $y' = 2xy$ (1)

$y'(x) = c \cdot 2x \cdot e^{x^2}$

$\overset{(*)}{\Rightarrow} \; c \cdot 2x \cdot e^{x^2} = 2x \cdot c \cdot e^{x^2}$ ✓

- $y(x) = c \cdot e^{-t}$ solves $y' + y = 0$ (2)

$y'(x) = -c \cdot e^{-t}$

$\Rightarrow -c \cdot e^{-t} + c \cdot e^{-t} = 0$ ✓

Ex 3

What to do: • calculate the derivatives of φ up to the order of the differential equation

- relate φ to the derivatives (does not have to be linear)

Example: • $y = c \cdot e^{-x}$, 1^{st} order

$$y'(x) = -\underbrace{c \cdot e^{-x}}_{= y(x)} \quad \Rightarrow \quad y'(x) = -y(x)$$

• $y = c \cdot e^{x^2}$ 1^{st} order

$$y'(x) = c \cdot 2x \cdot e^{x^2} = 2x \cdot \underbrace{c \cdot e^{x^2}}_{= y} \longrightarrow y' = 2x \cdot y$$

Regarding Example 2.1.4 in the script

$$y'' = x(x+1)y' - 3y \quad \Longleftrightarrow \quad Y' = \begin{pmatrix} 0 & 1 \\ -3 & x(x+1) \end{pmatrix} Y \quad \text{with } Y = \begin{pmatrix} y \\ y' \end{pmatrix} \Rightarrow Y' = \begin{pmatrix} y' \\ y'' \end{pmatrix}$$