Class 2

Monday, 28 September 2020 10:21

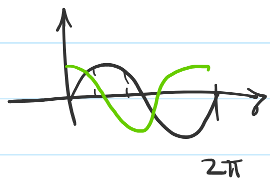
Old exercise sheet

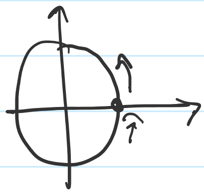
Most common errors:

- MC 1.1 (b) (f inj → at least on f_i injective)

Counterexample: $f: [0,2\pi) \to \mathbb{R}^2, \ x \mapsto \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}$

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$





- MC 1.2 $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ $\boxed{n \geq 1}$

Counterexample:

$f: [0,2\pi] \to \mathbb{R}^2, \ x \longmapsto \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}$

$f'(x) = \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix} \to |f'(x)| = \sin^2(x) + \cos^2(x) = 1 \quad \forall \in [0,2\pi]$

$f(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f(2\pi) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$f'(c) \neq \dfrac{f(2\pi) - f(0)}{2\pi} = 0$

- Ex 1.1 a), a) term "homogeneous" is only used for <u>linear</u> ODEs

b) $\underline{a_2}(x) y'' + a_1 y' + a_0 y = b(x)$ (can be constant)

→ degree of ODE depends on the prefactors

$a_2(x) \neq 0$ for at least one $x \in I \Rightarrow 2^{nd}$ order

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d) $y(x) = x \cdot y'(x) + f(y'(x))$

- linearity, homogeneity depended of f

e.g. $f(x) = x \rightarrow y(x) = x \cdot y'(x) + y'(x)$ linear, homog.

$f(x) = \cos(x) \rightarrow y(x) = x \cdot y'(x) + \underbrace{\cos(y'(x))}_{\text{non-linear}}$

- It is an ODE $f(y'(x))$

$\rightarrow$ the eq is linear $\Leftrightarrow f(x) = a \cdot x + b$

Rmk: • If $f(y''(x))$ instead of $f(y'(x))$

it would be 2^{nd} order!

- If $f \equiv$ const $\neq 0$ the ODE is linear & inhomog.

Rmk (Ex 1.2) $y'' + 2y' + y = 0$

- 2^{nd} order ODE has two linearly independent solutions ($\rightarrow$ theorem)
- you are given two solutions in (a) $\rightarrow e^{-x}, x \cdot e^{-x}$

$\rightarrow$ general solution is a linear combination of them both

$y(x) = a \cdot e^{-x} + b \cdot x \cdot e^{-x}$

Repitition

$y' + ay = b$

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- Integrating factor method (homog. & particular)

(a) determine integrating factor (IF)

$$\exp\left(\int a(x)\,dx\right) = \exp(A(x))$$

(b) multiply the ODE with IF

$$\underbrace{\exp(A(x))\left(\frac{dy}{dx} + a(x)\,y\right)} = b(x)\cdot\exp(A(x))$$

$$= \frac{d}{dx}\left(y\cdot\exp(A(x))\right)$$

$$= \frac{dy}{dx}\cdot\exp(A(x)) + y\cdot\underbrace{A'(x)}_{=a(x)}\cdot\exp(A(x))$$

$$\Rightarrow \frac{d}{dx}\big(\underbrace{y\cdot\exp(A(x))}_{\equiv z(x)}\big) = b(x)\cdot\exp(A(x)) \quad \rightarrow \text{Integrating}$$

(c) Check: same solution as with variation of constants

(i) $z(x) = y\cdot\exp(A(x)) \Rightarrow y(x) = z(x)\cdot\exp(-A(x))$

(ii) $z'(x) = b(x)\cdot\exp(A(x)) \Rightarrow z(x) = \int dx\ b(x)\cdot\exp(A(x))$

Example:

(1) $y' - 2y = x$

- IF $\quad \exp(-\int 2\,dx) = \exp(-2x)$

- ODE $\quad \underbrace{\exp(-2x)}_{IF} \cdot \underbrace{(y' - 2y)}_{\text{LHS of ODE}} = \underbrace{\exp(-2x)}_{IF} \cdot \underbrace{x}_{\text{RHS of ODE}}$

$$\Longleftrightarrow \quad \frac{d}{dx}\left(y \cdot \exp(-2x)\right) = \exp(-2x)\cdot x$$

- Integrate $\quad \int dx \, \frac{d}{dx}\left(y \cdot \exp(-2x)\right) = \underbrace{\int dx \, \exp(-2x)\cdot x}_{\text{Integration by parts}}$

$$\Longrightarrow \quad y \cdot \exp(-2x) = \left(-\frac{x}{2} - \frac{1}{4}\right)\cdot \exp(-2x) + C \quad \Big| \cdot \exp(+2x)$$

$$\Longleftrightarrow \quad y(x) = -\frac{x}{2} - \frac{1}{4} + C \cdot \exp(+2x) \quad , \; c \in \mathbb{C}$$

(2) $\quad y' + 4y = 0$

(3) $\quad y' - \frac{2}{x} y = x^2 \cdot \cos(x)$

Exercise sheet

MC 2.1 $\quad$ This is essentially solving $y'' - \sigma^2 y = 0$, $\sigma > 0$

Def: $f: X \to \mathbb{R}$ is called bounded if $\exists M \in \mathbb{R}$ s.t.

$$|f(x)| \leq M \quad \forall x \in X$$

MC 2.2

Ex 2.1 $\quad$ • Important: $\omega^2 \equiv \frac{k}{m}$.

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• linear ODE with constant coeff. → characteristic polynomial

Ex 2.2

$$\begin{cases} y'(t) = k \cdot y(t), & t > 0 \\ y(0) = y_0 \end{cases}$$

(a) –

(b) $$\begin{cases} \frac{y(t+\varepsilon) - y(t)}{\varepsilon} = k \cdot y(t) \\ y(0) = y_0 \end{cases}$$

What to

$$y(t) = \dots\ y(t-\varepsilon)$$
$$= \dots\ y_0$$

(c) $\exp(x) = \lim_{n \to \infty} \left(\frac{x}{n} + 1\right)^n$

Ex 2.3

(a) $y(x) = \frac{1}{u(x)}$, $y'(x) = ?$ (quotient rule) $y' \neq u' \neq (u^{-1})'$

(b) Integration factor method