Class 3

Monday, 5 October 2020 09:27

Short summary of the course: www.overleaf.com/read/tsstbfjkzbgq

Old exercise sheet

Most common errors:

MC 2.1 $y''(x) - \sigma^2 y(x) = 0$, $\sigma > 0$ characteristic polynomial

- general solution: $y(x) = c_1 \cdot e^{\sigma x} + c_2 \cdot e^{-\sigma x}$ $P(\lambda) = \lambda^2 - \sigma^2 = 0$

(c) $\forall a,b \in \mathbb{R}$ there's only one solution with $y(0) = a$, $y(1) = b$

- $y(0) = a \rightarrow c_1 + c_2 = a$
- $y(1) = b \rightarrow c_1 \cdot e^{\sigma} + c_2 \cdot e^{-\sigma} = b$

$$\Leftrightarrow \underbrace{\begin{pmatrix} 1 & 1 \\ e^{\sigma} & e^{-\sigma} \end{pmatrix}}_{A} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad \det A = e^{-\sigma} - e^{\sigma} \neq 0$$

Ex 2.1

$$m\ddot{x}(t) = -k \cdot x(t) \Leftrightarrow \ddot{x}(t) = -\omega^2 x(t), \quad \omega^2 = \frac{k}{m}$$

$$P(\lambda) = \lambda^2 + \omega^2 \rightarrow \lambda_{1,2} = \pm i \cdot \omega$$

general solution: $x(t) = c_1 \cdot e^{i\omega t} + c_2 \cdot e^{-i\omega t}$

$$= c_1 \cdot \underbrace{(\cos(\omega t) + i\sin(\omega t))}_{} + c_2 (\cos(\omega t) - i \cdot \sin(\omega t))$$

$$= \underbrace{(c_1 + c_2)}_{} \cdot \cos(\omega t) + \underbrace{i \cdot (c_1 - c_2)}_{} \sin(\omega t)$$

A B

Ex 2.2 $y' = k\cdot y \rightarrow e^{kt}\cdot c$

(b) $\frac{y(t+\varepsilon)-y(t)}{\varepsilon} = k\cdot y(t) \iff y(t+\varepsilon) = \overbrace{(1+k\cdot\varepsilon)}^{\text{independent of } t!}\, y(t)$ $\quad t \rightsquigarrow t-\varepsilon$

$y(t) = (1+k\cdot\varepsilon)\, y(t-\varepsilon)$

$\vdots$

$y(t) = (1+k\cdot\varepsilon)^n \underbrace{y(\underbrace{t-\varepsilon\cdot n}_{=0})}_{y_0}$

$\Rightarrow t-\varepsilon\cdot n = 0 \Rightarrow n = \frac{t}{\varepsilon}$

$y(t) = (1+k\cdot\varepsilon)^{t/\varepsilon}\, y_0$

c) $\lim_{\varepsilon\to 0}(1+k\cdot\varepsilon)^{t/\varepsilon} = \lim_{m\to\infty}\left(1+\frac{k}{m}\right)^{m\cdot t} = \exp(k\cdot t)$

Ex 2.3

a) $x\,y'(x) = 2y(x) - 3x\,y^2(x)$

$y(x) = \frac{1}{u(x)}, \quad y'(x) = \frac{-u'(x)}{u^2(x)} \neq \frac{1}{u'(x)}$ $\qquad \left(\frac{b(x)}{c(x)}\right)' = \frac{b'(x)\cdot c(x) - b(x)\cdot c'(x)}{c^2(x)}$

$\Rightarrow u'(x) + \frac{2}{x}u(x) = 3$

b) $u(x) = \frac{x^3+c}{x^2} \longrightarrow y(x) = \frac{x^2}{x^3+c}$

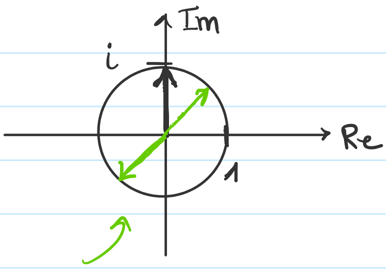
c) Beware: $y(x) = \frac{1}{u(x)} \iff u(x) = \frac{1}{y(x)} \rightarrow y(x) \neq 0$

This substitution does not give $y=0$ as a solution

Exercise sheet

MC 3.2

$\sqrt{i} = ?$ $\qquad i = e^{i\cdot\frac{\pi}{2}}$



$\sqrt{i} = e^{i\frac{\pi}{4}}, \; e^{i\frac{\pi}{4}\cdot 5}$

$e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + i\cdot\frac{\sqrt{2}}{2}$

$e^{i\frac{\pi}{4}} = \cos(\frac{\pi}{4}) + i\cdot\sin(\frac{\pi}{4})$

Ex 3.1

(a) $\to$ Skript Ex 2.1.4 $\quad y'' = x(x+1)y' - 3y \quad$ 2nd order ODE

$Y = \begin{pmatrix} y \\ y' \end{pmatrix} \quad Y' = \begin{pmatrix} y' \\ y'' \end{pmatrix}$

$Y' = AY$ with $A = \begin{pmatrix} 0 & 1 \\ -3 & x(x+1) \end{pmatrix}$

(b) Characteristic polyn of ODE $\Leftrightarrow$ characteristic poly of A

$\det(A - \mathbb{1}\cdot\lambda) = 0$

$\to$ prove this statement via induction

(c) u is eigenvector with eigenvalue λ $\Leftrightarrow$ $A\cdot\vec{u} = \lambda\cdot\vec{u}$

Revision

- particular solution to $y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1 y' + a_0 y = b(x)$

1. Ansatz

Solution y_p is similar to the disturbance function $b(x)$

$b(x)$	$y_p(x)$
$a \cdot \exp(\alpha x)$	$b \cdot \exp(\alpha \cdot x)$
$a \cdot \sin(\beta x)$ $b \cdot \cos(\beta \cdot x)$	$c \cdot \sin(\beta x) + d \cdot \cos(\beta x)$
$a \cdot \exp(\alpha x) \cdot \sin(\beta x)$ $b \cdot \exp(\alpha \cdot x) \cdot \cos(\beta x)$	$\exp(\alpha x) \cdot [c \cdot \sin(\beta x) + d \cdot \cos(\beta x)]$

Note: If the RHS solves the homog. ODE, then you to multiply the Ansatz with x^m, where m is multiplicity of the eigenvalue

Example: (i) $y' - 2y = x$ $\rightarrow$ homog. : $y_h(x) = c \cdot e^{2x}$

$$\left.\begin{aligned} y_p &= a \cdot x + b \\ y_p' &= a \end{aligned}\right\} \quad y_p' - 2y_p = a - 2ax - 2b \overset{!}{=} x$$

$\rightarrow a = -\frac{1}{2}$ $(-2ax = x)$

$b = -\frac{1}{4}$ $(a - 2b = 0 \overset{a=-\frac{1}{2}}{\Rightarrow} -2b = \frac{1}{2})$

$\Rightarrow y_p = -\frac{x}{2} - \frac{1}{4}$

(ii) $y' + 4y = e^{-4x}$ $\rightarrow$ homog sol: $y_h(x) = e^{-4x} \cdot d$

$y' + 4y = 0 \Rightarrow P(\lambda) = \lambda + 4 \Rightarrow \lambda = -4$

$y_p(x) = c \cdot x \cdot e^{-4x}$

$\hookrightarrow$ RHS equals to some homog. solution!

$y_p'(x) = c \cdot e^{-4x} - 4 \cdot c \cdot x \cdot e^{-4x}$

$$\Rightarrow y_p' + 4y_p = c \cdot e^{-4x} \underline{- 4 \cdot c \cdot x \cdot e^{-4x}} \underline{+ 4 \cdot c \cdot x \cdot e^{-4x}} \overset{!}{=} e^{-4x}$$

$$\Rightarrow c \overset{!}{=} 1$$

2. Variation of constants (dim = 2) → Skript p. 12–13

$y''(x) + a_1 \cdot y'(x) + a_0 \cdot y(x) = b(x)$

- homog. sol. $y_h = z_1 \cdot f_1(x) + z_2 \cdot f_2(x)$
- particular sol. $y_p = z_1(x) \cdot f_1(x) + z_2(x) \cdot f_2(x)$
  - impose a <u>constraint</u>: $z_1' \cdot f_1 + z_2' \cdot f_2 = 0$ (*)
  - find a second eq.: $z_1' f_1' + z_2' \cdot f_2' = b$ (**)

  How to get there: • calculate y_p', y_p'' and plug into ODE

  • using (*) we can solve for this eq.

- solve the system of equations

$$\begin{cases} z_1' f_1 + z_2' \cdot f_2 = 0 \\ z_1' f_1' + z_2' f_2' = b \end{cases} \iff \underbrace{\begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix}}_{=A} \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

→ A is invertible! $\det(A) = f_1 f_2' - f_1' f_2 \neq 0$

$f_1, f_2 \propto e^{\lambda_{1,2} x}$

$f_1', f_2' \propto \lambda_{1,2} \cdot e^{\lambda_{1,2} x}$

- solution: $\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ b \end{pmatrix}$ → gives z_1, z_2 by integration

Example 2.4.9 in script