Class 4

Monday, 12 October 2020 10:53

Short summary:
www.overleaf.com/read/tsstbfjkzbgq

Old exercise sheet

MC1.1 $y'(x) = e^{y(x)} > 0$

1) Fundamental theorem of calculus

$$y(t) = y(0) + \int_0^t y'(\tau)\,d\tau$$

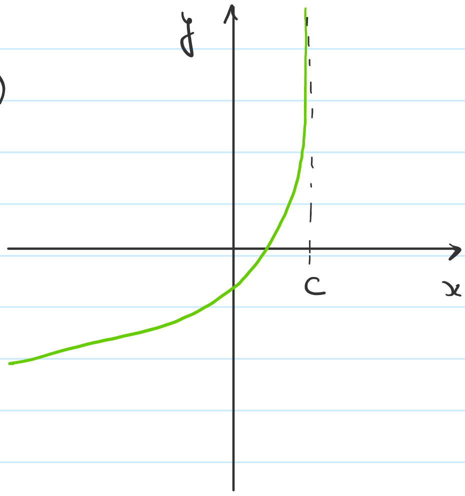
$$= \underbrace{y(0)}_{>0} + \int_0^t \underbrace{\exp(y(\tau))}_{>0}\,d\tau \geq 0 \text{ for } t>0$$

a) $y(t) > 0$ $t > 0$? c) y is increasing

b) $y(0) = a$ $y(1) = b$

$a > 0$, $b < 0$ ↯

2.) Solving the ODE $y(x) = -\ln(c-x)$



- $y(0) = a \Rightarrow -\ln(c) = a \Rightarrow c = \exp(-a)$

b) ⇝ solution only has one unknown variable, but two initial condition

↯ This can cause contradictions!

Remark: $y(x) = -\ln(c-x) = \ln\left(\frac{1}{c-x}\right)$

$\ln(x^a) = a \cdot \ln(x)$ $a = -1$

Ex 3.1

a) $y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = b(x) \qquad (*)$

Want to rewrite this into $z' = A \cdot z$

$$A = \left[\begin{array}{c|ccc} 0 & & & \\ \vdots & & \mathbb{1}_{n-1} & \\ 0 & & & \\ \hline -a_0 & -a_1 & \dots & -a_{n-1} \end{array}\right] \qquad \mathbb{1}_{n-1} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

b) Characteristic polyn. of $(*)$ equals the charac. poly. matrix A

Proof by induction:

1.) $n = 1$: $\quad y' = -a_0 y \quad \to P(\lambda) = \lambda + a_0$

$A = (-a_0) \quad \to \det(\lambda \cdot \mathbb{1}_1 - A) = \lambda + a_0$

$(\lambda \mathbb{1}_1 - A) = (\lambda + a_0)$

2.) Assume you know answer for $n \xrightarrow{\text{proof}} n+1$

Inductive step: $n \to n+1$

Assume we know $p_A(\lambda) = \lambda^n + a_n \cdot \lambda^{n-1} + \dots + a_2 \cdot \lambda + a_1$

(this is charac. poly of A for dimension n)

$p_{\tilde{A}}(\lambda) = \det(\lambda \cdot \mathbb{1}_{n+1} - \hat{A})$

$$= \det \begin{pmatrix} \lambda & -1 & 0 & \dots & \\ 0 & \lambda & \ddots & & 0 \\ 0 & & \ddots & -1 & \\ \vdots & & & \lambda & -1 \\ a_0 & a_1 & \dots & a_{n-1} & \lambda + a_n \end{pmatrix}$$

Calculate det

$$= \lambda \cdot \det \begin{pmatrix} \lambda & -1 & & \\ & \lambda & -1 & 0 \\ 0 & & \ddots & \\ & & & -1 \end{pmatrix} + \overbrace{(-1)^{n-1} \cdot (-1)^{n-1}}^{= (-1)^{2(n-1)} = 1} \cdot a_0$$

$$= \lambda \cdot \det \begin{pmatrix} 0 & \ddots & -1 \\ a_1 \; a_2 & \cdots & a_n + \lambda \end{pmatrix} + \underbrace{(-1)^{n+1}}_{\text{change of sign due to det}} \cdot \underbrace{(-1)^{n-1}}_{\det(-\mathbb{1}_{n-1})} \cdot a_0$$

inductive step

$$= \lambda \cdot \overbrace{(\lambda^n + a_n \cdot \lambda^{n-1} + \ldots + a_1)}^{\det(\tilde{A} - \lambda \mathbb{1}_{n+1})} + a_0$$

$$= \lambda^{n+1} + a_n \lambda^n + \ldots + a_1 \lambda + a_0$$

Ex 3.2

(a) $y^{(4)} + 1 = 0 \rightarrow$ inhomog ODE

$P(\lambda) = \lambda^4$ not $\lambda^4 + 1$!

$\overset{P(\lambda)=0}{\Longrightarrow} \lambda = 0$ with multiplicity 4! → you have to multiply by x^m where $m \in \{0,1,2,3\}$

$y_h(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$

$y_p(x) = a_4 \cdot x^4 \rightarrow$ prefactor to be determined by inserting into ODE

Note: in the solution we are just integrating y over x

- $\int y^{(4)} \, dx = y^{(3)} + a_4$
- $\int y^{(3)} + a_4 \, dx = y'' + a_4 x + a_3$

$\vdots$

$$y(x) = a_4 \cdot x^4 + a_3 \cdot x^3 + a_2 \cdot x^2 + a_1 \cdot x + a_0$$

Exercise sheet

MC 4.1

- Try to solve this without solving ODE (using theorem/propositions from lecture)

MC 4.2

Def: $f: (a,b) \to \mathbb{R}^n$, $f(t) = (f_1(t), \ldots, f_n(t))$ is differentiable if & only if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

exists.

$P \Rightarrow Q$, P is sufficient for Q and Q is necessary for P

Ex 4.1 → solve 2^{nd} order ODE

c) Use the results from a) + b)

Ex 4.2 → use variation of constant (→ last weeks tutorial)

Ex 4.3

→ use separation of variables for non-linear ODEs

Note: Cauchy problem → initial value problem / boundary value problem

Revision

- Separation of variables (first order ODE)

ODE: $\frac{dy}{dx} = h(x) \cdot g(y)$

Separation of variables (first order ODE)

ODE: $\quad \frac{dy}{dx} = b(x) \cdot g(y)$

Solve: $\quad \Leftrightarrow \underbrace{\frac{dy}{g(y)}}_{\text{only dependent on } y!} = \underbrace{b(x) \cdot dx}_{\text{only dependent on } x!}$

$$\int \frac{dy}{g(y)} = \int b(x)\, dx$$

Examples

1. $\frac{dy}{dx} = 8y \quad \Leftrightarrow \quad \frac{dy}{y} = 8 \cdot dx \quad \rightarrow$ integrate

$$\int \frac{dy}{y} = \int 8 \cdot dx$$

$$\ln|y| = 8 \cdot x + c \quad \Big| \exp$$

$$\Rightarrow |y(x)| = \exp(8x) \cdot \exp(c)$$

(we can drop the absolute value by exchanging $\exp(c)$ with $c_1 \in \mathbb{R}$!)

2. $\frac{dy}{dx} = \frac{1}{3y^2}$

$\rightarrow y(x) = \sqrt[3]{x+c}$

3. $\frac{dy}{dt} + \frac{1}{\sin(t)} = 0$

$\rightarrow y(t) = \arccos(t+c)$

We sometimes have to bring the ODE in a separable form by using substitution:

Example:

1. $\frac{dy}{dx} = \frac{y}{x} + 2$

$u = \frac{y}{x} \Leftrightarrow y = u \cdot x$

$y' = u + u' + x$ (derivative of y, using product rule)

$$\Rightarrow \quad u + u' \cdot x = u + 2$$

$$\Rightarrow \quad u' \cdot x = 2 \quad \Leftrightarrow \quad \frac{du}{dx} = \frac{2}{x}$$

$$\int du = \int dx \, \frac{2}{x}$$

$$u(x) = \ln(x^2) + c$$

$$\Rightarrow \quad y(x) = x \cdot u(x) = x \cdot \ln(x^2) + c \cdot x$$

2. $\frac{dy}{dx} = x + y$

$u = x + y \quad \Leftrightarrow \quad y = u - x$ substitution

derive!

$$\frac{dy}{dx} = \frac{du}{dx} - 1$$

$$\Rightarrow \quad \frac{du}{dx} - 1 = u \quad \Leftrightarrow \quad \frac{du}{dx} = u + 1$$

$$\int \frac{du}{u+1} = \int dx$$

$\ln|u+1| = x + c \quad \Big| \exp$

$u + 1 = \exp(x) \cdot c_1$ (Note that c_1 can be negative as we dropped the absolute value on the LHS)

$$u = \exp(x) \cdot c_1 - 1$$

$$\Rightarrow \quad y = c_1 \cdot \exp(x) - x - 1$$