Class 5

Monday, 19 October 2020 12:11

Cauchy problem:

Differential equation with certain conditions in domain of the solution

- e.g. initial value problem — initial conditions $y(t) = \_\_$ $y'(t) = \_\_$ $y''(t) = \_\_$
- boundary value problem — $y(0) = \_\_$ $y(\pi) = \_\_$ when condiseing the domain $[0, \pi]$

Old exercise sheet

MC 4.1

$$(*) \begin{cases} y' = \sqrt{|y|} & \text{for } t > 0 \\ y(0) = 0 \end{cases}$$

Solutions: $y(x) \equiv 0$, $y(x) = \frac{x^2}{4}$

Most solved ODE $\rightarrow$ not "goal" of exercise!

Theorem (1^{st} lecture)

For any initial condition, there exists a unique solution to the homog. problem if the ODE is linear!

$\rightarrow$ that theorem does not apply here as ODE is non-linear!

Ex 4.1

a) $y^{(4)} + 2y'' + y = \sin(x)$

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$P(\lambda) = \lambda^4 + 2\lambda^2 + 1 = (\lambda^2+1)^2 \xrightarrow{P(\lambda)=0}$ eigenvalue $\lambda_{1,2} = \pm i$ with multiplicity 2

!! If the eigenvalues of the charach. polyn. are imaginary, write down the solution in terms of sin & cos functions!!

$$y_h(x) = a \cdot \sin(x) + b \cdot \cos(x) + c \cdot x \cdot \sin(x) + d \cdot x \cdot \cos(x)$$

$$= (a + c \cdot x) \cdot \sin x + (b + d \cdot x) \cdot \cos(x)$$

$$f(x) = z_1 \cdot e^{ix} + z_2 \cdot e^{-ix} \quad , \quad e^{ix} = \cos(x) + i \cdot \sin(x)$$

$$= z_1 \cdot (\cos(x) + i \cdot \sin(x)) + z_2 \cdot (\cos(x) - i \cdot \sin(x))$$

$$= \underbrace{(z_1 + z_2)}_{A} \cdot \cos(x) + \underbrace{i(z_1 - z_2)}_{B} \cdot \sin(x)$$

$$\lambda = \pm a \cdot i \quad \rightarrow \quad y_h(x) = z_1 \cdot \sin(a \cdot x) + z_2 \cdot \cos(a \cdot x)$$

$$y_p = (z_1 \cdot \sin(x) + z_2 \cdot \cos(x)) \cdot x^2$$

→ as $\sin x / \cos x$ and $x \cdot \sin x / x \cdot \cos x$ are solution to the homog. problem, we have to multiply by x^2!!

→ solve for the right z_1, z_2 $\left(y_p^{(4)} + 2y_p'' + y_p \overset{!}{=} \sin(x) \right)$

Ex 4.2

Inverse matrix of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

[• Gaussian elimination]

• Formula $A^{-1} = \frac{1}{\det A} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

– change diagonal entries
– multiply offdiagonal entries by -1

Revision : Continuity

• Definition: f is continuous at x_0 $\overset{def}{\iff}$ $\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in X : \|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \varepsilon$

• Prop f is continuous at x_0 $\iff$ for every sequence $(x_k)_k \to x_0$ as $k \to \infty$ the $(f(x_k))_k \to f(x_0)$

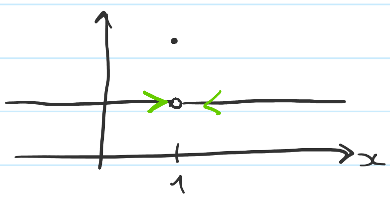
"$\lim\limits_{k\to\infty} f(x_k) = f(\lim\limits_{k\to\infty} x_k)$"

→ you should be able to approach x_0 in every way & get same limit

– use polar coordinates $(x,y) = (r \cdot \sin\theta, r \cdot \cos\theta)$ to find counterexample

• Prop f is continuous at x_0 $\iff$ $\lim\limits_{\substack{x \to x_0 \\ x \neq x_0}} f(x) = f(x_0)$

→ used as definition last lecture. Very useful to prove things!



$$f(x) = \begin{cases} 1 & x \in \mathbb{R} \setminus \{1\} \\ 2 & x = 1 \end{cases}$$

• Prop The composition, sum & product of continuous functions is continuous!

→ you can easily see whether some function are cont. or not!

Example:

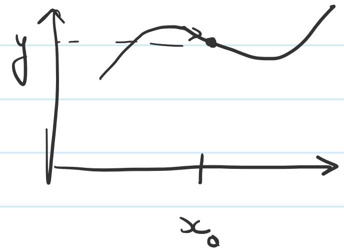
1. $\exp(\sin(x)+x) \to$ continuous on $\mathbb{R}$

2. $\log(x^2+1) \to$ continuous on $\mathbb{R}$

3. $\dfrac{\exp(x)}{x^2-1} = \underbrace{\exp(x)}_{\text{cont.}} \cdot \underbrace{\dfrac{1}{x^2-1}}_{\text{cont. on } \mathbb{R}\setminus\{\pm1\}} \to$ cont. on $\mathbb{R}\setminus\{\pm 1\}$

Revision: Limits

• **Definition**: $\lim\limits_{\substack{x\to x_0 \\ x\neq x_0}} f(x) = y \overset{\text{def}}{\Longleftrightarrow} \forall \varepsilon>0\, \exists \delta>0\, \forall x\in X\setminus\{x_0\}: \|x-x_0\|<\delta \Rightarrow |f(x)-y|<\varepsilon$



• **Prop** $\lim\limits_{\substack{x\to x_0 \\ x\neq x_0}} f(x) = y \Longleftrightarrow \forall (x_k)_{k\in\mathbb{N}} \in X \longrightarrow x_0 : (f(x_k))_k \to y$

$\to$ we can use limit & convergence interchangably, useful to prove things

• **Prop** f is continuous at $x_0 \Longleftrightarrow$ for every sequence $(x_k)_k \to x_0$ as $k\to\infty$

the $(f(x_k))_k \longrightarrow f(x_0)$

" $\lim\limits_{k\to\infty} f(x_k) = f(\lim\limits_{k\to\infty} x_k)$ "

Example:

1. $\lim\limits_{x\to 1} (x+1) = 1+1 = 2$

2. $\lim\limits_{x\to\pi} \sin(x+\pi) = \sin(\lim\limits_{x\to\pi} x+\pi) = \sin(2\pi) = 0$

3. $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x} = 1 \neq \dfrac{\sin(0)}{0}$ as $\dfrac{\sin x}{x}$ is not cont. at $x=0$!

3. $\lim\limits_{x \to 0} \frac{\sin(x)}{x} = 1 \neq \frac{\sin(0)}{0}$ as $\frac{\sin x}{x}$ is not cont. at $x=0$!

• **Lemma** (Sandwich) $\quad g, f, h : \mathbb{R}^n \to \mathbb{R}$, $a \in \mathbb{R}^n$

$$g(x) \leq f(x) \leq h(x) \quad \forall x \in \mathbb{R}^n , \quad \lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L \in \mathbb{R}$$

$\Rightarrow \lim\limits_{x \to a} f(x)$ exists and is equal to L!

Example: 1. $g(x,y) = \frac{x^2 \cdot y}{x^2 + y^2}$

Estimate: $\quad 0 \leq \frac{x^2}{x^2+y^2} \leq 1$, $-|y| \leq y \leq |y|$

$$\Rightarrow \underbrace{-|y|}_{\to 0} \leq \underbrace{\frac{x^2}{x^2+y^2} \cdot y}_{g(x,y)} \leq \underbrace{|y|}_{\to 0}$$

$$\xrightarrow{\text{lemma}} \lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^2+y^2} = 0$$

2. $f(x) = x^2 \cdot \underbrace{\sin\left(\frac{1}{x}\right)}_{\lim\limits_{x\to 0} ?}$

$\lim\limits_{x \to 0} f(x) = ?$

Estimation: $\quad -1 \leq \sin\left(\frac{1}{x}\right) \leq 1$

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

$$\lim_{x \to 0} x^2 = \lim_{x \to 0} (-x^2) = 0 \quad \xrightarrow{\text{Lemma}} \quad \lim_{x \to 0} x^2 \cdot \sin\left(\frac{1}{x}\right) = 0$$

• **Substitution**

Let $g: A \to B$, $f: B \to C$ with limit $\lim_{x \to a} g(x) = b$, $\lim_{x \to b} f(x) = c$

and f is continuous at b, then

$$\lim_{x \to a} f(g(x)) = \lim_{x \to b} f(x) = c$$

Example: 1. $\lim_{x \to 1} (x-1)^2 + 5 \overset{y = x+1}{=} \lim_{y \to 0} y^2 + 5 = 5$

Exercise sheet

MC 5.1

$L(a \cdot x + c) = a \cdot L(x) + L(c)$

Two ways to prove this:

① Look at the limit $\lim_{x \to 0} \| L(x+a) - L(a) \| \overset{\text{show}}{=} 0$ (use linearity)

Why does that prove the statement? Use $\lim_{x \to x_0} L(x) = L(x_0)$!

② Look at limit $\lim_{x \to y} L(x) = \lim_{t \to a} L(g(t)) \overset{\text{show}}{=} L(y)$

with $\lim_{t \to a} g(t) = g(a) = y$. How should you choose $g(t)$?

Use that $\lim_{x \to x_0} L(x) = L(x_0)$!

MC 5.2

Use what we discussed about continuity. Follow the hint!

If f is continuous, do these statements hold?

If these statement hold, does f have to be continuous?

Ex 5.1

Use what we discussed about continuity.

Use what we discussed about continuity.

Ex 5.2

a) 1. $\lim_{t \to 0} \frac{\sin(t)}{t} = 1$ using Taylor expansion $\sin(t) = t + O(t^2)$

2. Polar coordinates + $\underbrace{\text{l'hospital's rule}}$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \text{"}\frac{0}{0}\text{"} \text{ or } \text{"}\frac{\pm\infty}{\pm\infty}\text{"}$$

$$\Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

b) Sandwich theorem, variable substitution, $\lim_{x \to 1} \frac{\log(x)}{x-1} = 1$

c) Find two sequences that give different results!

Ex 5.3

Use the definitions:

Def: A subset $X \subset \mathbb{R}^n$ is

- bounded if the set of $\|x\|$ for $x \in X$ is bounded in $\mathbb{R}$
- closed if for every sequence $(x_k)_k$ in X that converges to $y \in \mathbb{R}^n$, we have that $y \in X$.
- compact if it is bounded and closed.