# Class 6

Monday, 26 October 2020 08:40

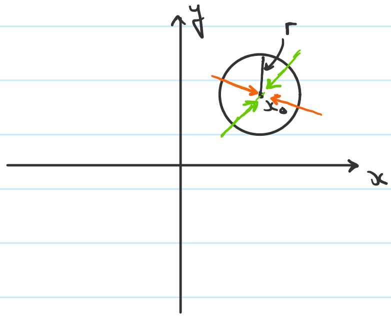
## Correction of last exercise class

Polar coordinates CANNOT be used in general to calculate limits!

This is only allowed if certain conditions are fulfilled (see Appendix of Summary), which is NOT part of the course!

However, polar coordinates CAN be used to show that certain limits do not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x\cdot y}{x^2+y^2} \overset{\substack{x=\cos\vartheta\cdot r\\ y=\sin\vartheta\cdot r}}{=} \lim_{r\to 0} \frac{\cancel{r^2}\cdot\sin\vartheta\cdot\cos\vartheta}{\cancel{r^2}\cdot(\sin^2\vartheta+\cos^2\vartheta)} = \sin\vartheta\cdot\cos\vartheta \quad \text{(dependend on } \vartheta\text{!)}$$




## Old exercise sheet

MC 5.1

$L: \mathbb{R}^m \to \mathbb{R}^n$ linear $\Delta$ continuous at $x_0 \in \mathbb{R}^n \Rightarrow L$ is cont. on $\mathbb{R}^m$

Proof: Let $y \in \mathbb{R}^m$ arbitrary

$$\lim_{x\to y} L(x) = \lim_{t\to x_0} L(t+y-x_0) \qquad \mid \text{subst. } x = t+y-x_0$$

$$= \underbrace{\lim_{t\to x_0} L(t)}_{} + L(y) - L(x_0) \qquad \mid \text{Linearity of } L$$

$$= \cancel{L(x_0)} + L(y) - \cancel{L(x_0)} \qquad \mid \text{Continuity at } x_0, \ \lim_{t\to x_0} L(t) = L(x_0)$$

$$= L(y)$$

$\rightarrow L$ is cont. for every point $y \in \mathbb{R}^m$ $\boxtimes$

Ex 5.2

Determining the limit of function using polar coordinates is only valid when showing that limit does not exist (in this course!)

$\rightarrow$ use Taylor expansion (a)) & sandwich theorem here!

b) $\lim_{(x,y)\to(1,0)} \frac{y^2 \log(x)}{(x-1)^2+y^2}$

$$0 \leq \frac{y^2}{(x-1)^2+y^2} \leq 1 \quad , \quad -|\log(x)| \leq \log(x) \leq |\log(x)|$$

$$\Rightarrow -|\log(x)| \leq \frac{y^2 \log x}{(x-1)^2+y^2} \leq |\log(x)|$$

$$\lim_{x\to 1} \pm|\log(x)| = 0 \xrightarrow[\text{theorem}]{\text{sandwich}} \lim_{(x,y)\to(1,0)} \frac{y^2\log(x)}{(x-1)^2+y^2} = 0$$

c) $\lim_{(x,y,z)\to\infty} (x^4+y^2+z^2-x^3+xyz-x+4) = ?$

- sequence $(x,0,0)$: $\lim_{(x,0,0)\to\infty} f(x,0,0) = \lim_{x\to\infty} x^4-x^3-x+4 = +\infty$

- sequence $(\sqrt{k},-k,k)$: $\lim_{k\to\infty} f(\sqrt{k},-k,k) = \lim_{k\to\infty} 3k^2-k^{3/2}-k^{5/2}-k^{1/2}+4 = -\infty$

$\Rightarrow$ the limit does NOT exist!

Ex 5.3

b) Finite sets are always closed!

c) $C = \{(x, f(x)) \in \mathbb{R}^2 \mid x \in (0,1],\ f(x) = \sin(\frac{1}{x})\}$ compact?

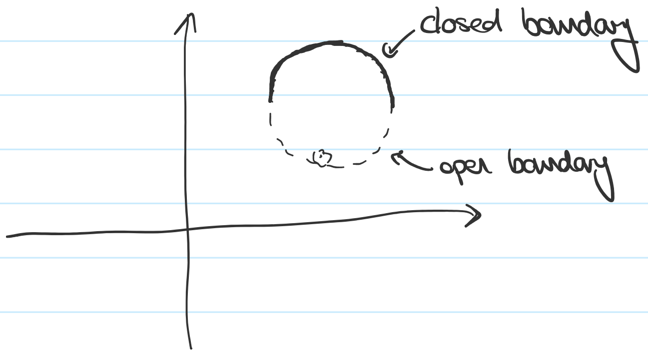
C is closed if for every sequence in C that converges to $y \in \mathbb{R}^n$, we have $y \in C$.

→ you have to find a converging sequence whose limit is not in C in order to prove that C is not closed! E.g.

$$(x_k, f(x_k))_k = \left(\frac{1}{\pi\cdot k}, \underbrace{\sin(\pi\cdot k)}_{=0}\right)_k \in C \longrightarrow (0,0) \text{ for } k\to\infty$$

⇒ C is not closed and hence not compact!

Set that's neither closed nor open:




$X \in \mathbb{R}^n$ is open $\Rightarrow$ $\mathbb{R}^n \setminus X$ is closed!

## Partial derivative

Def: $X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}^m$, $1 \leq i \leq n$. We say that $f$ has a partial derivative on $X$ w.r.t the i-th variable, if the function

$$g(t) = f(x_{0,1}, \dots, x_{0,i-1}, t, x_{0,i+1}, \dots, x_{0,n})$$

is differentiable at $t = x_{0,i}$ $\forall x_0 = (x_{0,1}, \dots, x_{0,n}) \in X$. Its derivative $g'(x_{0,i})$ is denoted by

$$\frac{\partial f}{\partial x_i}(x_0), \quad \partial_{x_i} f(x_0) \quad \text{or} \quad \partial_i f(x_0).$$

## Examples

1. $f(x,y) = x^2 \cdot y + \cos(x+y)$

$$\frac{\partial f}{\partial x}(x,y) = 2x \cdot y - \sin(x+y) \cdot 1$$

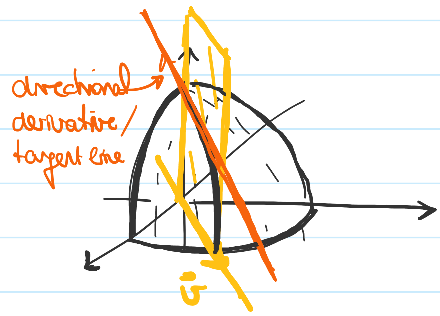
$$\frac{\partial f}{\partial y}(x,y) = x^2 - \sin(x+y) \cdot 1$$

2. $f(x,y) = e^{x^2} + y \cdot \log(x)$

$$\frac{\partial f}{\partial x}(x,y) = 2x \cdot e^{x^2} + \frac{y}{x}$$

$$\frac{\partial f}{\partial y}(x,y) = 0 + 1 \cdot \log(x)$$

## Directional derivatives




Def: $f: \mathbb{R}^n \to \mathbb{R}$, $v \in \mathbb{R}^n$.

$$D_v f(x) = \lim_{h \to 0} \frac{f(x+hv) - f(x)}{h}$$

is called directional derivative of $f$ along $v$ at point $x \in \mathbb{R}^n$.

→ generalisation of partial derivative $D_{(1,0)} f = \partial_{x_1} f$

### Evaluating directional derivatives:

1. Using differential quotient $D_v f(x) = \lim_{h \to 0} \frac{f(x+hv) - f(x)}{h}$ → MC 6.2

2. Define $\varphi: [-\delta, \delta] \to \mathbb{R}$, $t \longmapsto \varphi(t) = f(x + t \cdot v)$ and derive w.r.t $t$!

$$\varphi'(t)\big|_{t=0} = \varphi'(0) = D_v f(x)$$ → Ex 6.2

(3. Using $D_{\vec{v}} f(x_0) = \vec{v} \cdot \underbrace{\nabla f(x_0)}_{\text{gradient of } f}$ ) for the sake of completeness

$\vec{v} \cdot \nabla f(x_0) = v_1 \cdot \partial_1 f(x_0) + v_2 \cdot \partial_2 f(x_0) + \dots + v_n \cdot \partial_n f(x_0)$

Example:

1. $f(x,y) = \begin{cases} \frac{x^2 y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & \text{else} \end{cases}$ $\quad (x,y) = (0,0) \quad v = (1,1),\ w = (2,0)$

• Using differential quotient $\quad \vec{x} = (x,y) \rightarrow \vec{x} + h \cdot \vec{v} = \begin{pmatrix} x+h \\ y+h \end{pmatrix} \underset{\uparrow}{=} \begin{pmatrix} h \\ h \end{pmatrix}$

$(x,y) = (0,0)$

$$D_v f(0,0) = \lim_{h \to 0} \frac{f(\vec{x} + h \cdot \vec{v}) - f(\vec{x})}{h}$$

$$= \lim_{h \to 0} \frac{\frac{(0+h)^2 \cdot (0+h)^2}{(0+h)^2 + (0+h)^2} - 0}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h^4}{2h^2}}{h} = \lim_{h \to 0} \frac{h}{2} = 0$$

• Using $\varphi(t) = f(x + t \cdot v) \rightarrow$ way too much work!

2. $f(x,y) = \sin(x) \cdot y \qquad (x,y) = (\pi, 1)\ ,\ v = (1,1)$

• Using differential quotient (do this yourself)

• Using $\varphi(t) = f(\pi + t \cdot v_1, 1 + t \cdot v_2) = \sin(\pi + t \cdot v_1) \cdot (1 + t \cdot v_1)$

$$\varphi'(t) = v_1 \cdot \cos(\pi + t \cdot v_1)(1 + t \cdot v_2) + v_2 \cdot \sin(\pi + t \cdot v_2)$$

$$= \cos(\pi + t) \cdot (1 + t) + \sin(\pi + t)$$

$$\varphi'(t=0) = \cos(\pi) \cdot 1 + \sin(\pi) = -1 \quad = D_{(1,1)} f(\pi, 1)$$

Exercise sheet

MC 6.1 (sets open/closed/bound/compact) similar to Ex 5.3

Use definitions & proposition

Prop

$f: \mathbb{R}^n \to \mathbb{R}^m$ cont. For any open/closed set $Y \subset \mathbb{R}^m$, $f^{-1}(Y) = \{x \in \mathbb{R}^n \mid f(x) \in Y\} \subset \mathbb{R}^n$

is open/closed, as well.

MC 6.2 (directional derivatives sufficient for continuity?)

Think of examples of the lecture & determine the expressions for the directional derivatives (using what we discussed)!

Ex 6.1 (calculating partial derivatives)

The domain of a function is the set of all inputs of the function

$$f: \underbrace{X}_{\text{domain}} \subset \mathbb{R}^2 \to \mathbb{R}$$

c) $x^y = \exp(y \cdot \ln(x))$

Ex 6.2 (calculate directional derivatives)

Use what we discussed!

Ex 6.3 (partial derivatives $\frac{\partial^2}{\partial x \partial y}$, $\frac{\partial^2}{\partial y \partial x}$)

b) Case distinction: 1. $(x,y) \neq (0,0)$ $\partial_x f(x,y) = \dots$ $\partial_y f(x,y) = \dots$

$\partial_x \partial_y f(x,y) = \ldots \qquad \partial_y \partial_x f(x,y) = \ldots$

2. $(x,y) = (0,0)$ $\qquad \partial_x f(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$

from 1. !

$$\partial_y \partial_x f(0,0) = \lim_{h \to 0} \frac{\partial_x f(h,0) - \partial_x f(0,0)}{h}$$