Class 7

Saturday, 31 October 2020 12:21

Short summary:
www.overleaf.com/read/tsstbfjkzbgq

Corrected exercise sheets:

Please download your exercise sheets that I corrected as webbrowsers like chrome/edge cannot show all comments sometimes!

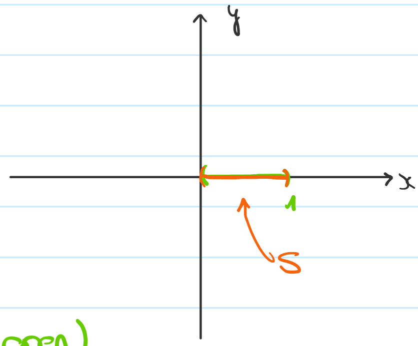
Rather, look at them with adobe acrobat for example! (it works for me with that, at least)

"Set theory"

Last class there were some questions about how we can prove that sets are open/closed. I made a little collection of definitions, propositions and procedures on my summary, please have a look at that if you are interested.

Old exercise sheet

MC 6.1



b) $S = (0,1) \times \{0\} \subset \mathbb{R}^2$

Neither closed nor open!

- It does not contain any **ball** in $\mathbb{R}^2$! (not open)
- **sequence** $(1-\frac{1}{n}, 0) \in S$, but $\lim_{n\to\infty} (1-\frac{1}{n}, 0) = (1,0) \notin S$! (not closed)

d) $S = \{(x, \sin(\frac{1}{x})) \mid x \in (0,2]\} \cup \{0\} \times [-1,1] \subset \mathbb{R}^2$

This set is closed, bounded and hence compact! **For proving that, let**

(x_n, y_n) be a sequence in S that is converging to $(x,y) \in \mathbb{R}^2$.

- $0 < x \leq 2$: $(x_n, y_n) = (x_n, \sin(\frac{1}{x_n})) \xrightarrow{n \to \infty} (x, \sin(\frac{1}{x})) \in S$
- $\underline{x = 0}$: y_n has to be bounded by -1 and 1 as $|\sin(\frac{1}{x_n})| \leq 1$ $(x_n > 0)$

  and $y_n \in [-1,1]$ for $x_n = 0$. $\Rightarrow (x,y) \in S$.

Ex 6.3

b) $f(x,y) = \begin{cases} \frac{xy}{2} \cdot \frac{x^2 - y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

- $\boxed{(x,y) \neq (0,0)}$ Simplify f → split into two summands

$$f(x,y) = \frac{xy}{2} \cdot \underbrace{\frac{x^2+y^2-2y^2}{x^2+y^2}}_{=1} = \frac{xy}{2} - \frac{xy^3}{x^2+y^2}$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{y}{2} - \frac{y^5 - x^2 y^3}{(x^2+y^2)^2} \quad (*)$$

$$\frac{\partial f}{\partial y}(x,y) = \ldots$$

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} = \ldots = \frac{\partial}{\partial x}\frac{\partial f}{\partial y}$$

- $\boxed{(x,y) = (0,0)}$ use differential quotients!

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0 \quad (\text{as } f(h,0) = 0 \ \forall h \neq 0)$$

$$\frac{\partial f}{\partial y}(0,0) = \ldots = 0$$

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{\partial_x f(0,h) - \partial_x f(0,0)}{h} \quad \Big| \quad \partial_x f(0,h) = \frac{h}{2} - \frac{h^5}{h^4} = -\frac{h}{2} \text{ (from } (*))$$

$$= \lim_{h \to 0} \frac{-\frac{h}{2} - 0}{h} = -\frac{1}{2}$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(0,0) = \ldots = \frac{1}{2} \neq \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(0,0) \quad !$$

→ derivatives do not commute as they are not continuous at $(x,y) = (0,0)$!

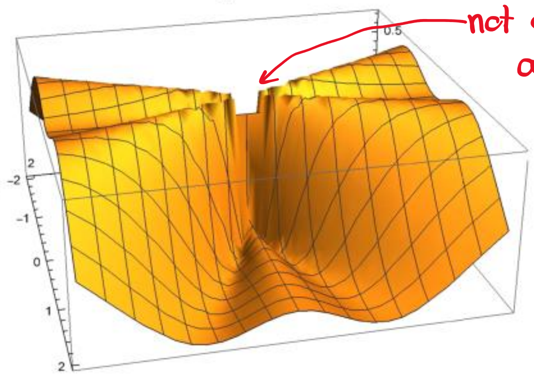
→ derivatives do not commute as they are not continuous at $(x,y)=(0,0)$!

In[25]:= `D[D[x * y / 2 * (x^2 - y^2) / (x^2 + y^2), x], y]`

Out[25]= $\frac{4x^2y^2(x^2-y^2)}{(x^2+y^2)^3} - \frac{x^2(x^2-y^2)}{(x^2+y^2)^2} - \frac{y^2(x^2-y^2)}{(x^2+y^2)^2} + \frac{x^2}{x^2+y^2} - \frac{y^2}{x^2+y^2} + \frac{x^2-y^2}{2(x^2+y^2)}$

In[26]:= Plot3D$\left[\frac{4x^2y^2(x^2-y^2)}{(x^2+y^2)^3} - \frac{x^2(x^2-y^2)}{(x^2+y^2)^2} - \frac{y^2(x^2-y^2)}{(x^2+y^2)^2} + \frac{x^2}{x^2+y^2} - \frac{y^2}{x^2+y^2} + \frac{x^2-y^2}{2(x^2+y^2)},\right.$
`{x, -2, 2}, {y, -2, 2}]`

Out[26]=



not continuous at (0,0)

Note:

Schwarz's theorem:

$\partial_x \partial_y f = \partial_y \partial_x f$

if $f: X \to \mathbb{R}^m$ with $X \subset \mathbb{R}^n$ open

& $\partial_x \partial_y$, $\partial_y \partial_x$ are continuous.

Printed by Wolfram Mathematica Student Edition

Revision: The differential

Motivation:

- We want to provide a way to approximate our functions by a linear map as we did in the 1-dim. case!

- Furthermore, partial (and directional) derivatives are not the correct analogue to differentiability (as we have seen in MC 6.2) as their existence do not imply continuity!

⇒ What we need is the differential!

Def: $X \subset \mathbb{R}^n$ open, $f: X \to \mathbb{R}^m$ a function. Let $u: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, $x_0 \in X$.
We say that f is differentiable at x_0 with differential u if

$$\lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{f(x) - f(x_0) - u(x - x_0)}{\| x - x_0 \|} = 0$$

and denote $df(x_0) = u$.

→ $df(x_0)$ is a linear map (has a matrix representation) and can be different for every point $x_0 \in X$!

Question: Is that the right analogue to differentiability? How would we calculate the differential?

Theorem

$X \subset \mathbb{R}^n$ open, $f: X \to \mathbb{R}^m$ differentiable at x_0. Then

1. f is continuous at x_0!

2. f has all partial derivatives at x_0 and the differential of f at x_0 is given by

$$df(x_0): \mathbb{R}^n \to \mathbb{R}^m, \quad x \mapsto Ax$$

where A is the Jacobi matrix of f at x_0, i.e.

where A is the Jacobi matrix of f at x_0, i.e.

$$A = J_f(x_0) = \left(\frac{\partial f_i}{\partial x_j}(x_0) \right)_{\substack{1 \le j \le m \\ 1 \le i \le n}}$$

Example: 1. $f(x,y) = x \cdot y$

$df(x,y) = (\partial_x f \quad \partial_y f) = (y \quad x)$ $\to$ f is differentiable

2. $f(x,y) = \begin{pmatrix} \cos x \\ \sin y \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$

$df(x,y) = \begin{pmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{pmatrix} = \begin{pmatrix} -\sin(x) & 0 \\ 0 & \cos(y) \end{pmatrix}$ $\to$ f is diff.

Answer: Yes! This is the correct analogue of differentiability (as it implies continuity and provides means to linearly approximate f)! We determine it by calculating the Jacobi matrix of f.

Question: How do wee see that f is differentiable?

Theorem

$X \subset \mathbb{R}^n$ open, $f: X \to \mathbb{R}^m$. If f has all partial derivatives $\partial_j f_i : X \to \mathbb{R}^m$ and they are continuous on X, then f is differentiable on X!

Note: It is not sufficient that the partial (or directional) derivatives exist!

Answer: The partial derivatives must exist AND be continuous for the function to be differentiable!

Question: Now, how do we determine the differential of composed functions? Is there an analogue to the chain rule in single variable calculus?

Theorem (Chain rule)

$X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ open, $f: X \to Y$, $g: Y \to \mathbb{R}^p$ differentiable functions. Then $g \circ f : X \to \mathbb{R}^p$ is differentiable on X with the differential

$$d(g \circ f)(x_0) = dg(f(x_0)) \circ df(x_0) \qquad \forall x_0 \in X.$$

The Jacobi matrix satisfies

$$J_{g \circ f}(x_0) = J_g(f(x_0)) \cdot J_f(x_0).$$

matrix product

Remark: Visualisation

$$X \xrightarrow{f} Y \xrightarrow{g} \mathbb{R}^p,$$

$$x_0 \longmapsto y_0 = f(x_0) \longmapsto g(f(x_0)) = g(y_0)$$

$$\mathbb{R}^n \xrightarrow{df(x_0)} \mathbb{R}^m \xrightarrow{dg(f(x_0))} \mathbb{R}^p$$

$$x \longmapsto y = A \cdot x \longmapsto B \cdot y = B \cdot A x$$

where $A = J_f(x_0)$ & $B = J_g(f(x_0))$.

Examples:

1. $f(x,y) = x^2 + y^2$, $g(x) = \exp(x)$ $\to h(x,y) \equiv (g \circ f)(x,y) = \exp(x^2+y^2)$

$df(x,y) = \begin{pmatrix} 2x & 2y \end{pmatrix}$ $\quad dg(x) = (\exp(x))$

$$d(g\circ f)(x_0) = dg(f(x,y))\circ df(x,y)$$

$$= (\exp(x^2+y^2))\cdot\begin{pmatrix}2x & 2y\end{pmatrix}$$

$$= \begin{pmatrix}2x\cdot\exp(x^2+y^2) & 2y\cdot\exp(x^2+y^2)\end{pmatrix}$$

prove yourself

$$= dh(x,y)$$

2. $f(x,y) = \boxed{xy}$ $\quad g(x) = x^2 \quad \to h(x,y) = (xy)^2$

$$df(x,y) = \begin{pmatrix}y & x\end{pmatrix} \qquad dg(x) = (2x)$$

$$d(g\circ f)(x,y) = dg(f(x,y))\circ df(x,y)$$

matrix mult.

$$= (2xy)\cdot\begin{pmatrix}y & x\end{pmatrix}$$

$$= \begin{pmatrix}2xy^2 & 2x^2y\end{pmatrix}$$

prove yourself

$$= dh(x,y)$$

Answer: Yes! There is a chain rule for multivariable calculus!

Exercise sheet

MC 7.1 (L linear map $\to$ dL known?) $\qquad L(a\cdot x+b) = a\cdot L(x) + L(b)$

Use the definition and remember that $dL(x_0)$ is a linear map itself!

linear map: $\quad L: \mathbb{R}^n \to \mathbb{R}^m,\ x \mapsto Ax,$

$$L\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} = \begin{pmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots\\ a_{m1} & \cdots & a_{mn}\end{pmatrix}\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} = \begin{pmatrix}a_{11}\cdot x_1 + \dots + a_{1n}\cdot x_n\\ \vdots\\ a_{m1}x_1 + \dots + a_{mn}x_n\end{pmatrix}$$

MC 7.2 (det cont. / differentiable?)

Remember how to calculate the determinante!

$$\begin{array}{c} j=1 \\ \begin{pmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots\\ a_{n1} & \cdots & a_{nn}\end{pmatrix}\end{array}$$

$\det A = \sum^n (-1)^{i+j} a_{ij} \det(\hat{A}_{ij})$ for any fixed $i \le n$

Reminder how to calculate the determinant:

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \cdot \det(\underset{\sim}{\hat{A}_{ij}}) \qquad \text{for any fixed } j \leq n \qquad a_{n1} \; \cdots \; a_{nn}$$

matrix a with removed i-th row & j-th column

Show that this is a polynomial. What does that tell us?

Ex 7.1 (Calculating differentials)

Use what we discussed. (Note that you are calculating directional derivatives!)

Ex 7.2

a) · Do not calculate $y(t)$!

· Use product rule $\left(\frac{d}{dt} x^2(t) = 2 \cdot x(t) \cdot \underbrace{\frac{d}{dt} x(t)}_{= \dot{x}(t)}\right)$

b) · Evaluate the integral $f(x,y,z) = \dots$

· Calculate $\partial_x f, \partial_y f$

· $\underbrace{\partial_z f = \lim_{h \to 0} \frac{f(\frac{\pi}{2}, \frac{\pi}{3}, h) - f(\frac{\pi}{2}, \frac{\pi}{3}, 0)}{h}}$ + Taylor expansion or l'Hospital's rule

→ hint in exercise sheet

c) · Case distinction $xy > 0$, $xy = 0$, $xy < 0$

· look at $\{(x,0) \in \mathbb{R}^2 \mid x \neq 0\}$ & $\{(0,y) \in \mathbb{R}^2 \mid y \neq 0\}$ & $x = y = 0$

so $\lim_{y \to 0}$ with x fixed $\qquad \lim_{x \to 0}$ with $y \neq 0$ fixed

· for $x = y = 0$, recall that $|xy| \leq \frac{1}{2}(x^2 + y^2)$ as $0 \leq (x-y)^2 = x^2 - 2xy + y^2$

$\Rightarrow 2xy \leq x^2 + y^2$

$\Rightarrow xy \leq \frac{x^2+y^2}{2}$

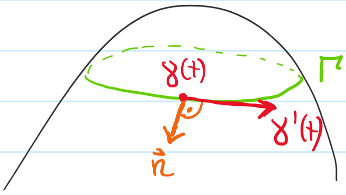
Ex 7.3

a) $\cdot\ f \circ \gamma(t) = ?$ $\rightarrow \frac{d}{dt} \ldots$

b) $D_{\gamma'(t)} f(\gamma(t))$ is the directional derivative along $\gamma'(t)$ at point $\gamma(t)$

$\rightarrow$ use the last theorem of last lecture!

c) $\cdot$ $\gamma'(t)$ points in the direction of Γ (part a))



$\cdot$ every vector $v \in \mathbb{R}^2$ can be written as

$$v = a \cdot \gamma'(t) + b \cdot \vec{n}$$

where $\vec{n} \cdot \gamma'(t) = 0$, namely $\vec{n}$ is orthogonal to Γ!

$\rightarrow$ What should a, b be?