Class 8_2

Friday, 6 November 2020 16:36

Old exercise sheet

Ex 7.1

a) $f(x,y) = y$, $p_0 = (3, \frac{1}{3})$, $v = (-2,-4)$

1. We want to determine the differential of f at point p_0: $df(p_0)$

- We do this by determining the Jacobi matrix of f at point p_0:

$$J_f(p_0) = \left(\frac{\partial f}{\partial x}\boxed{(p_0)}, \frac{\partial f}{\partial y}(p_0)\right) = \underline{\underline{(0 \quad 1)}} \cdot \cancel{\begin{pmatrix} p_{0,1} \\ p_{0,2} \end{pmatrix}}$$

⇝ p_0 is the point at which the partial derivatives are evaluated!!

- The differential of f at point p_0 is therefore given by

$$df(p_0): \mathbb{R}^2 \to \mathbb{R}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto J_f(p_0) \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

⇝ the Jacobi matrix is the representation of the differential!

⇝ When asked for the differential, we sometimes just state the Jacobi matrix!

$$df(p_0) \neq J_f \overset{\text{matrix product}}{\cdot} \begin{pmatrix} p_{0,1} \\ p_{0,2} \end{pmatrix}$$

2. Calculate $df(p_0) \cdot v$ (which is essentially $\underbrace{df(p_0)}_{\text{linear map}}(v)$)

argument/point where to evaluate

- $df(p_0) \cdot v = (0 \quad 1) \cdot \begin{pmatrix} -2 \\ -4 \end{pmatrix} = 0 \cdot (-2) + 1 \cdot (-4) = \underline{\underline{-4}}$

Ex 7.2

b) $f(x,y,z) = \int_{\cos x + \sin y}^{z} e^{tz}\, dt$ ⊛ $\quad df(\frac{\pi}{2}, \frac{\pi}{3}, 0) = ?$

- Evaluate integral $\tilde{f}(x,y,z) \overset{z \neq 0}{=} \frac{1}{z} e^{z^2} - \frac{1}{z} e^{z \cdot (\cos x + \sin y)}$
- $\partial_x \tilde{f}(\frac{\pi}{2}, \frac{\pi}{3}, 0)$, $\partial_y \tilde{f}(\frac{\pi}{2}, \frac{\pi}{3}, 0)$
- 1. $\partial_z f(\frac{\pi}{2}, \frac{\pi}{3}, z) = -\frac{\exp(z^2)}{z^2} + 2e^{z^2} + \frac{\exp(z \cdot a)}{z^2} - \frac{a \cdot \exp(z \cdot a)}{z}$, $a = \frac{\sqrt{3}}{2}$

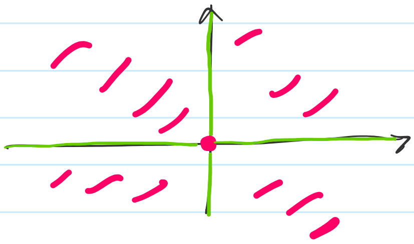
$$\lim_{z\to 0} \partial_z f(\tfrac{\pi}{2},\tfrac{\pi}{3},z) = \dots = \frac{5}{8}$$ (l'Hospital's rule)

2. $$\partial_z f(\tfrac{\pi}{2},\tfrac{\pi}{3},0) = \lim_{h\to 0} \frac{f(\frac{\pi}{2},\frac{\pi}{3},h)-f(\frac{\pi}{2},\frac{\pi}{3},0)}{h} = \dots = \frac{5}{8}$$

Note that you have to use ⊛ to evaluate $f(\frac{\pi}{2},\frac{\pi}{3},0)$!

→ you can also use Leibniz's rule

c) $f(x,y) = |xy|$ $\quad df(x,y) = ?$



- $xy > 0$ → $df(x,y) = (y \;\; x)$
- $xy < 0$ → $df(x,y) = (-y \;\; -x)$
- $xy = 0$ 1. $x \neq 0,\ y = 0$ $$\lim_{h\to 0} \frac{f(x,h)-f(x,0)}{h} = \frac{|x\cdot h|-0}{h} = \frac{|h|}{h}\cdot|x|$$

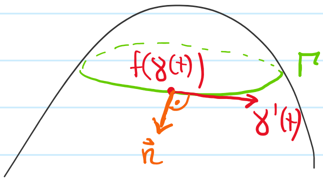
  → no limit!

  2. $y \neq 0,\ x = 0$ (as part 1.)

  3. $x = 0,\ y = 0$: $$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)}{\|(x,y)\|} \overset{\sqrt{x^2+y^2}}{=} 0 \quad \rightarrow df(0,0) = 0$$

Ex 7.3

c) To show: $D_v f(x,y)$ for $(x,y) \in \Gamma$ and $\|v\| = 1$ assumes its maximum if v is orthogonal to Γ.



Proof: Idea: We want to write every vector v as a linear comb. of vectors pointing along Γ $(\gamma'(t))$ and orthogonal to Γ (n) and show that the coefficient of the latter has to be maximised!

Let $v \in \mathbb{R}^2$ arbitrary with $|v| = 1$. Then we can write it as

$$v = a\cdot\gamma'(t) + b\cdot n$$

where $\gamma'(t)$ is the tangent of Γ at point $\gamma(t) = (x,y)$ and n is the where $\gamma'(t)$ is the tangent of Γ at point $\gamma(t) = (x,y)$ and n is the vector orthogonal to $\gamma'(t)$ with $\|n\| = 1$, meaning $\boxed{n \cdot \gamma'(t) = 0.}$

Then we have (using (a))

$$D_v f(\gamma(t)) = J_f(\gamma(t)) \cdot v$$

$$\overset{\boxed{\square}}{=} a \cdot \underbrace{\nabla f(\gamma(t)) \cdot \gamma'(t)}_{=0 \quad (a)} + b \cdot \nabla f(\gamma(t)) \cdot n$$

$\Rightarrow$ a has to be zero as its contribution vanishes!

b has to be ± 1 to maximise the expression.

→ whatever vector we take, its normal component to Γ maximises the gradient! ▨

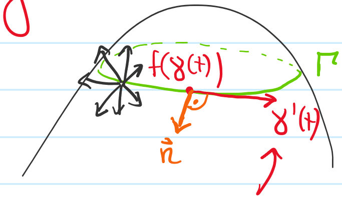
a) $\Gamma = f^{-1}(\{c\})$ $\qquad$ $\gamma((a,b)) = \Gamma$

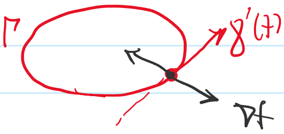
$$\underbrace{\frac{d}{dt} f(\gamma(t))}_{} = \frac{d}{dt} c = 0$$

$$\nabla f(\gamma(t)) \cdot \gamma'(t)$$

$\gamma'(t)$ is pointing along Γ

$$\gamma' \perp v$$
$$\downarrow$$
$$\gamma'(t) \cdot v = 0$$





Revision

- Taylor approximation

functions whose partial derivatives of order k exist & are continuous!

Def: $X \subset \mathbb{R}^n$ open, $x_0 \in X$, $k \geq 1$ $f \in C^k(X;\mathbb{R})$. Then the k-th Taylor polynomial of f at point x_0 is a polynomial in n variables of degree $\leq k$ given by

$$T_k f(y; x_0) = f(x_0) + \underbrace{\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_0) \cdot y_i}_{= df(x_0)\cdot y} + \underbrace{\frac{1}{2} \cdot \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) \cdot y_i y_j}_{= \frac{1}{2} y^T \cdot \mathrm{Hess}_f(x_0) \cdot y} + \dots$$

point to look at

point around which we are approximating

$$+ \sum_{m_1 + \dots + m_n = k} \frac{1}{m_1! \dots m_n!} \cdot \frac{\partial^k f(x_0)}{\partial x_1^{m_1} \dots \partial x_n^{m_n}} y_1^{m_1} \cdot y_2^{m_2} \cdots y_n^{m_n}$$

Question: Can we use these polynomials to approximate f?

Answer: Yes!

Theorem (Taylor approximation)

$X \subset \mathbb{R}^n$ open, $x_0 \in X$, $k \geq 1$, $f \in C^k(X;\mathbb{R})$. Then

$$f(x) = T_k f(\underbrace{x - x_0}_{\uparrow}; x_0) + \underbrace{E_k(f, x, x_0)}_{\text{Error of approximation}}$$

where

$$\lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{E_k(f, x, x_0)}{\| x - x_0 \|^k} = 0 .$$

Rmk: The second order Taylor approximation of a function $f: \mathbb{R}^2 \to \mathbb{R}$ is given by:

$$T_1 f\left[\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}, \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right] = f(x_0, y_0) + df(x_0, y_0) \boxed{\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}} \leftarrow = y$$

$$T_2 f\left[\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}, \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right] = f(x_0, y_0) + df(x_0, y_0) \cdot \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} + \frac{1}{2} \overbrace{(x-x_0 \quad y-y_0)}^{y^T} \cdot \text{Hess}_f(x_0, y_0) \cdot \overset{y}{\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}}$$

where $\text{Hess}_f(x,y) = \begin{pmatrix} \partial_x^2 f & \partial_x \partial_y f \\ \partial_y \partial_x f & \partial_y^2 f \end{pmatrix}$ is the Hessian matrix of f

Examples:

(1) $f(x) = \frac{1}{1-x}$, $x_0 = 0$, $k = 1,2$

- $\frac{df}{dx} = \frac{1}{(1-x)^2}$ $\quad \frac{d^2 f}{dx^2} = \frac{2}{(1-x)^3}$ $\quad \frac{d^2 f}{dx^2}(0) = \frac{2}{(1-0)^2} = 2 \otimes$

- $T_1 f(x; 0) = f(0) + \frac{df}{dx}(0) \cdot x = 1 + \underbrace{\frac{1}{(1-x)^2}\Big|_{x=0}}_{=1} \cdot x$ (affine linear approximation)

  point to look at (at x) — point around which we are approximating (at 0)

- $T_2 f(x; 0) = T_1 f(x; 0) + \frac{1}{2} \underbrace{\frac{d^2 f}{dx^2}(0)}_{=2 \otimes} \cdot x^2$

$= 1 + x + x^2$

$\Rightarrow$ Geometric series: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x|<1$

(2). $g(x,y) = x\cdot y$, $(x_0,y_0) = (1,1)$ $k=1,2$

- $\frac{\partial g}{\partial x} = y \quad \frac{\partial g}{\partial y} = x \quad \rightarrow dg(x,y) = (y \;\; x)$

$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 g}{\partial y \partial x} = 1, \quad \frac{\partial^2 g}{\partial x^2} = \frac{\partial^2 g}{\partial y^2} = 0 \quad \rightarrow \text{Hess}_g(x,y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

- $T_1 g\left[\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] = g(1,1) + dg(1,1)\cdot\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}$ (affine linear approximation)

$= 1 + (1 \;\; 1)\cdot\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}$

$= 1 + (x-1) + (y-1)$

$= \underline{\underline{-1+x+y}}$

- $T_2 g\left[\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] = g(1,1) + dg(1,1)\cdot\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} + \frac{1}{2}\cdot(x-1, y-1)\cdot \text{Hess}_g(1,1)\cdot\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}$

$= -1+x+y + \frac{1}{2}(x-1 \;\; y-1)\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}$

$= -1+x+y + \frac{1}{2}(x-1 \;\; y-1)\cdot\begin{pmatrix} y-1 \\ x-1 \end{pmatrix}$

$= -1+x+y + 2\cdot\frac{1}{2}\cdot(x-1)\cdot(y-1)$

$= \underline{-1}+\underline{x}+\underline{y}+xy\underline{-x}\underline{-y}\underline{+1}$

$= x\cdot y$ $= g(x,y)$! → 2nd order polynomial!

(3). $h(x_1,x_2,x_3) = x_1x_2x_3$, $x_0 = (2,2,2)$

→ • $\partial_1 f = x_2x_3 \quad \partial_2 f = x_1x_3 \quad \partial_3 f = x_1x_2 \quad \rightarrow dh(x_1,x_2,x_3) = (x_2x_3 \;\; x_1x_3 \;\; x_1x_2)$

$\partial_1^2 f = \partial_2^2 f = \partial_3^2 f = 0, \quad \partial_1\partial_2 f = x_3, \quad \partial_1\partial_3 f = x_2, \quad \partial_2\partial_3 f = x_1$

$\rightarrow \text{Hess}_h(x_1,x_2,x_3) = \begin{pmatrix} 0 & x_3 & x_2 \\ x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{pmatrix}$

→ • $T_2 h\left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}\right] = \underbrace{h(2,2,2) + dh(2,2,2)\cdot\begin{pmatrix} x_1-2 \\ x_2-2 \\ x_3-2 \end{pmatrix}}_{\text{affine linear approximation}}$

$+ \frac{1}{2}(x_1-2 \;\; x_2-2 \;\; x_3-2)\cdot \text{Hess}_f(2,2,2)\cdot\begin{pmatrix} x_1-2 \\ x_2-2 \\ x_3-2 \end{pmatrix}$

$$= 2^3 + 2^2 \cdot [(x_1-2)+(x_2-2)+(x_3-2)]$$

$$+ 2 \cdot \sum_{i<j}^{3} (x_i-2)\cdot(x_j-2)$$

Exercise sheet

MC 8.1 (Fct dis- /continuous, diff, C^1)

- differentiability ⇒ continuity (use differential quotient at $(x,y)=(0,0)$)
- Def: $X \subset \mathbb{R}^n$ open, $f: X \to \mathbb{R}^n$ differentiable. We say that f is of class C^1 (or $f \in C^1(X;\mathbb{R}^n)$) if it has all partial derivatives and they are continuous!

→ are the partial derivatives continuous?

MC 8.2 (Critical point, tangent plane)

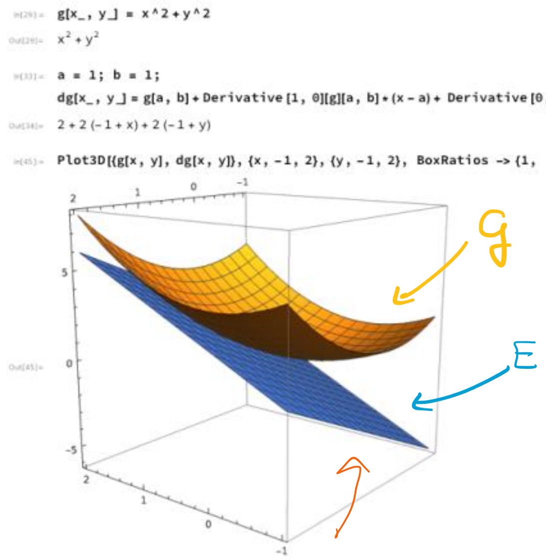
Have a look at the definition of the tangent space. What's the equation of the affine linear approximation?

Ex 8.1 (Fct → tangent plane)

a) $\mathcal{G} = \{(x,y,z) \in \mathbb{R}^3 \mid z = f(x,y)\}$

$E = \{(x,y,z) \in \mathbb{R}^3 / z = ?\}$

→ affine linear approximation



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In[29]:= g[x_, y_] = x^2 + y^2
Out[29]= x² + y²

In[33]:= a = 1; b = 1;
         dg[x_, y_] = g[a, b] + Derivative[1, 0][g][a, b] * (x - a) + Derivative[0, 1][g][a, b] * (y - b)
Out[34]= 2 + 2 (-1 + x) + 2 (-1 + y)

In[45]:= Plot3D[{g[x, y], dg[x, y]}, {x, -1, 2}, {y, -1, 2}, BoxRatios -> {1,  1,  1}]
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b) Use mathematica (like this plot)

or other programs like pgfplots, octave, sagemath

c) Whats the condition for $df(x,y)$ such that the tangent plane is

c) Whats the condition for at (x,y) such that the tangent plane is parallel to the x-y-plane?

Ex 8.2 (Scd order Taylor approx, error estimation)

a) Use what we discussed!

b) It's all about estimation!

$$R_1 f(x,y) = \frac{1}{2} \underbrace{\frac{\partial^2 f}{\partial x^2}(x_s,y_s)}_{\leq \dots} \underbrace{(x-x_0)^2}_{\leq \dots} + \dots$$

Ex 8.3

a) $\sum_{n=0}^{\infty} b^n = \frac{1}{1-b}$ if $|b|<1$ (Geometric series)

b) $\arctan(t) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1} = t + \dots$

c) $\log(1+t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} t^k$ for $|t|<1$

d) Consider the simplified example (3)