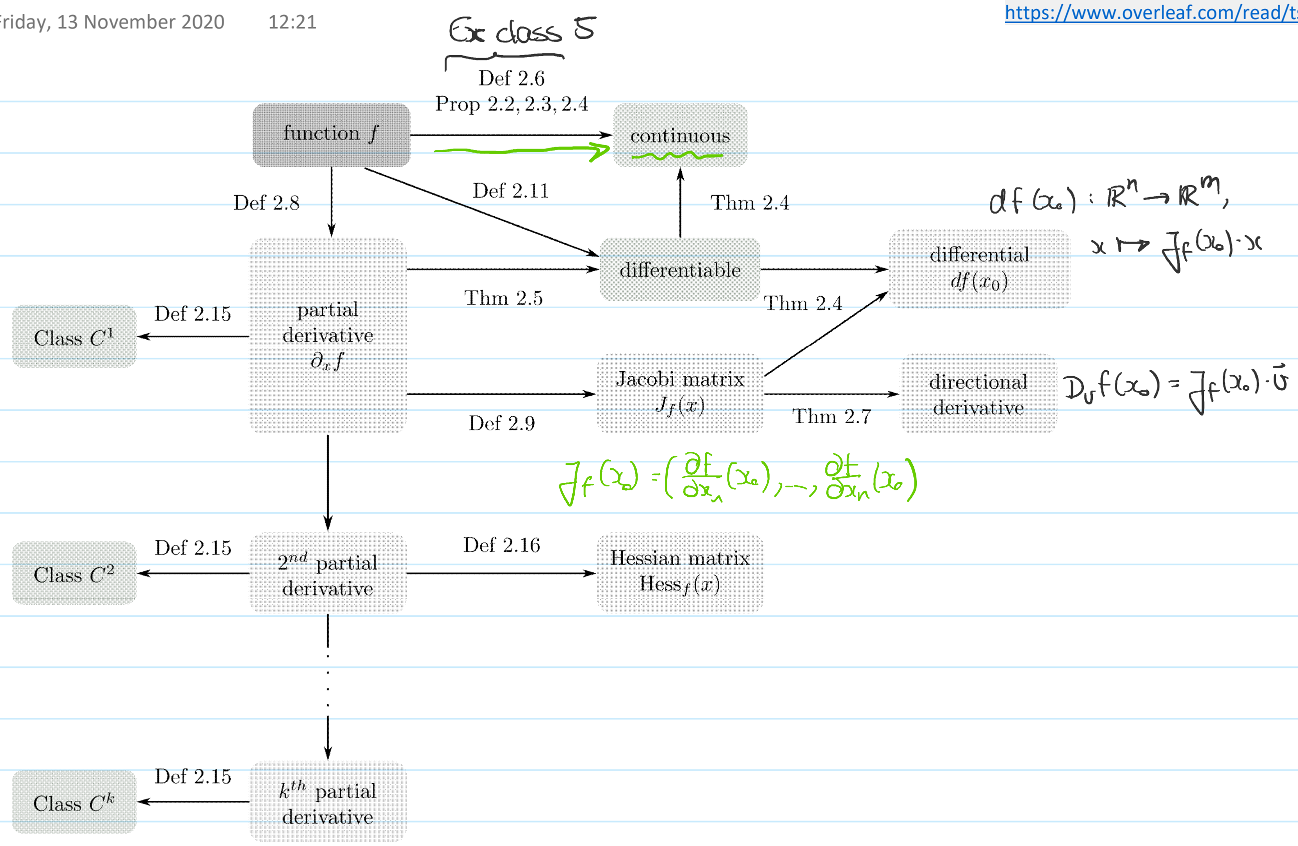
Class 9

Friday, 13 November 2020 12:21

Summary
https://www.overleaf.com/read/tsstbfjkzbgq



Old exercise sheet

MC 8.1

We show that f is differentiable at $(0,0)$ using the differential quotient

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)}{\|(x,y)\|} = 0 \quad \to df(0,0) = 0$$

This implies that f is also continuous at $(0,0)$!

To see that $f \notin C^1$, we have to calculate $\partial_x f, \partial_y f$ and show that they are not continuous at $(0,0)$.

Ex 8.1

$f(x,y) = e^{-(x^2+y^2-2x+3y+2)}$

affine linear approx $\Leftrightarrow T_1 f\left(\begin{pmatrix} x-0 \\ y-0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$

• Tangent plane $E = \left\{ (x,y,A(x,y)) \in \mathbb{R}^3 \mid A(x,y) = f(0,0) + df(0,0) \cdot \begin{pmatrix} x-0 \\ y-0 \end{pmatrix} \right\}$

In[1]:= `g[x_, y_] = Exp[- (2 - 2 x + x^2 + 3 y + y^2)]`

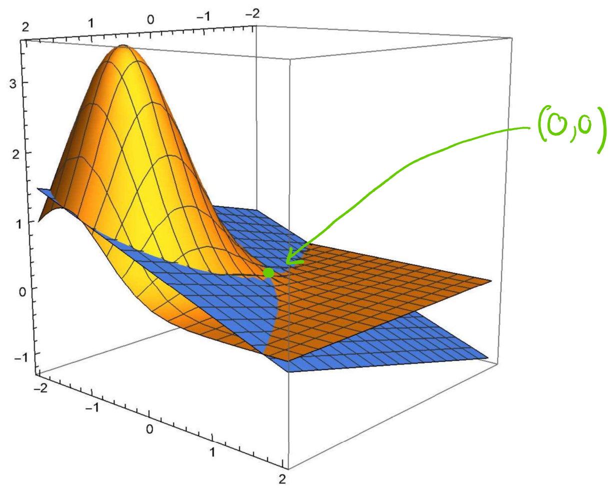
Out[1]= $e^{-2+2x-x^2-3y-y^2}$

In[2]:= `g[0, 0] + Derivative[1, 0][g][0, 0] * x + Derivative[0, 1][g][0, 0] * y`

Out[2]= $\frac{1}{e^2} + \frac{2x}{e^2} - \frac{3y}{e^2}$

In[3]:= `Plot3D[{`$e^{-2+2x-x^2-3y-y^2}$`, `$\frac{1}{e^2} + \frac{2x}{e^2} - \frac{3y}{e^2}$`}, {x, -2, 2}, {y, -2, 2}, BoxRatios → {1, 1, 1}, PlotRange → All]`

Out[3]=



In[4]:=

• parametric form $\varphi(s,t) = (s, t, A(s,t))$

$$= s \cdot \begin{pmatrix} 1 \\ 0 \\ \text{--} \end{pmatrix} + t \cdot \begin{pmatrix} 0 \\ 1 \\ \text{--} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ e^{-2} \end{pmatrix}$$

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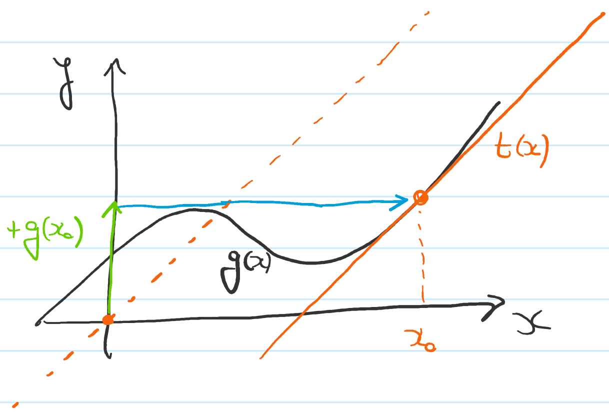
Ex 8.2

$f(x,y) = e^x \cdot \sin(y)$

$\neq \begin{pmatrix}x\\y\end{pmatrix}$!! $\to$ only for $(x_0,y_0)=(0,0)$

a) $T_1 f\left(\underbrace{\begin{pmatrix}x-0\\y-\frac{\pi}{2}\end{pmatrix}}_{v=\binom{x}{y}-\binom{x_0}{y_0}}, \underbrace{\begin{pmatrix}0\\\pi/2\end{pmatrix}}_{\binom{x_0}{y_0}}\right) = f\underbrace{\begin{pmatrix}0\\\pi/2\end{pmatrix}}_{\binom{x_0}{y_0}} + df\underbrace{\begin{pmatrix}0\\\pi/2\end{pmatrix}}_{\binom{x_0}{y_0}}\cdot\underbrace{\begin{pmatrix}x-0\\y-\frac{\pi}{2}\end{pmatrix}}_{v\,!} = \tilde{f}(x,y)$

$T_2 f\left[\underbrace{\begin{pmatrix}x-0\\y-\frac{\pi}{2}\end{pmatrix}}_{v}, \begin{pmatrix}0\\\frac{\pi}{2}\end{pmatrix}\right] = f\underbrace{\begin{pmatrix}0\\\frac{\pi}{2}\end{pmatrix}}_{\binom{x_0}{y_0}} + df\underbrace{\begin{pmatrix}0\\\frac{\pi}{2}\end{pmatrix}}_{\binom{x_0}{y_0}}\cdot\underbrace{\begin{pmatrix}x-0\\y-\frac{\pi}{2}\end{pmatrix}}_{v} + \frac{1}{2}\underbrace{\begin{pmatrix}x-0 & y-\frac{\pi}{2}\end{pmatrix}}_{v^T} \mathrm{Hess}_f\underbrace{\begin{pmatrix}0\\\frac{\pi}{2}\end{pmatrix}}_{\binom{x_0}{y_0}}\cdot\underbrace{\begin{pmatrix}x-0\\y-\frac{\pi}{2}\end{pmatrix}}_{v}$



$t(x) = \underline{g'(x_0)}\,\underbrace{(x-x_0)}_{\text{shift to } x_0} + \underline{g(x_0)}$

Revision : Critical points

Function f

$df(p_0) \overset{!}{=} 0$ (Def 2.18) $\to$ tangent plane is parallel to x-y plane

critical point p_0

$\det(\mathrm{Hess}_f(p_0)) = 0$

degenerate critical point

cannot use Hessian matrix to make a statement :(

$\det(\mathrm{Hess}_f(p_0)) \neq 0$ (Def 2.19)

non-degenerate critical point

"2nd derivative test"

$\mathrm{Hess}_f(p_0) > 0$ pos. def. → local minimum

$\mathrm{Hess}_f(p_0) < 0$ neg. def. → local maximum

Hessian matrix indefinite (Def 2.20, Theorem 2.13) → saddle point

For a function $f: \mathbb{R}^n \to \mathbb{R}$, $x \mapsto f(x)$, the differential at some point $x_0 \in \mathbb{R}^n$

is given by

$$df(x_0) : \mathbb{R}^n \to \mathbb{R}^n,$$

$$x \longmapsto \left(\frac{\partial f}{\partial x_1}(x_0), \frac{\partial f}{\partial x_2}(x_0), \dots, \frac{\partial f}{\partial x_n}(x_0) \right) \cdot x$$

partial derivatives

and the Hessian matrix of f at the same point is

$$\mathrm{Hess}_f(x_0) = \begin{pmatrix} \partial^2_{x_1} f & \partial_{x_1}\partial_{x_2} f & \dots & \partial_{x_1}\partial_{x_n} f \\ \vdots & & \ddots & \vdots \\ \partial_{x_n}\partial_{x_1} f & \dots & & \partial^2_{x_n} f \end{pmatrix}$$

Examples:

1. $f(x,y) = x^2 \pm y^2$ $\quad (x^2 + y^2,\ x^2 - y^2)$

i) Differential: $\quad \partial_x f(x,y) = 2x \qquad \partial_y f(x,y) = \pm 2y$

$\to df(x,y) = (2x \ \ \pm 2y) \quad (J_f(x,y))$

ii) Critical point: $df(x_0,y_0) \overset{!}{=} 0 \ \to \ (x_0,y_0) = (0,0) \ \to df(0,0) = (0,0)$

iii) Hessian matrix: $\quad \partial_x^2 f(x,y) = 2, \qquad \partial_x\partial_y f(x,y) = 0$

$\partial_y^2 f(x,y) = \pm 2, \qquad \partial_y\partial_x f(x,y) = 0$

$$\to \mathrm{Hess}_f(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & \pm 2 \end{pmatrix}$$

iv) $\det(\mathrm{Hess}_f(0,0)) = \pm 4 \neq 0$!

$\to$ Hessian matrix is positive definite (indefinite)

as it has only positive eigenvalues (both pos. & negative eigenvalues)

v) $\to$ critical point is a local minimum (saddle point)

v) → critical point is a local minimum (saddle point)

Def: A matrix $A \in M_{n \times n}(\mathbb{R})$ is called

- positive definite if the following equivalent conditions hold:

  1. $x^T \cdot A \cdot x > 0 \quad \forall x \in \mathbb{R}^n$

  2. A has only positive eigenvalues

  3. the determinant of the leading principal minors A_i of A are positive

     $\det(A_i) > 0 \quad \forall i \leq n$

$$\begin{array}{c} \quad A_1 \quad A_2 \quad A_k \\ A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix} \end{array}$$

- negative definite if the following equivalent conditions hold:

  1. $x^T \cdot A \cdot x < 0 \quad \forall x \in \mathbb{R}^n$

  2. A has only negative eigenvalues

  3. the determinant of the leading principal minors $-A_i$ of $-A$ are positive

     $\det(-A_i) > 0$

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2. $g(x,y) = x^2 - 2xy$

i) Differential: $dg(x,y) = \begin{pmatrix} 2x-2y & -2x \end{pmatrix}$

ii) Critical points $(x_0, y_0) = (0,0)$

iii) Hessian matrix $\text{Hess}_g(x,y) = \begin{pmatrix} 2 & -2 \\ -2 & 0 \end{pmatrix}$

exactly one negative & one pos. eigenvector! (2x2 matrix)

iv) $\det(\text{Hess}_g(0,0)) = -4 < 0$ → Hessian matrix indefinite!

exactly one negative & one pos. eigenvector! (2x2 matrix)

iv) $\det(\text{Hess}_g(0,0)) = -4 < 0$ → Hessian matrix indefinite!

v) Critical point is a saddle point!

3. Degenerate critical point which is a local minimum/maximum

$$m_{\pm}(x,y) = \pm (x^2 + y^4)$$

4. Degenerate critical point which is a saddle point

$$s(x,y) = x^2 + y^3$$

Exercise sheet

MC 9.1 (Hessian matrix → critical point?)

- have a look at the restriction $\varphi(y,z) = f(x_0,y,z)$. What can you say about the critical point (y_0,z_0) of this function using $\text{Hess}_{\varphi}(y_0,z_0)$?
- find examples or show some reasoning

MC 9.2 (loc min/max on a square)

→ Theorem 2.12 (on my summary)

→ Theorem 2.13 (+ remark)

Ex 9.1 (determine critical points, extrema/saddle points)

Use what we discussed (→ recipe)

Ex 9.2

Note: I is the moment of inertia (you should know this from your physics courses)

The centre of mass is by definition the point where the MoI is smallest!

→ determines the torque required to apply the desired angular acceleration

- $x \in \mathbb{R}^n$ & $a_i \in \mathbb{R}^n$, $m_i > 0$
- $|x-a|^2 = ((x_1-a_1)^2 + \ldots + (x_n-a_n)^2)$
- solution should look like $C = \frac{\sum \ldots}{\sum \ldots}$

Ex 9.3 (function attains global max/min) $f(x) = \frac{a \cdot x}{|x|^2+1}$

a)

- whats the limit $\lim_{x \to \pm\infty} |f(x)| = ?$
- Can the function diverge?

b) (this is sort of hard)

- Don't try to determine the gradient, this will become very messy and you'll get a system of equations that's not easy to solve!
- use $x = \overset{\in \mathbb{R}}{t} \cdot a + u$ (with $a \cdot u = 0$) and determine an upper bound for f in the half space $H_p = \{x \in \mathbb{R}^n \mid x \cdot a \geq 0\}$

→ Show that the upper bound is reached for $u = 0$ and $\frac{2}{t \cdot |a| + \frac{1}{\ldots}} = 1$

→ Show that the upper bound is reached for $u=0$ and $\frac{2}{t\cdot|a| + \frac{1}{t\cdot|a|}} = 1$

- use $x = -t\cdot a + u$ and determine a lower bound for f in the half space

  $H_n = \{x \in \mathbb{R}^n \mid x\cdot a < 0\}$

  ...