Class 10

Friday, 20 November 2020 16:15

## Evaluation of my teaching activity

Thanks for all of you that gave me feedback on my teaching!

Regarding the corrections of the exercise sheet (and of all other things as well, of course): please give some feedback on how I can improve the correction. A mere mark does not help me much (unless it's already good as it is).

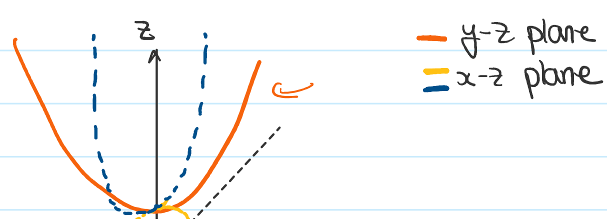
## Old exercise sheet

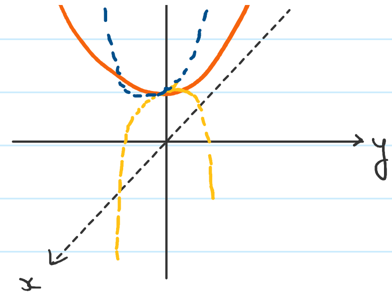
### MC 9.1

$$\mathrm{Hess}_f(p_0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad p_0 = (x_0, y_0, z_0)$$

- $\det(\mathrm{Hess}_f(p_0)) = 0$ $\rightarrow$ cannot apply $2^{nd}$ derivative test
- we can show that $f$ cannot be a local maximum ($\rightarrow$(a) is true):
  - restricting $f$ to $x = x_0$, we define $\varphi(y,z) = f(x_0,y,z)$
  - then we have $\mathrm{Hess}_\varphi(y_0,z_0) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $\longrightarrow$ $(y_0,z_0)$ is a local min of $\varphi$

$\Rightarrow$ $p_0$ cannot be a local max as the restriction $\varphi$ has a local min at $(y_0, z_0)$






Ex 9.1

a) Determining whether $A \in M_{2\times2}(\mathbb{R})$ is pos/neg. definite or indefinite, ...

— $A = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$ is positive definite   ($A_1$ = 6)

- leading principal minors:

$$A = \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_k & A_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix} \end{array}$$

  - $\det(A_1) = \det(6) = 6 > 0$

  - $\det(A_2) = \det(A) = 27 > 0$

  $\Rightarrow \det(A_i) > 0 \quad \Rightarrow A$ is pos. def.

- eigenvalues: $\lambda_1 = 3$ corresponding to $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

  $\lambda_2 = 9$ " $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

  $\Rightarrow$ only positive eigenvalues $\rightarrow$ $A$ is pos. def.

  (this is showing that $x^T A x > 0 \quad \forall x \in \mathbb{R}^2$ )

  $(a\cdot v_1 + b\cdot v_2)^T A\, (a\cdot v_1 + b\cdot v_2)$   $v_1^T A v_1 > 0$  $v_2^T A v_2 > 0$

— $B = \begin{pmatrix} \boxed{0} & -3 \\ -3 & 0 \end{pmatrix}$ is indefinite   ($B_1$)

- leading principal minors

- $\det(B_1) = 0$

$\Rightarrow$ neither pos/neg. definite as $\det(B_1) = B_1 = 0$ !

- $\det(B_2) = -9 < 0$

$\Rightarrow$ as $\det(B) < 0$ this means that it has to have exactly one positive and one negative eigenvalue (as it's a 2 by 2 matrix)
$\rightarrow$ only two eigenvalues

$\Rightarrow$ B is indefinite

- eigenvalues $\lambda_1 = -3$ corresponding to $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 3$ " $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow$ both pos. & negative eigenvalues $\Rightarrow$ B indefinite

$\rightarrow$ (this is showing that $x^T B x$ can take any value in $\mathbb{R}$)

(showing that $x^T B x = 0$ for one $x \in \mathbb{R}^2$ does not show that B is indefinite!!)

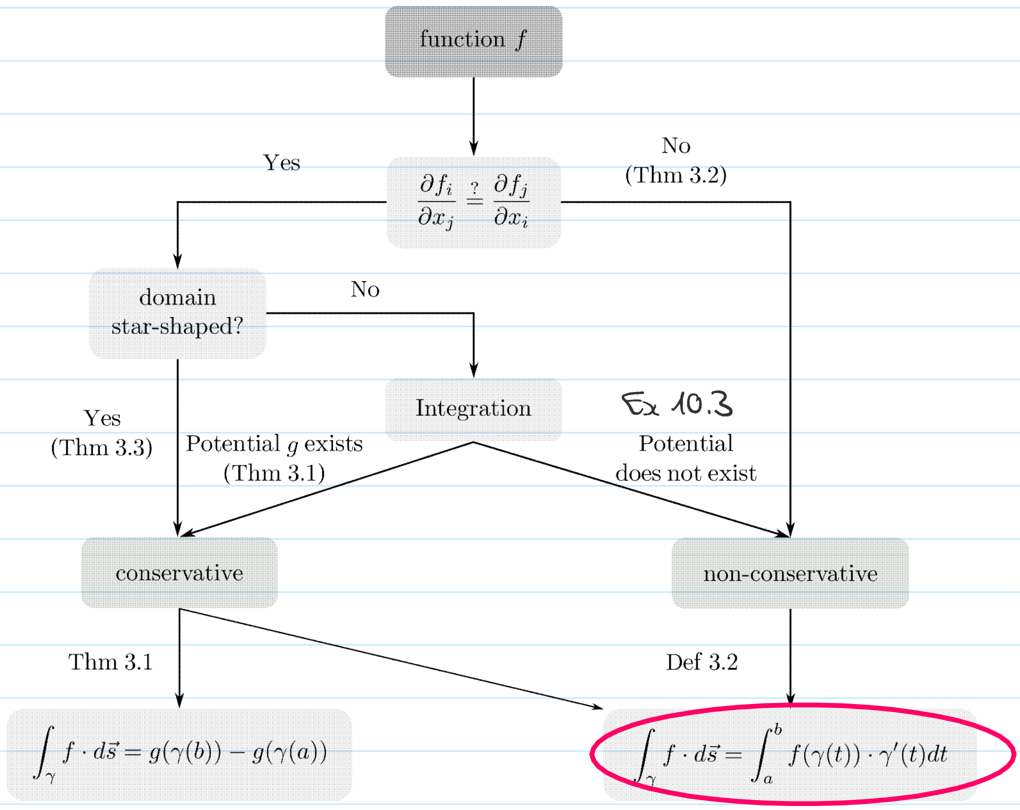
- If a matrix is pos (neg) semi-definite $(x^T A x \geq 0 \ \forall x \in \mathbb{R}^n)$ then the extrema cannot be maxima $\rightarrow$ MC 9.1

$$\begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 2 \end{pmatrix}$$

- If $B_{11} = 0 \Rightarrow$ B cannot be neg/pos. definite

it still can be pos/neg. semi-definite or indefinite

## Revision : line integrals



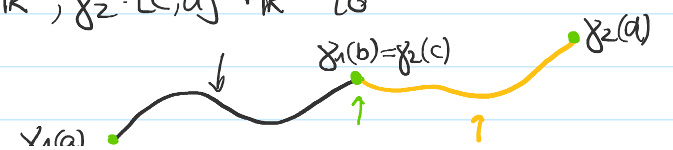

### Def. 3.2

Let $\gamma : [a,b] \to \mathbb{R}^n$ parametrisation of a curve, $X \subset \mathbb{R}^n$ subset which contains image of $\gamma$, $f : X \to \mathbb{R}^n$ continuous map. Then the line integral of $f$ along $\gamma$ is given by

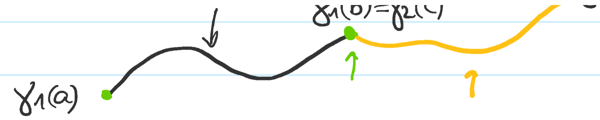
$$\int_\gamma f(s) \cdot d\vec{s} = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt$$

### Properties:

- you can connect paths $\gamma_1 : [a,b] \to \mathbb{R}^2$, $\gamma_2 : [c,d] \to \mathbb{R}^2$ to $\gamma = \gamma_1 + \gamma_2$ if $\gamma_1(b) = \gamma_2(c)$



$\gamma = \gamma_1 + \gamma_2$ if $\gamma_1(b) = \gamma_2(c)$




- the line integral is invariant under <u>orientation preserving parametrisation</u> of the curve (Prop. 3.1)

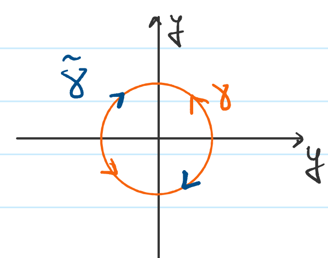
$\gamma : [0, 2\pi] \to \mathbb{R}^2, \; t \mapsto \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$

$h : \mathbb{R} \to \mathbb{R}, \; t \mapsto 2t-1 \qquad h'(t) > 0 \qquad \Rightarrow \int_\gamma d\vec{s} = \int_{\tilde\gamma} d\vec{s}$

$\tilde\gamma = \gamma \circ h : [\tfrac{1}{2}, \pi + \tfrac{1}{2}] \to \mathbb{R}^2, \; t \mapsto \begin{pmatrix} \cos(2t-1) \\ \sin(2t-1) \end{pmatrix}$

- going along the parametrised curve $\gamma$ backwards (orientation changing parametrisation $-\gamma \equiv \gamma(a+b-t)$) leads to the negated value of the line integral (Prop. 3.2) $\quad \int_{-\gamma} f \, d\vec{s} = -\int_\gamma f \, ds$

Example:




1) $f : \mathbb{R}^2 \to \mathbb{R}^2, \; (x,y) \mapsto (-y, x)$ with paths

$\gamma : [0, 2\pi] \to \mathbb{R}^2, \; t \mapsto \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$ and $\tilde\gamma(t) = \begin{pmatrix} \cos(-t) \\ \sin(-t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}$

$\gamma'(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} \qquad \tilde\gamma'(t) = \begin{pmatrix} -\sin(t) \\ -\cos(t) \end{pmatrix}$

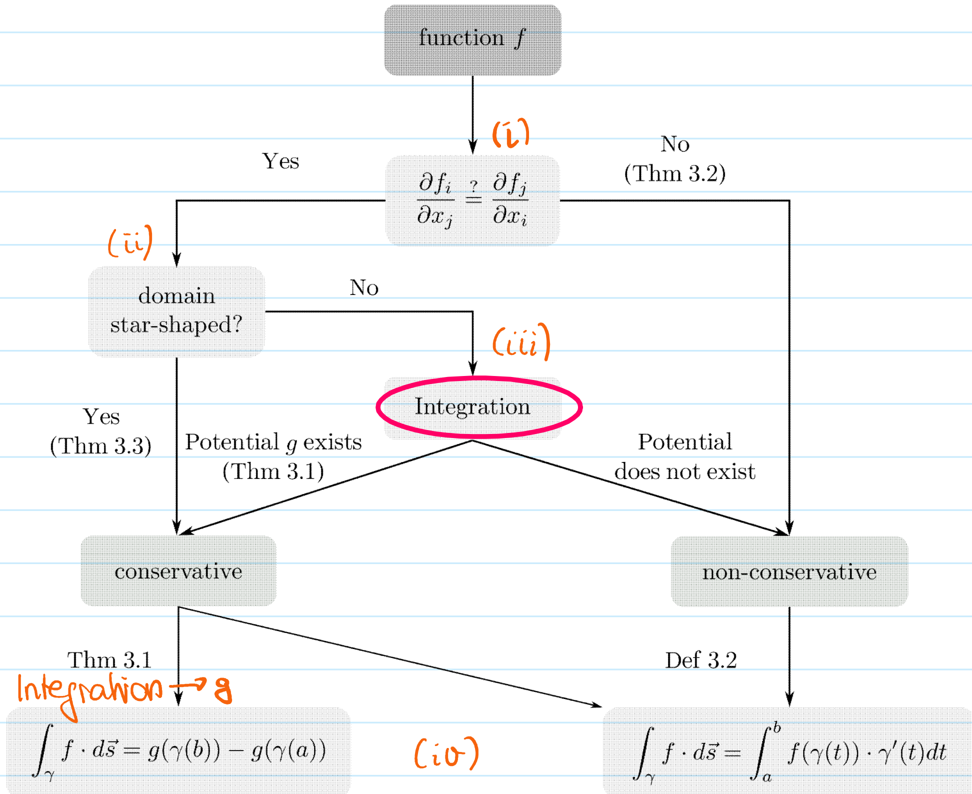
$$\int_\gamma f \cdot d\vec{s} = \int_0^{2\pi} f(\gamma(t)) \cdot \gamma'(t) \, dt = \int_0^{2\pi} \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} \cdot \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} dt = \int_0^{2\pi} \underbrace{\sin^2(t) + \cos^2(t)}_{=1} \, dt = 2\pi$$

$$\int_{\tilde\gamma} f \, d\vec{s} = \int_0^{2\pi} f(\tilde\gamma(t)) \tilde\gamma'(t) \, dt = \int_0^{2\pi} \underbrace{\begin{pmatrix} +\sin(t) \\ \cos(t) \end{pmatrix} \cdot \begin{pmatrix} -\sin(t) \\ -\cos(t) \end{pmatrix}}_{-1} dt = -2\pi$$

$$\Rightarrow \int_{\tilde{\gamma}} f \cdot d\vec{s} = - \int_{\gamma} f \cdot ds \qquad \text{as expected!}$$

## The potential $\quad f : \mathbb{R}^n \to \mathbb{R}^n$

→ this is sort of the integral of multi-variable vector-valued functions!!



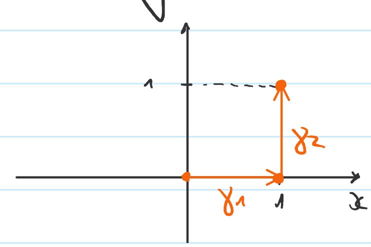

→ note that a function on a domain that is not star-shaped can still be conservative

Example:

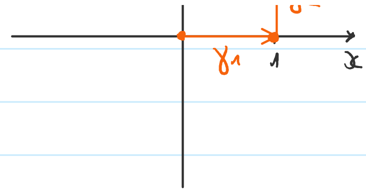
1) Determine the line integral of $f : \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto (2xy, x^2)$ along

$\gamma = \gamma_1 + \gamma_2$ with $\gamma_1 : [0,1] \to \mathbb{R}^2, t \mapsto \begin{pmatrix} t \\ 0 \end{pmatrix}$ and

$\gamma_2 : [0,1] \to \mathbb{R}^2, t \mapsto \begin{pmatrix} 1 \\ t \end{pmatrix}$



$\gamma_2 : [0,1] \to \mathbb{R}^2, \; t \mapsto \binom{1}{t}$




i) mixed partial derivatives $\left(\partial_x f_2 \overset{?}{=} \partial_y f_1\right)$

$\partial_x f_2 = 2x \quad , \quad \partial_y f_1 = 2x \quad \longrightarrow$ they match!

ii) domain ($\mathbb{R}^2$) is star-shaped $\longrightarrow$ the potential exists!

iii) obtaining the vector potential:

$$\nabla g = f \iff \partial_x g = f_1 \; , \; \partial_y g \overset{\otimes}{=} f_2$$

1. $\xrightarrow{\text{Integrate}}$ first condition

$$g(x,y) = \underbrace{\int f_1(x,y)\,dx}_{\text{this includes all terms depending on } x!} + \underbrace{h(y)}_{\text{integration constant}}$$

$$= \int 2xy\,dx + h(y)$$

$$= x^2 y + h(y)$$

$g(x,y) = x^2y + h(y)$

$$\Rightarrow \partial_y g = x^2 + \underbrace{\partial_y h}_{\overset{!}{=}0} \overset{!\,\otimes}{=} x^2$$

$\to$ h is unimportant constant!

$$\Rightarrow g(x,y) = x^2 y + c$$ is the vector potential of $f$!

iv) line integral:

$$\int_\gamma f \cdot ds = g(\gamma(1)) - g(\gamma(0))$$

$$= 1^2 \cdot 1 - 0^2 \cdot 0 = \underline{1}$$

Def. 3.2

$$\int f \cdot d\vec{s} \overset{\downarrow}{=} \int f(s) \cdot d\vec{s} + \int f(s) \cdot d\vec{s}$$

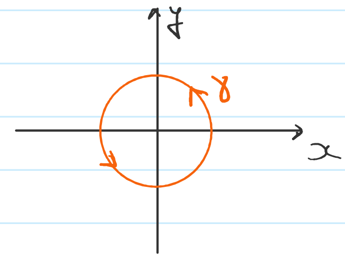
$$\int_\gamma f\cdot d\vec{s} = \int_{\gamma_1} f(s)\cdot d\vec{s} + \int_{\gamma_2} f(s)\cdot d\vec{s}$$

$$= \int_0^1 f(\gamma_1(t))\cdot\gamma_1'(t)\,dt + \int_0^1 f(\gamma_2(t))\cdot\gamma_2'(t)\,dt$$

$$= \int_0^1 \underbrace{\begin{pmatrix}0\\t^2\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}}_{=0}dt + \int_0^1 \underbrace{\begin{pmatrix}2t\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}}_{=1}dt$$

$$= 0 + 1$$

2) $f(x,y,z) = (y\cdot z\cdot e^x,\ z\cdot e^x,\ y\cdot e^x)$ , $\gamma : [0,2\pi]$, $t \mapsto \begin{pmatrix}\cos(t)\\ \sin(t)\\ 0\end{pmatrix}$




i) mixed partial derivatives $\left(\frac{\partial f_i}{\partial x_j} \overset{?}{=} \frac{\partial f_j}{\partial x_i}\right)$

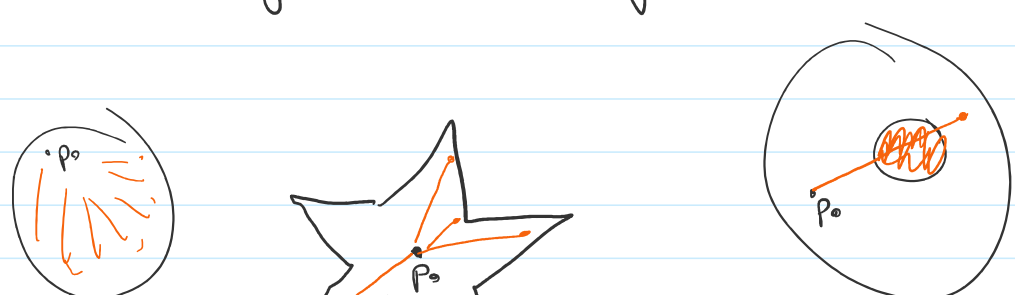
$\frac{\partial f_2}{\partial x} = \partial_x f_2 = z\cdot e^x$ $\quad\quad$ $\partial_y f_1 = z\cdot e^x$

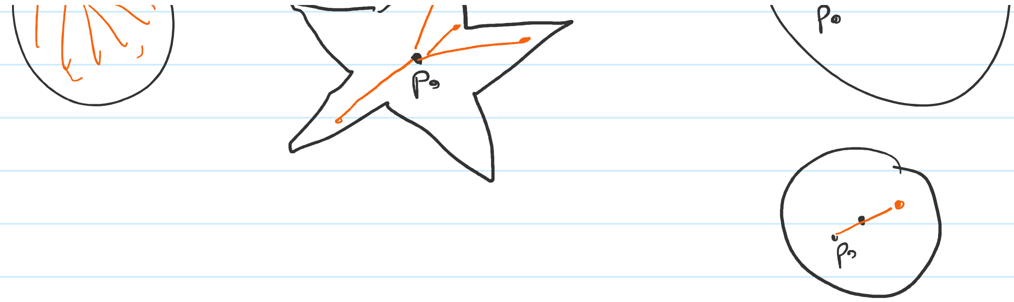
$\partial_x f_3 = y\cdot e^x$ $\quad\quad$ $\partial_z f_1 = y\cdot e^x$ $\quad\quad$ $\rightarrow$ they match!

$\partial_y f_3 = e^x$ $\quad\quad$ $\partial_z f_2 = e^x$ $\quad\quad$ (this is equivalent to saying $\mathrm{curl}(f) = \nabla\times f = 0$)

ii) domain ($\mathbb{R}^3$) is star-shaped! $\rightarrow$ potential exists!

<u>Def</u>: (star-shaped) $X \subset \mathbb{R}^n$ is star-shaped if $\exists p_0 \in X$ s.t. $\forall x \in X$ can be connected to $p_0$ with a line segment ("straight line") & this line segment is in $X$ itself.





iii) Integration

$$\nabla g = f \iff \overset{1.}{\partial_x g = f_1}, \quad \overset{2.}{\partial_y g = f_2}, \quad \overset{3.}{\partial_z g = f_3}$$

$$g(x,y,z) \overset{1.}{=} \underbrace{\int f_1(x,y,z)\,dx}_{\text{all terms dependent on } x} + h(y,z) \quad \leftarrow \text{additional terms dependent on } y,z \text{ only!}$$

$$= y \cdot z \cdot \exp(x) + h(y,z)$$

$$\Rightarrow \partial_y g = z \cdot \exp(x) + \partial_y h(y,z) \overset{!2.}{=} z \cdot \exp(x) \quad \Rightarrow \partial_y h(y,z) = 0$$

$$\partial_z g = y \cdot \exp(x) + \underbrace{\partial_z h(y,z)}_{\overset{!}{=}0} \overset{!3.}{=} y \cdot \exp(x) \quad \Rightarrow \partial_z h(y,z) = 0$$

$$\Rightarrow g(x,y,z) = y \cdot z \cdot \exp(x) + c \quad \text{is the potential of } f(x,y,z)!$$

iv) line integral

As $f$ has a potential, it is conservative and thus all line integral along a closed curve are zero!

$$\oint f \cdot ds \qquad \int_\gamma f \cdot ds = 0 \qquad (\text{sometimes written as } \oint f \, d\vec{s})$$

(you can calculate the line integral using Def. 3.2 to verify this)

3. $f(x,y) = \left( \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) \qquad \gamma : [0, 2\pi] \to (\cos(t), \sin(t), 0)$

3. $f(x,y) = \left( \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$   $\gamma : [0, 2\pi] \to (\cos(t), \sin(t), 0)$

→ this is exercise 10.3   $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$, but f not conservative!

## Exercise sheet

MC 10.1

- three distinct coinciding pairs of matrix elements, e.g.

$$\partial_{x_1} V_1 = \partial_{x_2} V_2, \ \partial_{x_2} V_3 = \partial_{x_2} V_1, \ \partial_{x_1} V_2 = \partial_{x_3} V_3$$

or

$$\partial_i V_j = \partial_j V_i$$

MC 10.2

- symmetric: $A_{ij} = A_{ji}$

Ex 10.1

- Simply use Def. 3.2)

b) look up some geometric identities ( $\sin(x) \cdot \cos(x) = \frac{1}{2} \sin(2x)$, $\sin^2(x) + \cos^2(x) = 1$, ... )

Ex 10.2

Use what we discussed!

Ex 10.3

Use what we discussed :)