Class 11

Friday, 27 November 2020 15:41

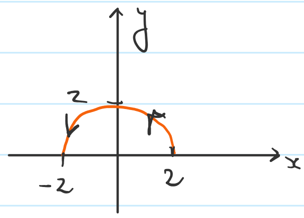
Old exercise sheet

Ex 10.1

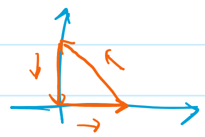
b) Parametrisation of the half circle

$\gamma : [0,\pi] \to \mathbb{R}^2,\ t \mapsto 2 \cdot \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$ ← go for this one!

$\hat{\gamma} : [-2,2] \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} -t \\ \sqrt{4-t^2} \end{pmatrix}$



c) $F(x,y) = \begin{pmatrix} x^2+y^2 \\ x^2-y^2 \end{pmatrix}$ has not potential as $\boxed{\partial_x F_2 \neq \partial_y F_1}$!



→ thus we cannot say that "$\int_\gamma F ds = 0$ because γ is closed".

Rather we have to evaluate the line integral!

Ex 10.2

a) $V(x,y,z) = \begin{pmatrix} 2xy^3 \\ 3x^2y^2 + 2yz \\ y^2 \end{pmatrix}$ has a potential $f(x,y,z) = x^2y^3 + y^2 \cdot z$

We can use the potential to calculate the line integral as

$$\boxed{\int_\gamma V(s) \cdot d\vec{s} = f(\gamma(1)) - f(\gamma(0)) = 12}$$

Ex 10.3

b) You had to calculate $\int_\gamma B \cdot d\vec{s}$ here, using the definition, which gives you $\mu_0 \cdot I \cdot m \neq 0$.

$\Rightarrow$ the potential doesn't exist!!

$B = \frac{\mu_0 I}{2\pi} \cdot \frac{1}{x^2+y^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$ ←

Some of you calculated

$\partial_x g = B_1$

Some of you calculated

$$\hat{g}(x,y,z) = \frac{\mu_0 I}{2\pi} \int -\frac{y}{x^2+y^2}\, dx \qquad (y \neq 0)$$

$\partial_x g = B_1$

$\partial_y g = B_2$

$\partial_z g = B_3 = 0$

$$= -\frac{\mu_0 \cdot I}{2\pi} \cdot \arctan\left(\frac{x}{y}\right) + h(y,z)$$

$$\frac{\partial \tilde{g}}{\partial y} = \frac{\mu_0 I}{2\pi} \cdot \frac{x}{x^2+y^2} + \underbrace{\partial_y h}_{\overset{!}{=} 0} \overset{!}{=} \frac{\mu_0 \cdot I}{2\pi} \cdot \frac{x}{x^2+y^2}$$

$\rightarrow h$ is unimportant constant as $\partial_z h = 0$, as well

The reason why this doesn't work is that the potential cannot be defined for $y=0$! (everywhere else, it just works fine)

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In[1]:= f[x_, y_] = ArcTan[x / y]
        Plot3D[f[x, y], {x, -2, 2}, {y, -2, 2}]
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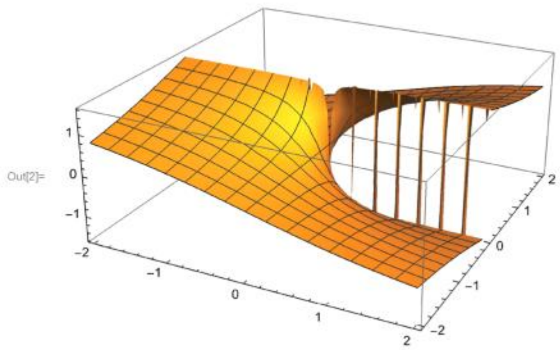
Out[1]= $\text{ArcTan}\left[\frac{x}{y}\right]$

Out[2]=



Note that g is indeed a potential of B for starshaped domains which do not include $y=0$ and $z=0$!

c) As $\oint_\gamma B \cdot d\vec{s} \neq 0$ (so the integral over the closed path γ is not zero) we know that there cannot be a potential on $\mathbb{R}^3 \setminus \{z\text{-axis}\}$

($\rightarrow$ Thm 3.1!)

Revision: Fubini's theorem

Theorem (Fubini)

The order of integration can be swapped

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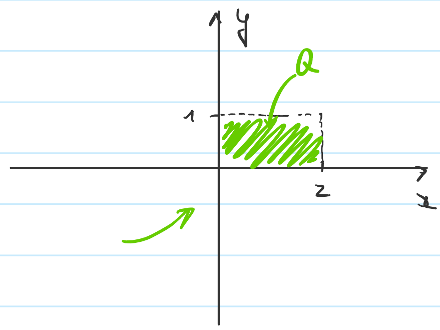
$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x,y)\,dx\,dy = \int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x,y)\,dy\,dx \qquad (\text{for } n=2)$$

if the function is integrable & continuous on its domain.

Example:

$f(x,y) = x\cdot\exp(y)$ on $Q = [0,2]\times[0,1]$.



a) w.r.t x at first

$$\int_0^1\int_0^2 x\cdot\exp(y)\,dx\,dy = \int_0^1 \left[\frac{x^2}{2}\cdot\exp(y)\right]_{x=0}^{2} dy$$

$$= \int_0^1 2\exp(y) = 2e-2$$

b) w.r.t. y first

$$\int_0^2\int_0^1 x\cdot\exp(y)\,dy\,dx = \int_0^2 [x\cdot\exp(y)]_{y=0}^{1}\,dx$$

$$= \int_0^2 (e-1)x\,dx$$

$$= \left[\frac{(e-1)x^2}{2}\right]_0^2 = 2e-2$$

$\circleddash$ =

Double integral over general regions (Corollary 3.6.1)

$$\int_{a(x)}^{b(x)}\int_{c(y)}^{d(y)} f(x,y)\,dx\,dy = \int_{?}^{?}\int_{?}^{?} f(x,y)\,dy\,dx + \dots$$

Recipe: 1.a) Draw the domain (if you are given a domain defined via functions)

1.b) Determine the equation of the functions (if the domain is drawn)

→ Rewrite the equations (if you want to swap order of integration)

$y = f(x) \iff x = g(y)$

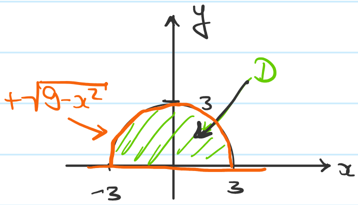
2) Find the intersection points of the functions

3) Write down the integral & evaluate

Examples:

i) $f(x,y) = x + y$ on D, the upper half disk with radius 3.

1. Equation of circle: $y^2 + x^2 = 9$

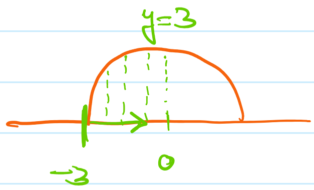


$\Rightarrow y_0 = +\sqrt{9-x^2}, \quad y_1 = 0,$

$x_0 = -3, \quad x_1 = +3$

2. Intersection points: $(-3,0), (3,0)$

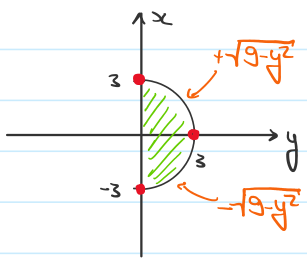
3. $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} x+y \, dy\,dx = \int_{-3}^{3} \left[xy + \frac{1}{2}y^2\right]_0^{\sqrt{9-x^2}} dx$



$\displaystyle = \int_{-3}^{3} x\cdot\sqrt{9-x^2} + \frac{1}{2}(9-x)^2 \, dx$

$= \ldots = \underline{\underline{18}}$

Swap order of integration (first w.r.t x)



1. $x_2 = -\sqrt{9-y^2}, \quad x_3 = +\sqrt{9-y^2}$

$y_2 = 0, \quad y_3 = 3$

2. Intersection point $(0,3), (\pm 3, 0)$
$\quad x \; y$

3. $\displaystyle\int_{0}^{3}\int_{-\sqrt{9-y^2}}^{+\sqrt{9-y^2}} x+y \, dx\,dy = \int_0^3 \left[\frac{1}{2}x^2 + x\cdot y\right]_{-\sqrt{9-y^2}}^{+\sqrt{9-y^2}} dy$

$\displaystyle = \int_0^3 \frac{1}{2}(9-y^2) + y\cdot\sqrt{9-y^2} - \frac{1}{2}(9-y^2) + y\cdot\sqrt{9-y^2} \, dy$

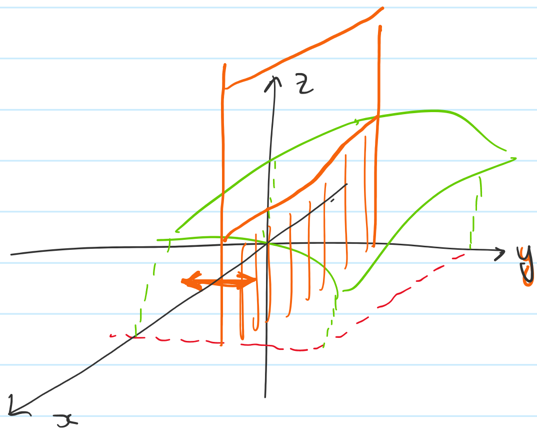
$$= \int_0 \frac{1}{2}(9-y^2)+y\cdot\sqrt{9-y^2} - \frac{1}{2}(9-y^2)+y\cdot\sqrt{9-y^2}\;dy$$

$$= 2\int_0^3 y\cdot\sqrt{9-y^2}\;dy \qquad u = y^2,\quad du = 2y\,dy$$

$$= \int_0^9 \sqrt{9-u}\;du$$

$$= \left[-\frac{2}{3}(9-u)^{\frac{3}{2}}\right]_0^9 = \frac{2}{3}\cdot 9^{\frac{3}{2}} = \underline{\underline{18}}$$

The integration can be visualised as integration in $\mathbb{R}^3$ where we cut our function with planes and integrate our the intersection of our function with this plane:



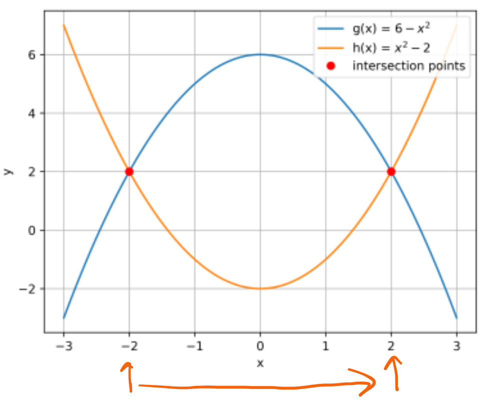
ii) Draw the domain enclosed by $\underline{g(x) = 6-x^2}$ and $h(x) = \underline{x^2-2}$ and give the coordinates of the intersection points!

- **Intersection points:**

$$\underline{\underline{x^2-2 = 6-x^2}}$$

$$\Leftrightarrow\; 2x^2 = 8 \;\rightarrow\; x_{1,2} = \underline{\pm 2}$$

$$(x_1, y_1) = (2,2)\;,\quad (x_2, y_2) = (-2,2)$$



Write down how you would integrate a function $f(x,y)$ over this domain, integrating both w.r.t x and y first:

integrating both w.r.t x and y first:

- first w.r.t y: → this is quite easy as our boundary functions are given in terms of x

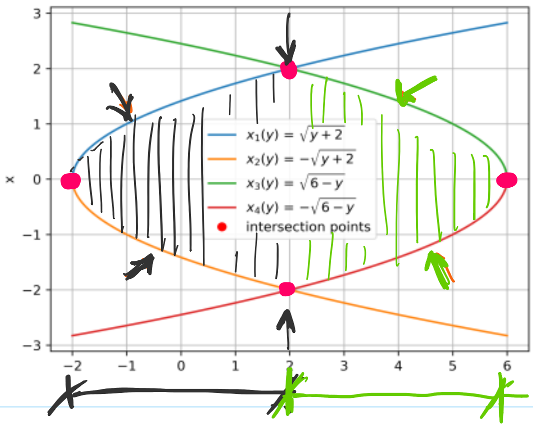
$$\int_{-2}^{2} \int_{x^2-2}^{6-x^2} f(x,y)\,dy\,dx$$

- first w.r.t. x:

1. rewriting the equations:

$$y_1 = x^2 - 2 \iff x_{1,2} = \pm\sqrt{y+2}$$

$$y_2 = 6 - x^2 \iff x_{3,4} = \pm\sqrt{6-y}$$



2. Intersection points:

- $x_1 = x_2 \Rightarrow (x,y) = (0,-2)$
- $x_2 = x_4 \Rightarrow (x,y) = (-2,2)$
- $x_1 = x_3 \Rightarrow (x,y) = (2,2)$
- $x_3 = x_4 \Rightarrow (x,y) = (0,6)$

3. Integral

$$\int_{-2}^{2} \int_{-\sqrt{y+2}}^{\sqrt{y+2}} f(x,y)\,dx\,dy + \int_{2}^{6} \int_{-\sqrt{6-y}}^{+\sqrt{6-y}} f(x,y)\,dx\,dy$$

Exercise sheet

MC 11.1

curl: $h: \mathbb{R}^3 \to \mathbb{R}^3$, $\operatorname{curl}(h) = \nabla \times h = \underbrace{\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \times \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}}_{\text{cross product}} = \boxed{\begin{pmatrix} \partial_2 h_3 - \partial_3 h_2 \\ \partial_3 h_1 - \partial_1 h_3 \\ \partial_1 h_2 - \partial_2 h_1 \end{pmatrix}}$

gradient: $k: \mathbb{R}^3 \to \mathbb{R}$, $\operatorname{grad}(k) = \nabla \cdot k = \begin{pmatrix} \partial_1 k \\ \partial_2 k \\ \partial_3 k \end{pmatrix}$ $\quad \nabla \cdot k : \mathbb{R}^3 \to \mathbb{R}^3$!

MC 11.2

→ Theorem 3.5

Ex 11.1 – 11.3

Use what we discussed!