Class 12_2

Friday, 4 December 2020 15:01

Today we will look at

- change of variables (+ spherical coordinates) sections 2.4 + 3.4
- how to calculate volumes / areas (+ Greens formula, Gauss formula)

(• double integrals → last week)

Old exercise sheet

MC 11.1

Schwarz' theorem

$$\operatorname{curl}(\operatorname{grad}(f)) = \nabla \times (\nabla f) = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \times \begin{pmatrix} \partial_1 f \\ \partial_2 f \\ \partial_3 f \end{pmatrix} = \begin{pmatrix} \boxed{\partial_2\partial_3 f - \partial_3\partial_2 f} \\ \partial_3\partial_1 f - \partial_1\partial_3 f \\ \partial_1\partial_2 f - \partial_2\partial_1 f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

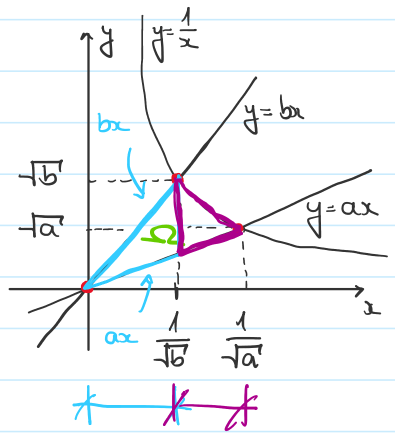
$f : \mathbb{R}^3 \to \mathbb{R}^1$

Ex 11.1

$$\int_\Omega x \cdot y \, d\mu = \int_0^{\frac{1}{\sqrt{b}}} \int_{ax}^{bx} x \cdot y \, dy\, dx + \int_{\frac{1}{\sqrt{b}}}^{\frac{1}{\sqrt{a}}} \int_{ax}^{\frac{1}{x}} x \cdot y \, dy\, dx$$

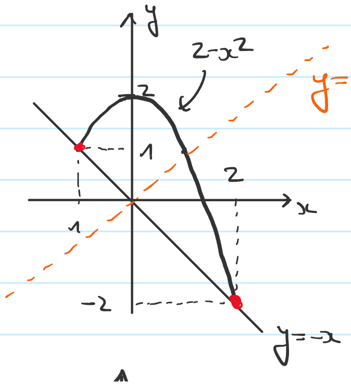
$= \ldots$

in this case, this is the Riemann measure (essentially equal to $dx\,dy$ or $dy\,dx$)

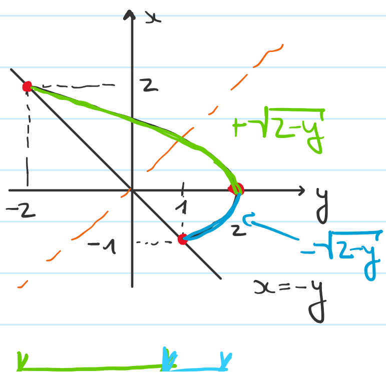


Ex 11.2

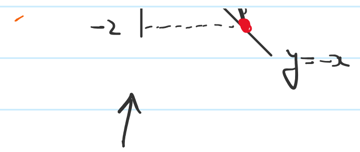
b) $\displaystyle\int_{-1}^{2} \int_{-x}^{2-x^2} f(x,y)\, dy\, dx$

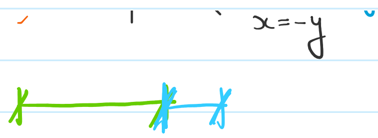


$\Longrightarrow$



Changing order:





Changing order:

1. determine functions

$$y = -x \quad \Rightarrow \quad x = -y$$

$$y = 2 - x^2 \quad \Rightarrow \quad x = \pm\sqrt{2-y}$$

2. Intersection points: $(2,-2), (0,2), (-1,1)$
   (x,y)

3. Integral

$$\int_{-2}^{1}\int_{-y}^{\sqrt{2-y}} f(x,y)\,dx\,dy + \int_{1}^{2}\int_{-\sqrt{2-y}}^{+\sqrt{2-y}} f(x,y)\,dx\,dy$$

Revision: Change of variables

Theorem 3.9 (Change of variables) ("mathematically incorrect")

Let $X, Y \subset \mathbb{R}^n$ be compact and $\varphi : X \to Y$ of class C_1 and bijective. Then for any continuous function f on Y we have

$$\int_Y f(y)\,dy = \int_X f(\varphi(x)) \cdot |\det J_\varphi(x)| \cdot dx$$

Note: the most common changes of variables are

$$\left.\begin{array}{l} \text{1. polar coordinates} \\[2em] \text{2. spherical coordinates} \\[2em] \text{3. cylindrical coordinates} \end{array}\right\} \text{section 2.4!} \quad \begin{array}{l} \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} r\cdot\cos\theta \\ r\cdot\sin\theta \end{pmatrix} \\[1em] \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} r\cdot\sin\varphi\cdot\cos\theta \\ r\cdot\sin\varphi\cdot\sin\theta \\ r\cdot\cos\varphi \end{pmatrix} \\[1em] \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} r\cdot\cos\theta \\ r\cdot\sin\theta \\ z \end{pmatrix} \end{array}$$

$\Rightarrow$ you should always check what sort of symmetry the domain has, e.g. if it is symmetric about a single point, use polar coordinates (2D) or spherical coordinates (3D)!

$$\sqrt{r^2} = |r| \overset{r \geq 0}{=} r$$

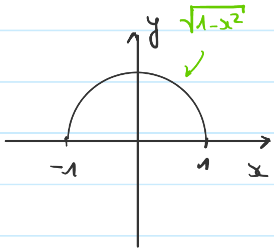
Examples:

1. $$\int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} \underbrace{\sqrt{x^2+y^2}}_{f(x,y)}\, dy\,dx = \int_{0}^{1}\int_{0}^{\pi} \underbrace{\sqrt{r^2}}_{f(\varphi(x,y))} \cdot \underbrace{r}_{|\det J_\varphi(r,\theta)|} \cdot d\theta\,dr = \pi \cdot \int_0^1 r^2\,dr = \frac{\pi}{3}$$

polar coordinates: $\varphi(r,\theta) = \begin{pmatrix} r\cdot\cos\theta \\ r\cdot\sin\theta \end{pmatrix} \hat{=} \begin{pmatrix} x \\ y \end{pmatrix}$

$$J_\varphi = \begin{pmatrix} \underset{\partial_r}{\cos\theta} & \underset{\partial_\theta}{-r\cdot\sin\theta} \\ \sin\theta & r\cdot\cos\theta \end{pmatrix}$$

$\rightarrow \det J_\varphi = r\cdot(\sin^2\theta + \cos^2\theta) = r$

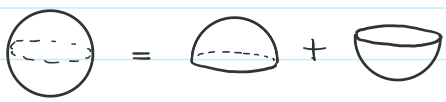


this is how to calculate volumes

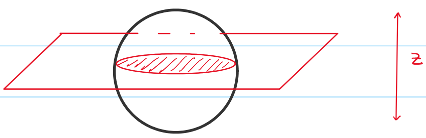
2. $\boxed{\mathrm{vol}\,(B_1(0) \subset \mathbb{R}^3) = \iiint_{B_1(0)} 1 \cdot dx\,dy\,dz}$ (Thm 3.5)

$\rightarrow$ you did this in the lecture in 2 different ways:

i) using half spheres $\rightarrow$



ii) integrating over the area of the disks along z-axis

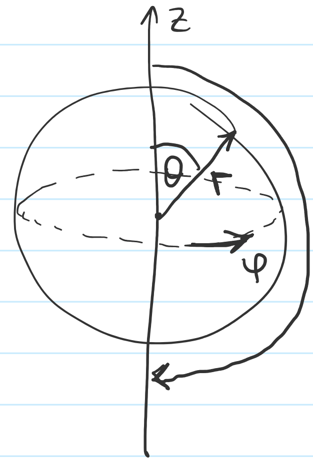


We want to use spherical coordinates here to do this:

$$f(r,\theta,\varphi) = \begin{pmatrix} r\cdot \boxed{\sin\varphi}\cos\theta \\ r\cdot \boxed{\sin\varphi \sin\theta} \\ r\cdot\cos\varphi \end{pmatrix}, \qquad J_f(r,\theta,\varphi) = \begin{pmatrix} \sin\varphi\cos\theta & \cdots \\ \vdots & \end{pmatrix}$$

$$\uparrow \quad \uparrow \quad \uparrow$$
$$r \quad \varphi \quad \theta$$

$$\rightarrow \det J_f(r,\theta,\varphi) = -r^2 \sin\theta$$



$$\Rightarrow \iiint_{B_1(0)} 1\cdot dx\,dy\,dz = \int_0^1\int_0^{2\pi}\int_0^{\pi} 1\cdot \underbrace{r^2\cdot\sin\theta}_{|\det J_f(r,\theta,\varphi)|}\, d\theta\, d\varphi\, dr$$

$h(x,y,z)=1$ $\qquad$ $h(f(r,\theta,\varphi))=1$

$$= \int_0^1\int_0^{2\pi} \underbrace{\left[-r^2\cdot\cos\theta\right]_0^{\pi}}_{=2\cdot r^2} d\varphi\, dr$$

$$= 2\cdot\int_0^1 \underbrace{\left[r^2\cdot\varphi\right]_0^{2\pi}}_{=2\pi\cdot r^2} dr$$

$$= 4\pi\cdot\left[\tfrac{1}{3}r^3\right]_0^1 = \underline{\underline{\frac{4\pi}{3}}}$$

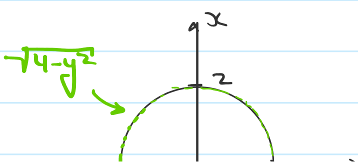
3. $$\int_{[0,2]\times[0,3]} f(x,y)\,dx\,dy = 6\cdot\int_0^1\int_0^1 f(2x,3y)\,dx\,dy \quad ?$$

$$\int_0^1\int_0^1 f(\varphi(x,y))\cdot \underbrace{|\det J_\varphi(x,y)|}_{=6\ ?}\,dx\,dy$$

$$\varphi: \mathbb{R}^2 \to \mathbb{R}^2,\ \begin{pmatrix} x\\ y\end{pmatrix} \mapsto \begin{pmatrix} 2x\\ 3y\end{pmatrix}$$

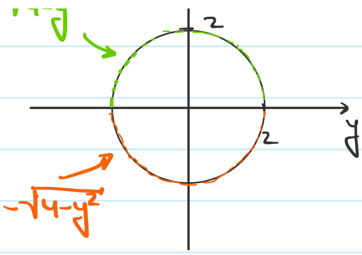
$$J_\varphi = \begin{pmatrix} 2 & 0\\ 0 & 3\end{pmatrix} \rightarrow \det J_\varphi(x,y) = 6 \qquad \Rightarrow \underline{\text{correct!}}$$

4. $$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2+y^2\,dx\,dy = \int_0^2\int_0^{2\pi} r^2\cdot \underbrace{r}_{|\det J_\varphi|}\, d\theta\, dr$$



$-2 \quad -\sqrt{4-y^2} \qquad v \qquad v \qquad 0 \qquad 0$

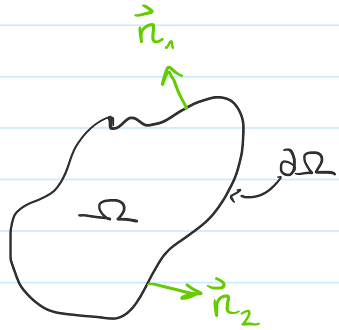
$$= \int_0^2 \underbrace{\left[r^3 \cdot \theta\right]_0^{2\pi}}_{2\pi \cdot r^3} dr$$



$$= 2\pi \cdot \left[\tfrac{1}{4} \cdot r^4\right]_0^2 = \underline{\underline{8\pi}}$$

Calculating volumes

1. $vol(\Omega) = \int\int_\Omega 1 \, dx_n dx_{n-1} \dots dx_1$



- describing the boundaries with functions and integrate (Fubini, → last lecture)
- change of variables $\int_X f(\varphi(x)) \, |\det J_\varphi(x)| \, dx = \int_Y f(y) \, dy$

→ $vol(B_1(0))$

2. (2D) Green's formula (next lecture?)

$$\int_{\partial\Omega} \vec{v} \cdot ds = \int\int_\Omega \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) dx \, dy \qquad (\to \text{Ex } 12.3)$$

You can use this formula to calculate volumes by choosing the vector field $\vec{v}$ in a way that satisfies $\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 1\right)$.

$$\Rightarrow \int_{\partial\Omega} \vec{v} \cdot d\vec{s} = \int\int_\Omega 1 \cdot dx dy = vol(\Omega)$$

any function that does not depend on y!

Possible v could be $(0, x), (x, x), (f(x), x)$.

3. (3D) Gauss formula (in two weeks?)

$$\iint_S (\vec{f}\cdot\vec{n})\,d\sigma = \iiint_V \underbrace{\nabla\cdot\vec{f}}_{=div(f)}\,dV$$ (S is a parametrised surface of the boundary ∂V of the set V, $\vec{n}$ is the normal vector at the surface)

Here, again, we can choose $\underline{\nabla\cdot f = 1}$ (e.g. $f=(x,0,0),(0,y,0),\ldots$)

and then calculate the surface integral $\iint_S f\cdot dA$!

Remark: Have a look at the online repository of old exams / summaries:

https://exams.vis.ethz.ch/category/Analysis#

Exercise sheet

MC 12.1

- Consider $\varphi: B_1(0)\to B_r(0),\ x\mapsto ?$ and $\det(J_\varphi(x))$

MC 12.2

- Use spherical coordinates to evaluate the integral!

Ex 12.1

→ use what we discussed last week

Ex 12.2

b) - $vol(K) = \int_{Z_1\cap Z_2} 1\cdot dx\,dy\,dz$

- parametrise the boundaries $\ldots\leq x\leq\ldots$ $\ldots\leq z\leq\ldots$

$$\int^1 \left(\int^{g(y)} \left(\int^{k(y)} 1\ dz\right)dx\right)dy$$

$$\int_{-1}^{1}\left(\int_{\underline{n(y)}}^{g(y)}\left(\int_{\underline{\ell(y)}}^{k(y)} 1 \, dz\right) dx\right) dy$$

→ boundaries should all be function of the same variable!

c) • $m(K) = \int_{-1}^{1}\left(\int_{n(y)}^{g(y)}\left(\int_{\ell(y)}^{k(y)} \underbrace{(1+x^2+z^2)}_{\rho(x,y,z) = 1+x^2+z^2} \, dz\right) dx\right) dy$

Ex 12.3

$$\operatorname{curl}(v) = \partial_x v_2 - \partial_y v_1$$

a) circle parametrisation $\gamma : [0, 2\pi] \to \mathbb{R}^2,\ t \mapsto 2\begin{pmatrix}\cos(t)\\ \sin(t)\end{pmatrix}$!!

b) use polar coordinates in one of the cases.

→ this is an application of Green's formula!

As usual: Study centre on Wednesday from 18:00 – 19:30!