Class 13

Friday, 11 December 2020 13:40

Old exercise sheet

MC 12.1

$$\int_{B_r(0)} f(x)\,dx = \int_{B_1(0)} \underbrace{?}_{f(\phi(x))\cdot|\det J_\phi|}\,dx$$

Find the change of variables $\underbrace{\phi : B_1(0) \to B_r(0)}_{\text{NOT } B_r(0)\to B_1(0) \text{ as domain of } f \text{ is } B_r(0)\text{ !}}$, $x \mapsto r\cdot x$

$$J_\phi(x) = \begin{pmatrix} r & & 0 \\ & r & \\ 0 & & \ddots \; r \end{pmatrix}, \quad \det J_\phi(x) = r^n$$

$$\Rightarrow \int_{B_r(0)} f(x)\,dx = \int_{B_1(0)} f(\phi(x))\cdot|\det J_\phi(x)|\,dx = r^n \int_{B_1(0)} f(r\cdot x)\,dx$$

MC 12.2 $\quad B_1(0) \subset \mathbb{R}^3$

$\int_{B_1(0)} \frac{1}{|x|^\alpha}\,dx$ converges for $\alpha < 3$!

$\rightarrow$ use spherical coordinates

$$\rightarrow \int_0^1\int_0^{2\pi}\int_0^{\pi} \frac{1}{r^\alpha}\cdot r^2\cdot \sin\varphi \, d\varphi\, d\theta\, dr = 4\pi\cdot \int_0^1 \underbrace{\frac{1}{r^{\alpha-2}}}\, dr$$

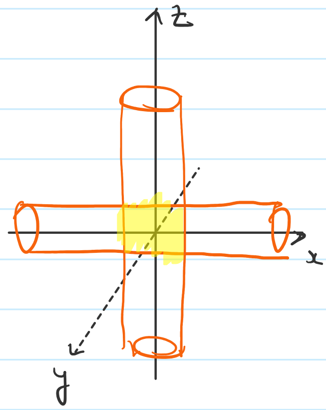
converges for $\alpha - 2 < 1 \Rightarrow \alpha < 3$
(given in the sheet)

without change of variables:

$$\int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{(x^2+y^2+z^2)^{\alpha/2}}\,dy\,dx\,dz$$ → really hard, not easy to see!

$$\int_{B_1(0)} \frac{1}{|x|^\alpha}\,dx \implies x = (x_1, x_2, x_3) \qquad |x| = \sqrt{x_1^2+x_2^2+x_3^2}$$

Ex 12.2



$$K = Z_1 \cap Z_2 = \{(x,y,z)\in\mathbb{R}^3 \mid x^2 \le 1-y^2 \wedge z^2 \le 1-y^2\}$$

→ boundaries: $-1 \le y \le 1$, $-\sqrt{1-y^2} \le x,z \le \sqrt{1-y^2}$

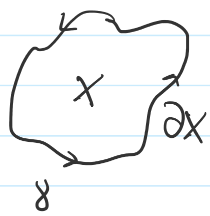
b) $$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 1\,dx\,dz\,dy = \int_{-1}^{1} 2\cdot\sqrt{1-y^2}\cdot 2\cdot\sqrt{1-y^2}\,dy = \dots = \frac{16}{3}$$

Revision: Green's formula

Theorem 3.10 (Green's theorem) (have a look at the summary for a more precise statement)

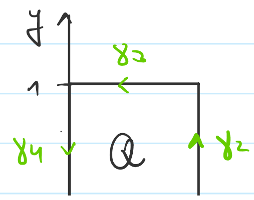
Let $X \subset \mathbb{R}^2$ be a compact set with boundary ∂X parametrised by $\gamma : [a,b] \to \mathbb{R}^2$. Let $f = (f_1, f_2)$ be a vector field of class C^1. Then

$$\iint_X \underbrace{\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)}_{\text{curl}(f)} dx\,dy = \int_{\partial X} f(\vec{s})\cdot d\vec{s}$$

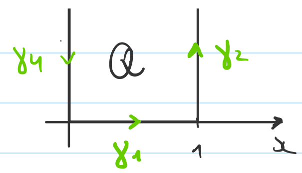


Examples

1. $g(x,y) = \begin{pmatrix} xy \\ x+y \end{pmatrix}$ $\qquad Q = [0,1]\times[0,1]$



$$\int_{\partial Q} g \cdot d\vec{s} = ?$$



i) line integral

$$\gamma_1(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}, \quad t \in [0,1], \quad \gamma_1'(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\gamma_2(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}, \quad t \in [0,1], \quad \gamma_2'(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\gamma_3(t) = \begin{pmatrix} 1-t \\ 1 \end{pmatrix}, \quad t \in [0,1], \quad \gamma_3'(t) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\gamma_4(t) = \begin{pmatrix} 0 \\ 1-t \end{pmatrix}, \quad t \in [0,1], \quad \gamma_4'(t) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\int_\gamma g(\vec{s}) \cdot d\vec{s} = \int_{\gamma_1} g\, ds + \int_{\gamma_2} g \cdot ds + \int_{\gamma_3} g \cdot ds + \int_{\gamma_4} g \cdot ds$$

$$g = \begin{pmatrix} xy \\ x+y \end{pmatrix} \quad \gamma_1(t) = \begin{pmatrix} t \\ 0 \end{pmatrix} \hat{=} \begin{pmatrix} x \\ y \end{pmatrix} \quad \Rightarrow \quad g(\gamma_1(t)) = \begin{pmatrix} 0 \\ t+0 \end{pmatrix}$$

$$= \int_0^1 \underbrace{\begin{pmatrix} 0 \\ t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{=0} dt + \int_0^1 \begin{pmatrix} t \\ 1+t \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\gamma_2'(t)} dt + \int_0^1 \begin{pmatrix} 1-t \\ 2-t \end{pmatrix} \cdot \underbrace{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}_{\gamma_3'(t)} dt$$

$$+ \int_0^1 \begin{pmatrix} 0 \\ 1-t \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} dt$$

$$= 0 + \int_0^1 1+t\, dt + \int_0^1 t-1\, dt + \int_0^1 t-1\, dt$$

$$= \int_0^1 3t - 1\, dt = \left[\frac{3}{2} t^2 - t \right]_0^1 = \frac{1}{2}$$

ii) Green's formula

$$\int_\gamma g \cdot ds \overset{\text{Green}}{=} \iint_Q \underbrace{\left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y} \right)}_{1-x} dx\, dy \qquad \frac{\partial g_2}{\partial x} = \frac{\partial (x+y)}{\partial x} = 1, \quad \frac{\partial g_1}{\partial y} = \frac{\partial (xy)}{\partial y} = x$$

$$= \int^1 \int^1 1-x\, dx\, dy$$

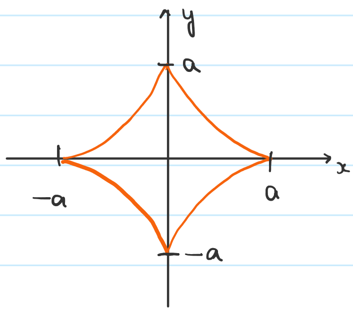
$$= \int_0^1 \int_0^1 1-x \, dx\,dy$$

$$= \int_0^1 \underbrace{\left[x-\tfrac{1}{2}x^2\right]_0^1}_{=\frac{1}{2}} dy$$

$$= \frac{1}{2}\int_0^1 dy = \underline{\underline{\frac{1}{2}}}$$

2. Compute the area of the ~~asteroid~~ christmas star whose boundary is given by

$$\partial S(a) = \{(x,y)\in\mathbb{R}^2 \mid x^{2/3}+y^{2/3}=a^{2/3}\}$$



$$\boxed{\mathrm{vol}(S) = \iint_S 1\cdot dx\,dy = ?}$$

$\Rightarrow$ we'll use green's theorem!

i) $$\iint_S \underbrace{\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)}_{=1} dx\,dy = \int_{\partial S} v\cdot d\vec{s}$$

We can choose $v=(0,x)$ and get: $\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y} = \frac{\partial (x)}{\partial x}-\frac{\partial (0)}{\partial y} = 1$!

ii) $$\gamma(t) = a\cdot(\underbrace{\cos^3(t)}_{\gamma_1}, \underbrace{\sin^3(t)}_{\gamma_2}), \quad t\in[0,2\pi]$$

$$\gamma_1^{2/3}+\gamma_2^{2/3} = a^{2/3}\cdot\cos^2(t)+a^{2/3}\cdot\sin^2(t) = a^{2/3}$$

$\Rightarrow$ parametrises $\partial S(a)$!

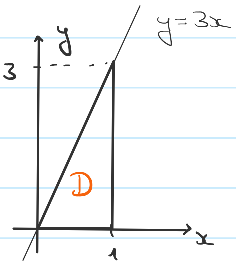
iii) $$\iint_S 1\cdot dx\,dy \overset{\text{Green}}{=} \int_\gamma \begin{pmatrix}0\\x\end{pmatrix}\cdot d\vec{s} = \int_0^{2\pi} \overbrace{\begin{pmatrix}0\\a\cdot\cos^3(t)\end{pmatrix}\cdot\begin{pmatrix}-a\cdot3\cos^2(t)\cdot\sin(t)\\a\cdot3\cdot\sin^2(t)\cdot\cos(t)\end{pmatrix}}^{v(\gamma(t))\,\cdot\,\gamma'(t)} dt$$

$$= 3\cdot a^2\cdot\int^{2\pi}\cos^4(t)\cdot\sin^2(t)\,dt$$

$$= \frac{\pi}{8}$$

$$= \frac{3\pi a^2}{8}$$

3. $h(x,y) = \begin{pmatrix} \sqrt{1+x^3} \\ 2xy \end{pmatrix}$ $\quad D = \{(x,y)\in\mathbb{R}^2 \mid 0 \leq y \leq 3x \leq 3\}$



$$\int_{\partial D} h \cdot ds = ?$$ → use Green's theorem

$$\int_{\partial D} h \cdot ds = \iint_D \underbrace{\left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y}\right)}_{=2y} dx\,dy$$

$\frac{\partial h_2}{\partial x} = \frac{\partial(2xy)}{\partial x} = 2y, \quad \frac{\partial h_1}{\partial y} = \frac{\partial(\sqrt{1+x^3})}{\partial y} = 0$

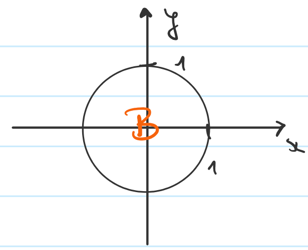
$$= \int_0^1 \int_0^{3x} 2y \, dy\, dx$$

$$= \int_0^1 [y^2]_0^{3x} dx = \int_0^1 9x^2 dx$$

$$= [3x^3]_0^1 = 3$$

4. $f(x,y) = \begin{pmatrix} -y \\ x \end{pmatrix}, \quad B = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 = 1\}$

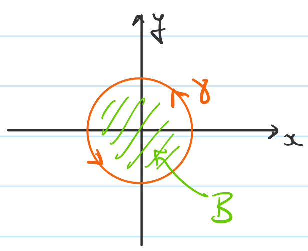
$$\int_{\partial B} f \cdot ds = ?$$



i) line integral

$\gamma : [0,2\pi] \to \mathbb{R}^2, t \mapsto \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$

$\gamma'(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$



$$\int f \cdot d\vec{s} = \int_0^{2\pi} f(\gamma(t)) \cdot \gamma'(t) dt$$

$$\int_\gamma f\cdot d\vec{s} = \int_0^{2\pi} f(\gamma(t))\cdot\gamma'(t)\,dt$$

$$= \int_0^{2\pi} \begin{pmatrix}-\sin(t)\\ \cos(t)\end{pmatrix}\cdot\begin{pmatrix}-\sin(t)\\ \cos(t)\end{pmatrix} dt$$

$$= \int_0^{2\pi} \underbrace{(\sin^2(t)+\cos^2(t))}_{=1}\,dt$$

$$= 2\pi$$

ii) Green's formula

$$\int_\gamma f\cdot d\vec{s} = \int_B \underbrace{\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)}_{=2} dx\,dy$$

$$\frac{\partial f_2}{\partial x} = \frac{\partial x}{\partial x} = 1,\quad \frac{\partial f_1}{\partial y} = \frac{\partial(-y)}{\partial y} = -1$$

polar coordinates

$$= \int_0^1\int_0^{2\pi} 2\cdot r\cdot d\theta\,dr$$

$$dx\,dy \longrightarrow \underbrace{r}_{|\det J_\phi(r,\theta)|}\cdot\, dr\,d\theta$$

$$= 2\pi\cdot\int_0^1 2r\,dr$$

$$= 2\pi\cdot\underbrace{\left[r^2\right]_0^1}_{=1} = 2\pi$$

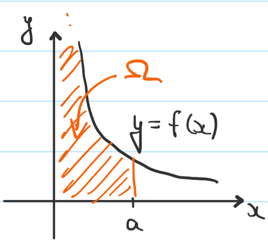
Exercise sheet

MC 13.1

Measurable means that you can evaluate that integral $\int_\Omega dx$.

In $\mathbb{R}^2$, its measure corresponds to its area,

in $\mathbb{R}^3$, its measure corresponds to its volume



$$\int_\Omega dy\,dx = \int_0^a\int_0^{f(x)} dy\,dx$$

- think of improper integrals (in $\mathbb{R}$) that are converging/diverging
- use contraposition $A \Rightarrow B \Leftrightarrow \neg B \Rightarrow \neg A$

MC 13.2

- think of converging infinite sums → b) and c)

Ex 13.1

- change of variables (Thm 3.9) → polar/spherical coordinates
- what we discussed two weeks ago

Ex 13.2

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Ex 13.3

- $\int_{E(a,b)} f(x,y)\,dx\,dy = \int_0^\infty \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_0^\infty f(x,y)\,dy\,dx$

$E(a,b) = [a,b] \times [0,+\infty)$