Sept 17. 20

Analysis II INFK

HS 2020

Lecturer: Özlem Imamoğlu

Course Assistent: Francesco Palmurella

webpage : https://metaphor.ethz.ch/x/2020/hs/401-0213-16L/

Outline.

1) Ordinary differential equations

2) Differential calculus in $\mathbb{R}^n$

3) Integral calculus in $\mathbb{R}^n$.

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

if $n=1$

$$f : \mathbb{R} \longrightarrow \mathbb{R}^m$$

$$x \longmapsto (f_1(x), f_2(x), \dots f_m(x)).$$

f is continuous

$\iff f_i(x) : \mathbb{R} \to \mathbb{R}$ $\forall i = 1, \dots m$ are cont.

§2. Ordinary differential equations.

Defn.

A diff. eqn is an equation for a function f that relates the values of f at x, $f(x)$ to the values of its derivatives, at the same point $f'(x)$, $f''(x)$, ...

Ex: 1) $f'(x) = f(x)$
has soln e^x, ce^x

2) $f''(x) = -f(x)$
$\sin x$ is a soln.

3) Consider a mass falling under the influence of gravity
Newton's law, $F = ma$
results in the equation

$$m \frac{d^2x}{dt^2} = -mg$$

x = height of the object above the ground

m = mass of the object

g = $9.8\,m/sec^2$ = gravitational acceleration.

Integrating once gives.

$$\frac{dx}{dt} = -gt + A$$

for some constant

Integrating again gives

$$x(t) = -\frac{1}{2}gt^2 + At + B$$

Notation: We usually just write $y, y', y'', y^{(3)}, \ldots, y^{(k)}$

instead of $f(x), f'(x), \ldots$

ODE.

Ex: a) $y' = 2xy$ order 1

b) $y^{(3)} + 2xy'' + e^x y + 1 = 0$ order 3

c) $(\sin x)y' = (\cos x)y^2 + 1$ order 1.

Defn. Order of a diff equation is the largest derivative present in the eqn.

$u(x,t)$

Ex 1) $a^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$

partial derivatives,

Partial diff eqn.

PDE

2) $f'(x+2) = f(x)$

This is not an ODE

it relates $f(x)$ to

its derivative at $x+2$.

In general ~~the~~ solution to an ODE is not unique. But if we are given "initial conditions" then we'll able to find unique soln.

ex: $y' = x+1$

$y = \dfrac{x^2}{2} + x + c$ is a soln for any c

if we are also given

$y(0) = 1$

$\Rightarrow c = 1$

---

§ 2.2 Linear differential equations

Defn. A linear ODE of order k on interval $I \subset \mathbb{R}$ is an eqn of the form

$$(*) \quad y^{(k)} + a_{k-1}(x)\, y^{(k-1)} + \dots + a_1(x) y' + a_0(x) y = b(x)$$

where $a_j(x)$ $j = 0, \dots, k$ and $b(x)$ are continuous functions from I to $\mathbb{C}$

if $b = 0$ then we say the equation is **homogeneous**.

otherwise inhomogeneous.

Rk 1) If $a_i(x)$, $b(x)$ are real-valued then we're often interested in solutions $f : I \to \mathbb{R}$

2) In <u>linear</u> diff eqn there are no products of the function $y(x)$ and its derivatives and neither its derivatives occur to any power other than 1.

Also neither the function nor its derivatives are "inside" another function e.g $\sqrt{y}$, $\sin y$ ...

e.g. $2yy' = \cos x$ is not linear

$$\left(\frac{d^2y}{dt^2}\right)^2 + (\sin y)\frac{dy}{dt} = 0,$$

is not linear.

where as

$$y' = y$$

$$y^{(3)} + 2y + 1 = 0$$

$\vdots$

Rk

Solving (*) linear ODE means finding all functions $f : I \to \mathbb{C}$ k times differentiable s.t. $\forall x \in I$

$$f^{(k)}(x) + a_{k-1}(x) f^{(k-1)}(x) + \dots + a_0(x) f(x) = b(x).$$

Defn

An initial condition for (*) is a set of equations

$$\left.\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y_1 \\ &\vdots \\ y^{(k-1)}(x_0) &= y_{k-1} \end{aligned}\right\} \text{specifying values of } y, y', \dots, y^{k-1} \text{ at some initial point } x_0.$$

---

eg.

$$\left.\begin{aligned} y^{(3)} + 2y + 1 &= 0 \\ y(0) &= 1 \\ y'(0) &= 2 \\ y''(0) &= -1 \end{aligned}\right\} \text{initial Value Problem.}$$

Question. Why are they called linear ODEs?

Let's denote the LHS of (*) with $D(f)$

where D

$$D: \frac{d^{(k)}}{dx^k} + a_{k-1}\frac{d^{k-1}}{dx^{k-1}} + \dots + a_0$$

$$Df = f^{(k)}(x) + \dots + a_0 f(x)$$

$$f \to \boxed{D} \to Df$$

Then D is linear.

$$D(\alpha f + \beta g) = \alpha D(f) + \beta D(g)$$

Main result about linear ODEs.

Thm (2.2.3) Let $I \subset \mathbb{R}$ open interval

$k \geq 1$ integer

Consider linear ODE

(*) $y^{(k)} + a_{k-1}(x) y^{(k-1)} + \cdots + a_0(x) y = b(x)$

where coefs $a_i(x)$, $b(x)$ are continuous functions.

① Let S_0 be the set of solutions when $b = 0$

Then S_0 is a vector space of dimension k.

② For any initial conditions, ie for any choice of $x_0 \in I$, and $(y_0, \ldots, y_{k-1}) \in \mathbb{C}^k$ there is a unique solution $f \in S_0$ s.t

$f(x_0) = y_0, \ldots, f^{(k-1)}(x_0) = y_{k-1}$

③ For an arbitrary b the set of solutions of (*) is

$$S_b = \{ f + f_p \mid f \in S_0 \}$$

where f_p is one "particular" solution of (*)

④ For any initial condition there is a unique soln.

Rk. This is very similar to the soln of lin. eqns $Ax = b$

Rk. To see for example ③ Note if f_p is a soln of *

$$Df = b \quad (*)$$

and if g is another soln of $Df = b$.

$$f_p^{(k)} + a_{k-1} f_p^{(k-1)} + \cdots + a_0 f_p = b(x)$$

$$g^{(k)} + a_{k-1} g^{(k-1)} + \cdots + a_0 g = b(x)$$

$$\left(f_p^{(k)} - g^{(k)}\right) + a_{k-1}\left(f_p^{k-1} - g^{k-1}\right) + \cdots + a_0 (f_p - g) = 0$$

$\Rightarrow$ $f_p - g$ satisfies the hom. Diff eqn

$$Df = 0$$

$$f_p - g \in S_0$$

$$f_p - g = f_h$$

for some $f_h \in S_0$.

$$g = f_p + f_h.$$

<u>Rk.</u> The linearity of the diff eqn (*) also implies a "superposition" principle

Suppose we have 2 different functions $b_1(x)$, $b_2(x)$ on the RHS, say solutions

f_1, f_2 : $Df_1 = b_1$

$Df_2 = b_2$

Then $f_1 + f_2$ solves $Df = b_1 + b_2$

Rk. given a diff eqn and a possible soln, we can always verify whether it is indeed a soln or not.

eg. $y' + 3x^2 y = 0$.

$y = e^{-x^3}$ is a soln.

Check: $y' = -3x^2 e^{-x^3}$

$$\begin{array}{rcl} & y' & = -3x^2e^{-x^3} \\ + & 3x^2y & = 3x^2 e^{-x^3} \\ \hline & y' + 3x^2 y & = 0 \end{array}$$

§ 2.3 Lin. diff eqns of order 1

We consider

$$y' + ay = b$$

a, b are continuous functions.

---

2 steps:

① Find solns of the corresponding hom. eqn $y' + ay = 0$.

Note if f is a soln

then so is zf

for any constant $z \in \mathbb{C}$

if $f' + af = 0$

$$(zf)' + a(zf) = z(\underbrace{f' + af}_{0}).$$

$$= 0.$$

② Find a particular soln $f_p : I \to \mathbb{C}$

s.t $f_p' + a f_p = b$

Homog soln:

$$y' + ay = 0.$$

$$y' = -ay$$

$$\frac{y'(x)}{y(x)} = -a(x)$$

$$\int \frac{y'(x)}{y(x)}\,dx = -\underbrace{\int a(x)\,dx}_{:= A(x)}$$

$\ln|y(x)|$ $\quad (A'(x) = a(x))$

$$\Rightarrow \ln|y(x)| = -A(x) + C$$

This leads to

$$y = z \exp(-A(x)).$$

for some constant $z \in \mathbb{C}$

Now we can check or verify that this indeed gives a soln.

I leave it as exercise

Prop. Any soln of $y' + a(x)y = 0$ is of the form $f(x) = z\exp(-A(x))$ where $A(x)$ is a primitive of $a(x)$, $z \in \mathbb{C}$.

Soln of inhom. eqn

$$y' + ay = b.$$

2 methods:

① "educated guess"

if $b(x)$ is a poly, we guess that fp is also a poly.

or if b is a trigon. func then we guess fp is also.

(2) More systematic.
"variation of constants"
parameters

a) Assume $f_p = z(x)\exp(-A(x))$
for some function $z : I \to \mathbb{C}$

b) We put this into the eqn
$$y' + ay = b$$
and see what it forces $z(x)$ to satisfy.

$$f_p = z(x)e^{-A(x)} = y$$

$$y' = z'(x)e^{-A(x)} + z(x)(-A(x))' e^{-A(x)}$$

$$= z'(x)e^{-A(x)} - z(x)a(x)e^{-A(x)}.$$

$$= e^{-A(x)}\left[z'(x) - z(x)a(x)\right]$$

$$+\ ay = a(x)z(x)e^{-A(x)}$$

$$y' + ay = \boxed{z'(x)e^{-A(x)} = b(x)}$$

$$\boxed{z'(x) = b(x)e^{A(x)}}$$

$z(x)$ is a primitive $b(x)e^{A(x)}$.