Last time

1.10.20

Linear Differential Equations with constant coefficients.

$$y^{(k)} + a_{k-1} y^{(k-1)} + \cdots + a_0 y = b(x) \qquad (*).$$

$a_0, \ldots, a_{k-1} \in \mathbb{C}$.

Defn: $P(\lambda) = \lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_0$ is called the characteristic polynomial of $(*)$. The zeroes of $P(\lambda)$ are called the eigenvalues.

Thm $y = e^{\lambda x}$ is a solution of $*$ with $b=0$ $\iff$ λ is an eigenvalue.

Thm. Let $\alpha_1, \ldots, \alpha_r$ be pairwise distinct eigenvalues of $*$ with corresponding multiplicities $m_1, \ldots, m_r$. Then the functions $f_{j,\ell} : \mathbb{R} \to \mathbb{C}$, $x \mapsto x^\ell e^{\alpha_j x}$, for $1 \le j \le r$, $0 \le \ell < m_j$ form a system of solutions of the homogeneous equation.

$$y^{(k)} + a_k y^{(k-1)} + \cdots + a_0 y = 0.$$

Rk if a_i's are real, it is natural to look at real solutions.

If $\alpha = \beta + i\gamma$ is a complex root $P(\lambda)$ then so is $\bar{\alpha} = \beta - i\gamma$.

Hence $f_1 := e^{\alpha x}$, $f_2 := e^{\bar{\alpha} x}$ are 2 solutions.

$$e^{\alpha x} = e^{\beta x} \cdot e^{i\gamma x}$$
$$e^{\beta x}[\cos \gamma x + i \sin \gamma x]$$

$$e^{\bar{\alpha} x} = e^{\beta x} \cdot e^{-i\gamma x}$$
$$= e^{\beta x}[\cos \gamma x - i \sin \gamma x]$$

We can replace any soln

$$a f_1 + b f_2$$

with a lin. combination of

$$\tilde{f}_1 = e^{\beta x} \cos x$$
$$\tilde{f}_2 = e^{\beta x} \sin x$$

Thm if $y^k + a_{k-1} y^{k-1} + \dots + a_0 y = 0$ has real coefs, then each pair of complex conjugate roots $\beta_j \pm i\gamma_j$ of $P(\lambda)$ with multiplicity m_j leads to solutions

$$x^{\ell} e^{\beta_j x} (\cos \gamma_j x + i \sin \gamma_j x)$$

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for $0 \le \ell < m_j$

which then can be replaced by solns

$$x^\ell e^{\beta_j x} \cos(\gamma_j x)$$

$$x^\ell e^{\beta_j x} \sin(\gamma_j x)$$

Ex. 1) $y'' - y = 0$

$$P(\lambda) = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$$

λ_j: $1, -1$, multiplicity 1.

$$y_h(x) = z_1 e^x + z_2 e^{-x}$$

2) $y'' + y = 0$ $\quad P(\lambda) = \lambda^2 + 1$

λ_j: $i, -i$

$$y_H = z_1 e^{ix} + z_2 e^{-ix}$$

$$= \tilde{z}_1 \cos x + \tilde{z}_2 \sin x$$

3) $y'' = 0$, $\lambda^2 = 0$

$\lambda = 0$ multiplicity 2.

Solns are $1, x$

$$y_h = z_1 + z_2 x$$

4) $y^{(4)} + 2y^{(2)} + y = 0$

$$P(\lambda) = \lambda^4 + 2\lambda^2 + 1 = 0.$$

$$= (\lambda^2 + 1)^2$$

$$= (\lambda - i)^2 (\lambda + i)^2$$

So eigenvalues are $\pm i$ w/ multip. 2.

We can use as solutions either

$$e^{ix}, \quad xe^{ix}$$
$$e^{-ix}, \quad xe^{-ix}$$

or

$$\cos x, \quad \sin x$$
$$x\cos x, \quad x\sin x$$

$$y_h = z_1 \sin x + z_2 \cos x + z_3 x\sin x + z_4 x\cos x.$$

Suppose we are given extra conditions on y

$$y(0) = 1, \quad y'(0) = 3$$
$$y(\pi) = 2 \quad y'(\pi) = 0$$

$$y_h(0) = z_1 \sin 0 + z_2 \cos 0 = 1$$
$$\Rightarrow z_2 = 1.$$

$$y(\pi) = z_1 \cancel{\sin\pi} + z_2 \cos\pi$$
$$z_3 \pi \cancel{\sin\pi} + z_4 \pi\cos\pi = 2$$

$$= -z_2 - z_4\pi = 2$$
$$= -1 - \pi z_4 = 2.$$
$$\pi z_4 = -3 \quad \Rightarrow z_4 = -3/\pi.$$

$$y'(x) = z_1 \cos x - z_2 \sin x$$
$$z_3[\sin x + x\cos x]$$
$$z_4[\cos x - x\sin x].$$

$$y'(0) = z_1 + z_4 = 3$$
$$\Rightarrow z_1 = 3 - z_4 = 3 + \frac{3}{\pi}$$

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Clicker Question

$$y'' - 4y' + 8y = 0.$$

$$\lambda^2 - 4y + 8 = 0.$$

$$\frac{4 \pm \sqrt{16-32}}{2} = 2 \pm 2i$$

$$e^{(2+2i)x}, \quad e^{(2-2i)x}$$

$$e^{2x}\cos 2x, \quad e^{2x}\sin 2x$$

$$y_h = e^{2x}\left[A\cos 2x + B\sin 2x\right].$$

$$e^{(\beta+i\gamma)x} = e^{\beta x}\left[\cos\gamma x + i\sin\gamma x\right]$$

$$e^{(\beta-i\gamma)x} = e^{\beta x}\left[\cos\gamma x - i\sin\gamma x\right]$$

$$A e^{(\beta+i\gamma)x} + B e^{(\beta-i\gamma)x}$$

$$\tilde{A} e^{\beta x}\cos x + \tilde{B} e^{\beta x}\sin x$$

$$A e^{\beta x}\left[\cos\gamma x + i\sin\gamma x\right] + B e^{\beta x}\left[\cos\gamma x - i\sin\gamma x\right].$$

$$= \underbrace{(A+B)}_{\tilde{A}} e^{\beta x}\cos\gamma x + \underbrace{i(A-B)}_{\tilde{B}} e^{\beta x}\sin\gamma x$$

Next, we look at the <u>inhomog.</u> eqn.

(*) $y^{(k)} + a_{k-1} y^{k-1} + \dots + a_0 y = b(x)$

Goal is to find a particular solution y_p

Then any soln of * will be of the form.

$$y = y_h + y_p$$

where y_h is a soln of hom. eqn.

Method 1:

"Methods of undetermined coefficients"

or "Ansatz" method.

Idea: Solution will be "similar" to the disturbance function $b(x)$.

If $b(x)$ is ~~[illegible]~~ a poly then y_p will want to be also a poly.

If $b(x)$, $\cos x$, $\sin x$, e^x

$b(x)$	Ansatz
$ae^{\alpha x}$	$be^{\alpha x}$
$a\sin(\beta x)$ $b\cos(\beta x)$	$c\sin(\beta x) + d\cos(\beta x)$
$ae^{\alpha x}\sin(\beta x)$ $be^{\alpha x}\cos(\beta x)$	$e^{\alpha x}\left[c\sin(\beta x) + d\cos(\beta x)\right]$
$P_n(x)e^{\alpha x}$	$R_n(x)\,e^{\alpha x}$
$P_n(x)e^{\alpha x}\sin(\beta x)$ $Q_n(x)e^{\alpha x}\cos(\beta x)$	$e^{\alpha x}\left[R_n(x)\sin\beta x + S_n(x)\cos\beta x\right]$

$P_n(x)$, $R_n(x)$, $Q_n(x)$, $S_n(x)$ are polynomials of degree n.

If $b(x) = ae^{\alpha x}$

we'll try

1) $y_p = Be^{\alpha x}$ for some B,

2) we'll put y_p into the diff eqn $(*)$.

This will give some conditions on B

3) we'll solve for B.

Rk 1) If $b(x)$ is a lin. combination of the above functions then we try the corresponding lin. comb. of "Ansatz" functions

Rk 2. If $\lambda = \alpha + \beta i$ is a zero of the char. poly $P(\lambda)$ of multiplicity m, then the "Ansatz" must be multiplied by x^m.

Ex: (*) $y'' + y' - 6y = 3e^{-4x}$

$\lambda^2 + \lambda - 6 = 0$

$(\lambda - 2)(\lambda + 3) = 0.$

$\lambda = 2, \quad \lambda = -3$

$y_h = z_1 e^{2x} + z_2 e^{-3x}$

To find y_p, we take as "Ansatz" $y_p = Be^{-4x}$

We put y_p into (*).

$$\underbrace{16Be^{-4x}}_{y_p''} - \underbrace{4Be^{-4x}}_{y'} - 6Be^{-4x} = 3e^{-4x}$$

$$6Be^{-4x} = 3e^{-4x}$$

$\Rightarrow 6B = 3 \Rightarrow B = 1/2.$

$\Rightarrow y_p = \frac{1}{2} e^{-4x}$

General soln of (*)

$$y = z_1 e^{2x} + z_2 e^{-3x} + \frac{1}{2} e^{-4x}.$$

Ex: $y'' + y' - 6y = 50 \sin x$ (**)

To find y_p, we try as

Ansatz $y_p = c_1 \sin x + c_2 \cos x$

put this into (**).

$$y_p' = c_1 \cos x - c_2 \sin x$$

$$y_p'' = -c_1 \sin x - c_2 \cos x$$

$$-6y_p = -6c_1 \sin x - 6c_2 \cos x$$

$+$

$$50 \sin x = (-7c_1 - c_2)\sin x + (c_1 - 7c_2)\cos x$$

$$\left.\begin{aligned} -7c_1 - c_2 &= 50 \\ c_1 - 7c_2 &= 0. \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= -7 \\ c_2 &= -1 \end{aligned}$$

$$y_p = -7 \sin x - \cos x.$$

Exercise try $y_p = c \sin x$

put it into (**), check

what you get.

Ex: $y'' + y' - 6y = 10 e^{2x}$

Note e^{2x} is also

a soln of the homog.

eqn. (2 is an e.value

of $P(\lambda)$ w/ mult. 1.)

As Ansatz we should try

$y_p = k x e^{2x}$

<u>Note</u> if we try instead

$y_p = k e^{2x}$ by MISTAKE!

$y'' = 4k e^{2x}$

$y' = 2k e^{2x}$

$-6y = -6k e^{2x}$

$+$

$10e^{2x} = 0$!!!!

$y_p' = k e^{2x} + 2k x e^{2x}$

$y_p'' = 2k e^{2x} + 2k[e^{2x} + 2x e^{2x}]$

$-6y = -6k x e^{2x}$

RHS $= y_p'' + y_p' - 6y_p = 10e^{2x}$.

LHS $= k e^{2x} + 2k e^{2x}$
$+ 2k e^{2x}$

$= 5k e^{2x}$

RHS $=$ LHS

$\Rightarrow 10e^{2x} = 5k e^{2x}$

$\Rightarrow \underline{k = 2}$.

$y_p = 2x e^{2x}$.

Method 2.

Variation of constants

Assume $k=2$.

$$y^{(2)} + a_1 y^{(1)} + a_0 y = b(x)$$

Assume homog eqn has soln

$$f = z_1 f_1 + z_2 f_2$$

f_1, f_2 are 2 lin indep. solutions.

Now we try for f_p a similar function

Namely $f_p = z_1(x) f_1 + z_2(x) f_2$

To determine the 2 unknown functions $z_1(x)$, $z_2(x)$ that'll work I need 2 equations for them.

We'll try

$$\left\{ \begin{array}{l} z_1'(x) f_1 + z_2'(x) f_2(x) = 0 \\ z_1'(x) f_1' + z_2'(x) f_2' = b. \end{array} \right\}$$

$$\begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix} \begin{pmatrix} z_1'(x) \\ z_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

Fact:

$$A = \begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix}$$ is

invertible

Hence

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ b \end{pmatrix}$$

This is an eqn for $z_1'(x)$ and $z_2'(x)$

Then solve for $z_1(x)$ $z_2(x)$

Then you found your particular soln $y_p = z_1(x) f_1 + z_2(x) f_2$

$\lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1)$

Ex: $y'' + 3y' + 2y = t + 4e^t$

$$y_h = z_1 e^{-t} + z_2 e^{-2t}$$

So we try $y_p = z_1(t) e^{-t} + z_2(t) e^{-2t}$

$$z_1'(t) e^{-t} + z_2'(t) e^{-2t} = 0$$

$$-z_1' e^{-t} - 2 z_2'(t) e^{-2t} = t + 4e^t$$

$$\begin{pmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{pmatrix} \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} 0 \\ t+4e^{t} \end{pmatrix}$$

$$\boxed{y_p = \frac{t}{2} - \frac{3}{4} + \frac{2}{3} e^{t}}$$

$$\rightsquigarrow \quad z_1' = te^{t} + 4e^{2t}$$

$$z_2' = -te^{2t} - 4e^{3t}$$

Look at Ex 2.4-9 in script.

$$\rightsquigarrow \quad z_1(t) = \int te^{t} + 4e^{2t}\,dt$$

$$z_2(t) = \int -te^{2t} - 4e^{3t}\,dt$$

$$z_1 = te^{t} - e^{t} + 2e^{2t}$$

$$z_2 = -\frac{te^{2t}}{2} + \frac{e^{2t}}{4} - \frac{4}{3} e^{3t}$$

$$y_p = z_1 e^{-t} + z_2 e^{-2t}$$

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