Defn: Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}^n$. Let $y\in\mathbb{R}^n$. $\left(\begin{array}{l} y=(y_1,\dots,y_n)\in\mathbb{R}^n \\ x_k=(x_{k,1},\dots,x_{k,n})\in\mathbb{R}^n \end{array}\right)$

We say (x_k) converges to y if

$\forall\, \varepsilon>0,\ \exists\, N\geq 1$ such that $\forall\, k\geq N$ we have $\|x_k-y\|<\varepsilon$.

We write $\lim\limits_{k\to\infty} x_k = y$ or $x_k \xrightarrow{k\to\infty} y$

Prop: $\lim x_k = y \iff$ For each i, the sequence $(x_{k,i})\in\mathbb{R}$ converge to y_i

ie $(x_{k,i})_k \longrightarrow y_i \qquad 1\leq i\leq m$

$\iff$ The sequence of real numbers $\|x_k-y\|$ converge to 0.

Defn. Let $f:X\to\mathbb{R}^m$, $X\subset\mathbb{R}^n$, $x_0\in X$, $y\in\mathbb{R}^m$. We say that f has a limit y, as $x\to x_0$ if $\forall\,\varepsilon>0,\ \exists\,\delta>0$ s.t $\forall\, x\in X,\ x\neq x_0$ such that $\|x-x_0\|<\delta$, we have $\|f(x)-y\|<\varepsilon$. We write $\lim\limits_{x\to x_0} f(x)=y$

Prop: $\lim\limits_{x\to x_0} f(x)=y \iff \forall$ sequence $(x_k)_k\subset X$ such that $\lim x_k = x_0$, we have $\lim\limits_{k\to\infty} f(x_k)=y$.

Defn: We say $f:X\to\mathbb{R}^m$ is continuous at $x_0\in X$ if $\lim\limits_{x\to x_0} f(x)$ exists and is equal to $f(x_0)$.

Prop f is cont. at $x_0 \iff \forall$ sequence (x_k) in X s.t $\lim x_k = x_0$ we have $\lim f(x_k) = f(x_0)$

ie $\boxed{\lim f(x_k) = f(\lim x_k)}$

Examples

1) $f : \mathbb{R}^n \to \mathbb{R}^m$
$x \longmapsto Ax$
Linear functions are continuous $\forall x \in \mathbb{R}^m$

2) Polynomials $P : \mathbb{R}^n \to \mathbb{R}$. are continuous $\forall x \in \mathbb{R}^n$

3) If $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^n \to \mathbb{R}^m$ are continuous at x_0, then $f \pm g$ are continuous at x_0

4) If $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$ are continuous then $f \cdot g$, f/g are continuous $\forall x \in \mathbb{R}^n$, s.t. $g(x) \neq 0$

5) Functions of separated variables are continuous $\Leftrightarrow$ Each factor is continuous.

6) $f : \mathbb{R}^n \to \mathbb{R}^m$
$x \longmapsto (f_1(x), \dots, f_m(x))$ is continuous $\Leftrightarrow$ For each i $f_i(x) : \mathbb{R}^n \to \mathbb{R}$ is continuous.

7) $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^m \to \mathbb{R}^s$ continuous then $g \circ f : \mathbb{R}^n \to \mathbb{R}^s$ is continuous.

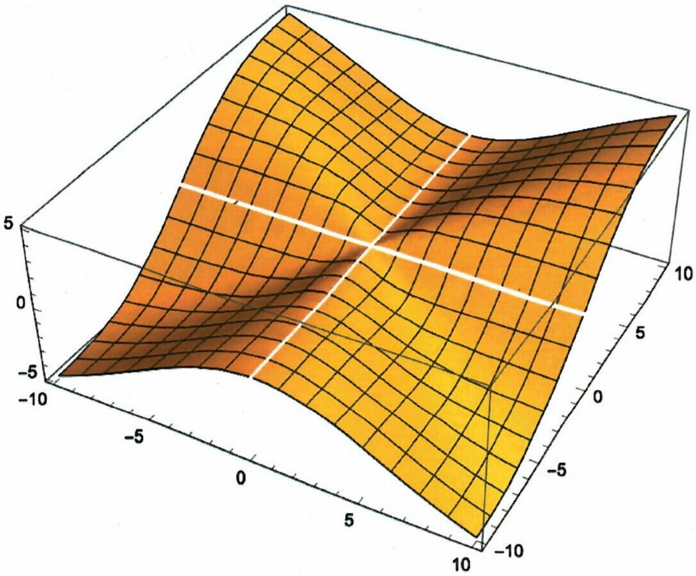
8) $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} xy/x^2+y^2 & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y) \to (0,0)} f(x,y) = ?$$

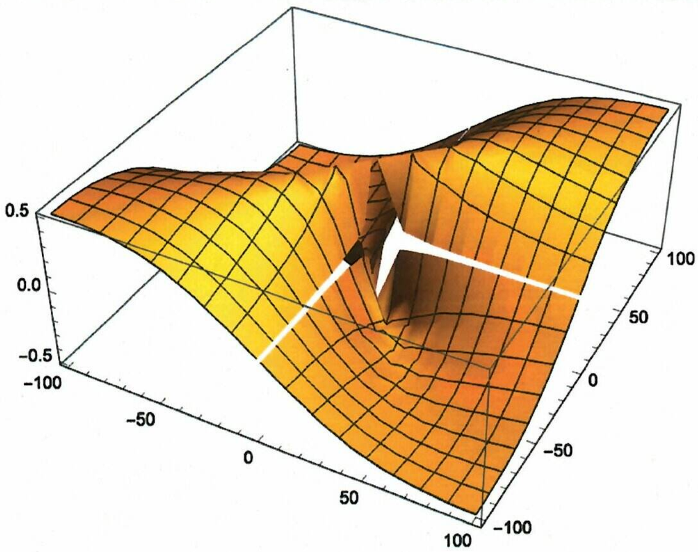
In[30]:=
```
Plot3D[x^2 * y / (x^2 + y^2), {x, -10, 10}, {y, -10, 10}, Exclusions → {x == 0, y == 0}]
```

Out[30]=

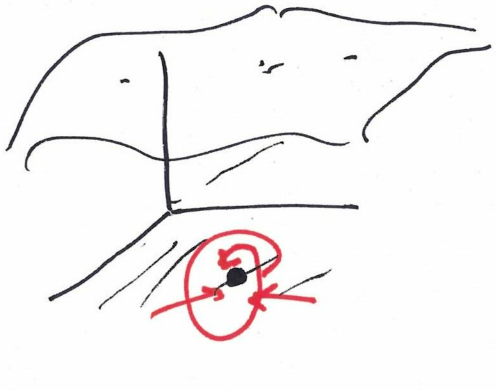


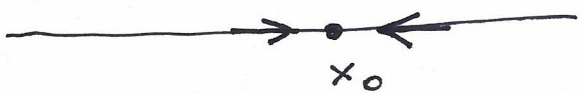
In[39]:=
```
Plot3D[x * y / (x^2 + y^2), {x, -100, 100}, {y, -100, 100}, Exclusions → {x == 0, y == 0}]
```

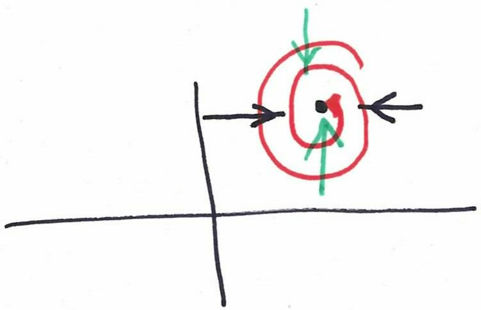
Out[39]=



One difficulty in working w/ functions of several variables is that the notion of the limit and the continuity is much stronger than for functions $f: \mathbb{R} \to \mathbb{R}$.

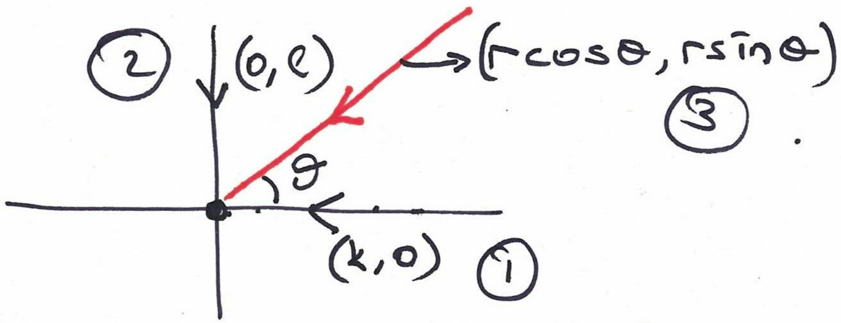






$f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} xy/x^2+y^2 & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$



① $(x_k, y_k) = (k, 0) \to (0,0)$

$f(x_k, y_k) = \frac{k \cdot 0}{k^2 + 0^2} = 0 \to 0.$

② $(x_\ell, y_\ell) = (0, \ell) \to (0,0)$

$f(x_\ell, y_\ell) = \frac{0 \cdot \ell}{0^2 + \ell^2} \to 0$

③ $(x_r, y_r) = (r\cos\theta, r\sin\theta) \to (0,0)$

$f(x_r, y_r) = \frac{(r\cos\theta)(r\sin\theta)}{r^2}$

$= \cos\theta \sin\theta$

for ex if $\theta = 45°$ it converges to $1/2$.

if $\theta = 60 \Rightarrow \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$

④. $\frac{\cos k}{k}, \frac{\sin k}{k} \to (0,0)$

$\Rightarrow f$ does not have a limit as $(x,y) \to (0,0)$.

For studying limits of functions $f : \mathbb{R}^n \to \mathbb{R}$ the following lemma is useful.

Lemma (Sandwich Lemma).

If $f, g, h : \mathbb{R}^n \to \mathbb{R}$ where $f(x) < g(x) < h(x)$ $\forall x \in \mathbb{R}^n$.

let $a \in \mathbb{R}^n$.

If $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$

then $\lim_{x \to a} g(x)$ also exists

and is equal to L.

<u>Ex</u>: $g(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$h(x,y) = \dfrac{x^2}{x^2+y^2}$ $\qquad g(x,y) = y\, h(x,y)$

$0 \leq \dfrac{x^2}{x^2+y^2} < 1$

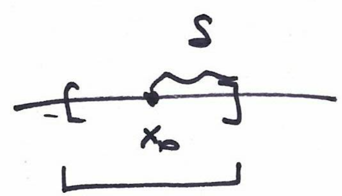
$\underset{\downarrow 0}{0} < |g(x,y)| < \underset{\downarrow 0}{|y|}$

Then by sandwich lemma.

$g(x,y) \to 0$

as $(x,y) \to (0,0)$

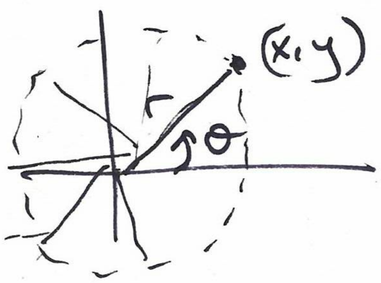
$\|x - x_0\| < \delta$.



$|x - x_0| \leq \delta$

<u>Rk</u>. Sometimes it is helpful to use polar coordinates especially with rational functions $f: \mathbb{R}^2 \to \mathbb{R}$.

$$f(x,y) = f(r\cos\theta, r\sin\theta)$$



$(x,y) = (r\cos\theta, r\sin\theta)$

$$f(x,y) = \frac{r^2\cos^2\theta \, r\sin\theta}{r^2(\cos^2\theta + \sin^2\theta)}$$

$(x,y) \neq (0,0)$

$$= r\cos^2\theta\sin\theta$$

$$\downarrow$$

$$0$$

Previous example.

$$f(x,y) = \frac{r^2\cos\theta\sin\theta}{r^2} = \cos\theta\sin\theta$$

Remember! Min-max thm.

If $f : [a,b] \to \mathbb{R}$.

compact interval

and continous.

then f takes its max and min. ie $\exists\, v^+ \in [a,b]$ s.t $f(x) \leq f(v^+)$ $\forall x \in [a,b]$

$\exists\, v^- \in [a,b]$ s.t

$f(x) \geq f(v^-)$ $\forall x \in [a,b]$

We need the analog of closed, compact intervals in $\mathbb{R}^n$.

Defn. 1) A sset $X \subset \mathbb{R}^n$ is **bounded** if the set $\{\|x\| \mid x \in \cancel{X}\}$ is bounded in $\mathbb{R}$.

2) A sset X of $\mathbb{R}^n$ is **closed** if for every sequence $(x_k)_k \subset X$ that converges in $\mathbb{R}^n$, converges to a point $y \in X$

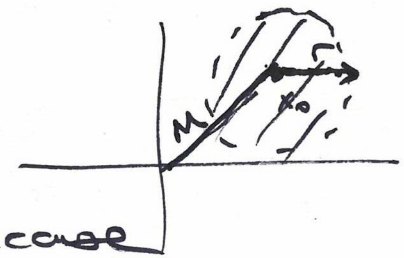
eg. $\left(\frac{1}{k}\right)_k \subset (0,1] = X \subset \mathbb{R}$.

$\frac{1}{k} \to 0$ but $0 \notin X$
X is not closed!

3) $X \subseteq \mathbb{R}^n$ called **compact** if it is closed and bounded.

Ex. 1) ϕ - empty set, $\mathbb{R}^n$ both closed.

2) $B_r(x_0) := \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$.



is bounded

Because if $x \in B_r(x_0)$

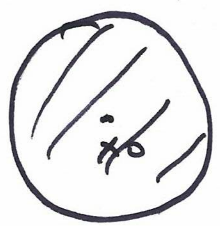
$$\|x\| \le \underbrace{\|x - x_0\|}_{<r} + \|x_0\| < r + \overbrace{\|x_0\|}^{M}.$$

It is not closed.

Because $(x_k) = \left(x_0 + \left(r - \frac{1}{k}, 0, \dots 0\right)\right)_k \subset B_r(x_0)$

$$x_k \xrightarrow[k\to\infty]{} x_0 + (\epsilon, 0, \dots, 0).$$

$$\overline{B_r(x_0)} := \{ x \in \mathbb{R}^n \mid \|x - x_0\| \leq r \}.$$



Rk. If $X \subset \mathbb{R}^n$, $Y \in \mathbb{R}^m$ are bounded (resp. closed resp. compact)

then $X \times Y = \{ (x,y) \in \mathbb{R}^{n+m} \mid x \in X, \ y \in Y \}$.

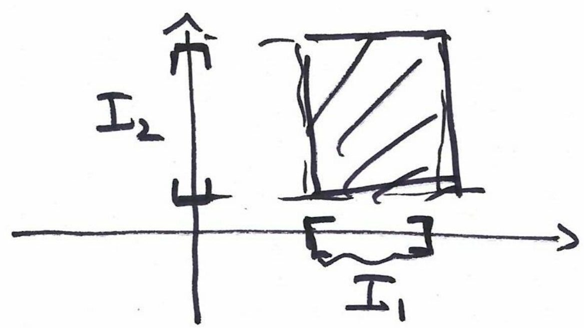
is bdd (resp. closed, compact) in $\mathbb{R}^{n+m}$.

In particular if

$$A = I_1 \times I_2 \times \dots \times I_n = \{ (x_1, \dots x_n) \in \mathbb{R}^n \mid x_i \in I_i \subset \mathbb{R} \}.$$

$I_i = [a_i, b_i]$. $1 \leq i \leq n$. intervals.

Then A is bdd and closed hence compact in $\mathbb{R}^n$.



Prop. If $f : \mathbb{R}^n \to \mathbb{R}^m$ continuous. Then for every $Y \subset \mathbb{R}^m$ closed the set $f^{-1}(Y)$ is closed.

$$f^{-1}(Y) = \{x \in \mathbb{R}^n \mid f(x) \in Y\} \subset \mathbb{R}^n$$

Inverse image of closed sets under continuous maps are closed.

Ex.: If $f : \mathbb{R}^n \to \mathbb{R}$ continuous then for any $a \leq b$

(*) $X : \{x \in \mathbb{R}^n \mid a \leq f(x) \leq b\}$ is closed.

$$X = f^{-1}([a,b])$$

Ex.: $X = \{(x,y,z) \in \mathbb{R}^3 \mid \cos(x^3 + e^{xy} + xyz) = 1\}$ is closed.

Let $f(x,y,z) = \cos(x^3 + e^{xy} + xyz)$

Then $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous.

$$X = f^{-1}(\{1\})$$

and single point $\{1\} \in \mathbb{R}$ is closed.

Same conclusion (*) also applies to ~~f(x~~

$\{x \in \mathbb{R}^n \mid f(x) \geq a\}$ or

$\{x \in \mathbb{R}^n \mid f(x) \leq b\}$.

<u>Warning !!</u> If f is continuous, then the set $\{x \in \mathbb{R}^n \mid a \leq f(x) \leq b\}$ is not always compact

$f^{-1}([a,b])$

inverse image of a closed interval hence closed.

but it is also inverse image of the compact interval $[a,b]$

But we can <u>NOT</u> say that it is compact.

eg. $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$

$(x,y,z) \longmapsto \sin(xyz)$

$\{(x,y,z) \mid -1 \leq f(x,y,z) \leq 1\}$.

$= f^{-1}([-1,1]) = \mathbb{R}^3$.

closed but not bounded

hence not compact.

Thm. (Min-Max thm for fnc of several variables.)

Let $X \subset \mathbb{R}^n$ compact set

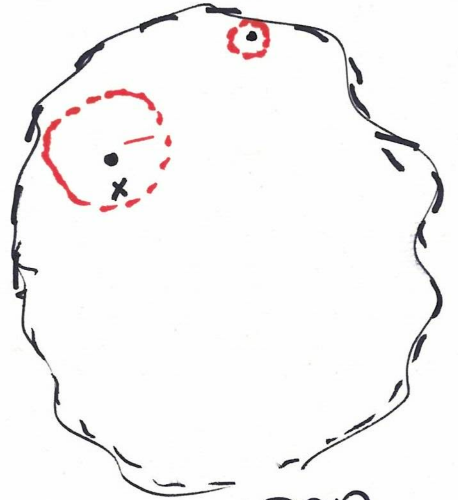
$f : X \longrightarrow \mathbb{R}$, a continuous function. Then f is bounded and attains its max and min. ie $\exists\ x^+ \in X$ and $x^- \in X$ s.t.

$$f(x^+) = \sup_{x \in X} f(x).$$

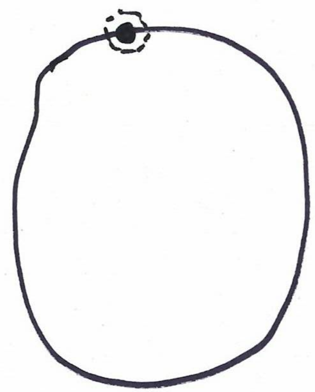
$$f(x^-) = \inf_{x \in X} f(x)$$

Defn. $X \subset \mathbb{R}^n$ is called open if its complement $\mathbb{R}^n \setminus X$ is closed.

This is equivalent to $\forall x \in X$, $\exists r > 0$ st the set $\{y \in \mathbb{R}^n \mid \|y-x\| < r\} = B_r(x) \subset X$



open.



not open

$\mathbb{R}$ (a,b) open interval is open.

Clicker question

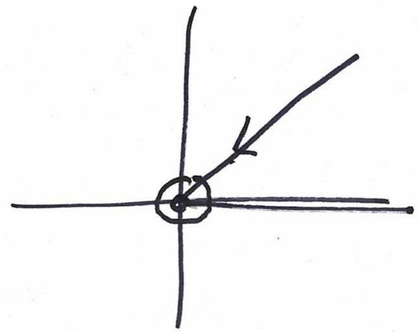
$f: \mathbb{R}^2 \to \mathbb{R}$.

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^3+y^3}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

True ?

False ?



$y = mx$

$m =$ slope.

If we approach $(0,0)$ along the line $y = mx$

$$f(x,y)\Big|_{\substack{\text{along the line} \\ y=mx}} = \frac{x^2 \cdot mx}{x^3 + (mx)^3} = \frac{m\cancel{x^3}}{(1+m^3)\cancel{x^3}} = \frac{m}{1+m^3}$$

In fact $\lim_{(x,y)\to(0,0)} f$ does <u>not</u> exist. $\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=mx}} f(x,y) = \frac{m}{1+m^3}$ depends on m.

Ex. 1) $(a,b) \subset \mathbb{R}$ is open

2) $[a,b) \subset \mathbb{R}$ neither open nor closed.

3) $\mathbb{R}^n$, $\emptyset$ both open.

4) $(a_1,b_1) \times (a_2,b_2) \subset \mathbb{R}^2$ open.

5) Prop
Inverse image of open sets under continous maps are open

§3.3. Partial derivatives.

Goal. is to define the analog of the derivative for $f: \mathbb{R}^n \to \mathbb{R}^m$. which we can then use to say something about

- how the function changes around a given point.
- How can we give an app. to the value of the function $f(x_0+h)$ if we know $f(x_0)$.

If $f: \mathbb{R} \longrightarrow \mathbb{R}^m$

$$x \longmapsto (f_1(x), \ldots, f_m(x))$$

$f_i: \mathbb{R} \longrightarrow \mathbb{R}$.

$$f(x) = (f_1(x), f_2(x), \ldots f_m(x)).$$

In this ~~the~~ derivative

$$f'(x) = \frac{df}{dx}(x) = (f_1'(x), f_2'(x), \ldots f_m'(x))$$

If $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

$$x \longrightarrow (f_1(x), \ldots f_m(x))$$

$f_i: \mathbb{R}^n \longrightarrow \mathbb{R}$.