22.10.20

- $f : \mathbb{R} \longrightarrow \mathbb{R}$.

  $$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$

- $f : \mathbb{R} \longrightarrow \mathbb{R}^n$

  $$x \longmapsto \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$

  Then $$f'(x_0) = \begin{pmatrix} f_1'(x_0) \\ f_2'(x_0) \\ \vdots \\ f_n'(x_0) \end{pmatrix}$$

  $f_i : \mathbb{R} \longrightarrow \mathbb{R}$.

- $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

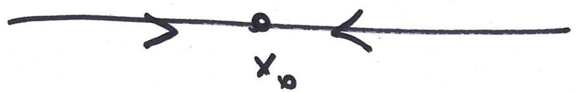
  $$x \longmapsto \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix}$$

  where each $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$. $1 \leq i \leq m$.

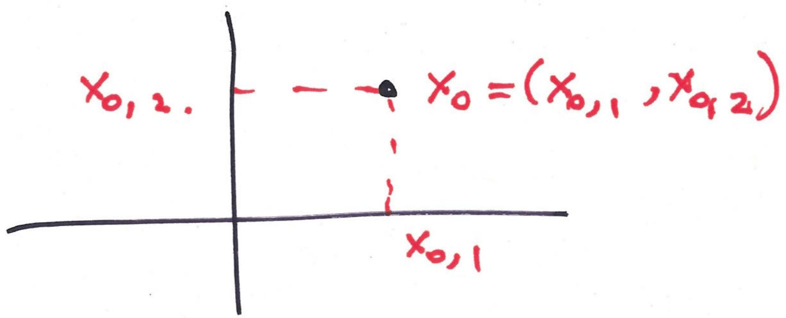
- How do we study a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ near a point $x_0 = (x_{0,1}, \dots, x_{0,n})$.

- How does f change near x_0?

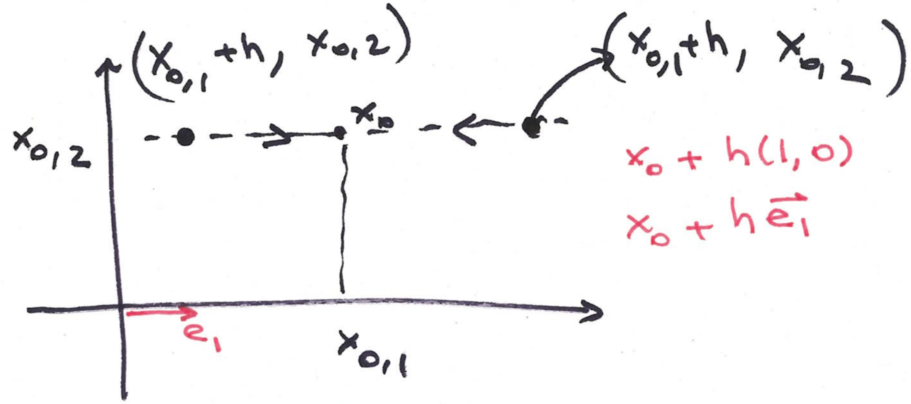
$f : \mathbb{R} \longrightarrow \mathbb{R}$



$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$



We can approach x_0 along the horizontal line



$$\frac{\Delta f}{\Delta x_1} = \frac{f(x_{0,1}+h, x_{0,2}) - f(x_{0,1}, x_{0,2})}{h}.$$

$$\| (x_{0,1}+h, x_{0,2}) - (x_{0,1}, x_{0,2}) \|.$$

$$= \| (h,0) \| = h.$$

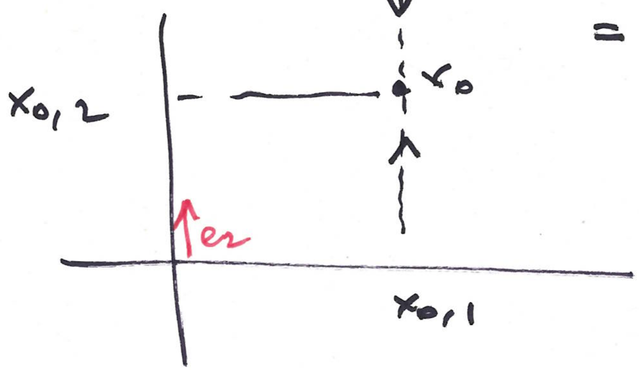
what happens if $h \to 0$.

$x^{o,l}$

$x^{o,r}$ $x^{o} = (x^{o,l}, x^{o,r})$

__or__ along the vertical line

$$(x_{0,1}, x_{0,2}+h) = x_0 + h\underbrace{(0,1)}_{e_2}$$



$$\frac{\Delta f}{\Delta x_2} = \frac{f(x_{0,1}, x_{0,2}+h) - f(x_{0,1}, x_{0,2})}{h}.$$

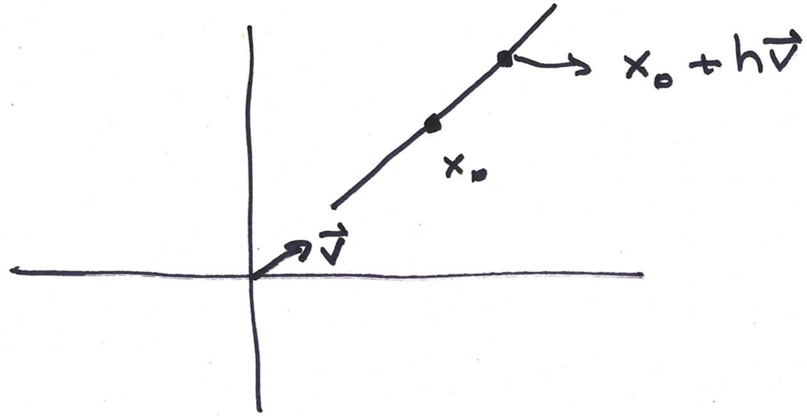
What we are doing is really to consider the following function of one variable:

$$g(t) := f(x_{0,1}, t)$$

and look at $g'(t)\Big|_{t = x_{0,2}}.$

$\overset{\text{or}}{=}$.

$$\frac{d}{dh} f(x_{0,1}, x_{0,2}+h)\Big|_{h=0}.$$



$$\frac{f(x_0 + h\vec{v}) - f(x_0)}{h} \xrightarrow[h\to 0]{} \;?$$

General case:

Let $f : \cancel{X} \subset \mathbb{R}^n \longrightarrow \mathbb{R}$

we want to study f around $x_0 = (x_{0,1}, \dots, x_{0,n})$.

For each j, we consider

$$g_j(t) := f(x_{0,1}, x_{0,2}, \dots x_{0,j-1}, t, x_{0,j+1}, \dots x_{0,n})$$

$g_j : \mathbb{R} \longrightarrow \mathbb{R}$

defined on the set

$$I := \{t \in \mathbb{R} \mid (x_{0,1}, \dots, t, \dots x_{0,n}) \in X\}.$$

Then we ask if

$\dfrac{dg_j}{dt}(x_{0,j})$ exists?

$$\frac{dg}{dt}(x_{0,j}) = \lim_{h \to 0} \frac{g(x_{0,j}+h) - g(x_{0,j})}{h}.$$

$$= \lim \frac{\left(f(x_{0,1}, \dots, x_{0,j-1}, x_{0,j}+h, x_{0,j+1}, \dots, x_{0,n}) - f(x_{0,1}, \dots x_{0,j}, \dots x_{0,n})\right)}{h}.$$

If the limit exists then we say that f has partial derivative with respect to x_j at the point x_0, and we write $\dfrac{\partial f}{\partial x_j}(x_0)$, $(\partial_{x_j} f)(x_0)$ or $(\partial_j f)(x_0)$

If $f : \mathbb{R}^n \to \mathbb{R}^m$, $x_0 \in \mathbb{R}^n$

then $g_j(t) := f(x_{0,1}, \dots, x_{0,j-1}, t, x_{0,j+1}, \dots x_{0,n})$

$$\begin{pmatrix} g_{j1} \\ \vdots \\ g_{jm} \end{pmatrix} = \begin{pmatrix} f_1(x_{0,1} \dots x_{0,j-1}, t, x_{0,j+1}, \dots x_{0,n}) \\ f_2(- \quad - \quad - \quad - \quad -) \\ \vdots \\ f_m(- \quad - \quad - \quad - \quad -) \end{pmatrix}$$

$$g_j : \mathbb{R} \to \mathbb{R}^m : \begin{pmatrix} g_{j1}(t) \\ \vdots \\ g_{jm}(t) \end{pmatrix}$$

$$g_j'(x_{0,j}) = \begin{pmatrix} g_{j1}'(x_{0,j}) \\ g_{j2}'(x_{0,j}) \\ \vdots \\ g_{jm}'(x_{0,j}) \end{pmatrix} =: \frac{\partial f}{\partial x_j}(x_0).$$

Rk. To evaluate partial derivatives of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ with respect x_j at a point $a = (a_1, \dots a_n)$ we differentiate

$$f(a_1, \dots a_{j-1}, x_j, a_{j+1}, \dots a_n)$$

wrt x_j treating all other variables as a constant with respect to x_j.

Ex: $f : \mathbb{R}^2 \to \mathbb{R}$

$(x, y) \mapsto (x^2 + xy)\sin y$

$$\frac{\partial f}{\partial x} = \sin y \,(2x + y)$$

$$\frac{\partial f}{\partial y} = x \sin y + (x^2 + xy)\cos y.$$

2) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$(x,y) \longmapsto \begin{pmatrix} x^2+y^2 \\ 2x \\ 2y \end{pmatrix}$$

$f_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$(x,y) \longmapsto x^2+y^2$$

$f_2(x,y) = 2x \qquad f_3(x,y) = 2y$

$$\frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x} \\ \frac{\partial f_2}{\partial x} \\ \frac{\partial f_3}{\partial x} \end{pmatrix} = \begin{pmatrix} 2x \\ 2 \\ 0 \end{pmatrix} \qquad \frac{\partial f}{\partial y} = \begin{pmatrix} 2y \\ 0 \\ 2 \end{pmatrix}$$

Say at $(x,y) = (1,1)$ $\quad \frac{\partial f}{\partial x}(1,1) = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$

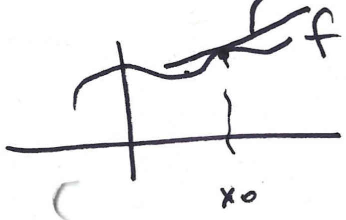
What does the partial derivatives mean geometrically?

Say: $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

graph f : $\{(x,y,z) \mid z = f(x,y)\}$

Recall $f : \mathbb{R} \longrightarrow \mathbb{R}$

graph f : $\{(x,y) \mid y = f(x)\}$.

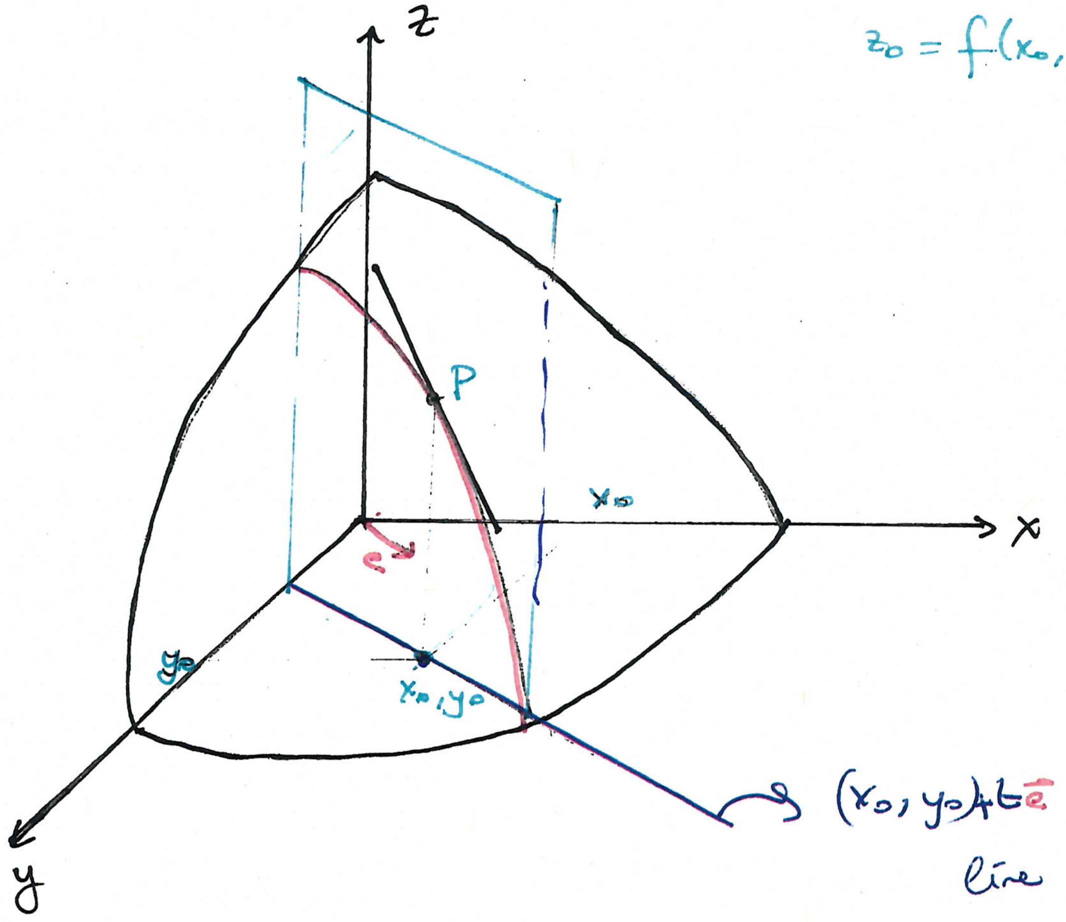


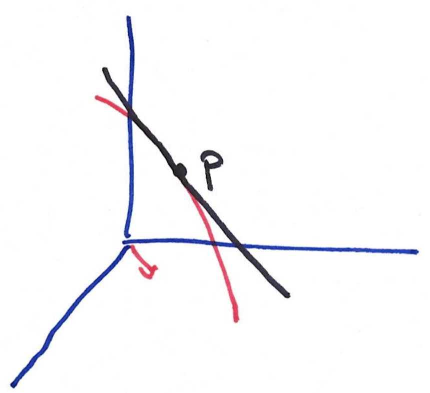
slope of the tangent line is $f'(x_0)$.

Directional derivative of f at x_0 in the direction of the vector $\vec{e}$.

$P = (x_0, y_0, z_0)$

$z_0 = f(x_0, y_0)$





$$\text{slope} = f_e(x_0, y_0) = d_e f(x_0, y_0) = \frac{d}{dt} f\big((x_0, y_0) + t\vec{e}\big)\Big|_{t=0}$$

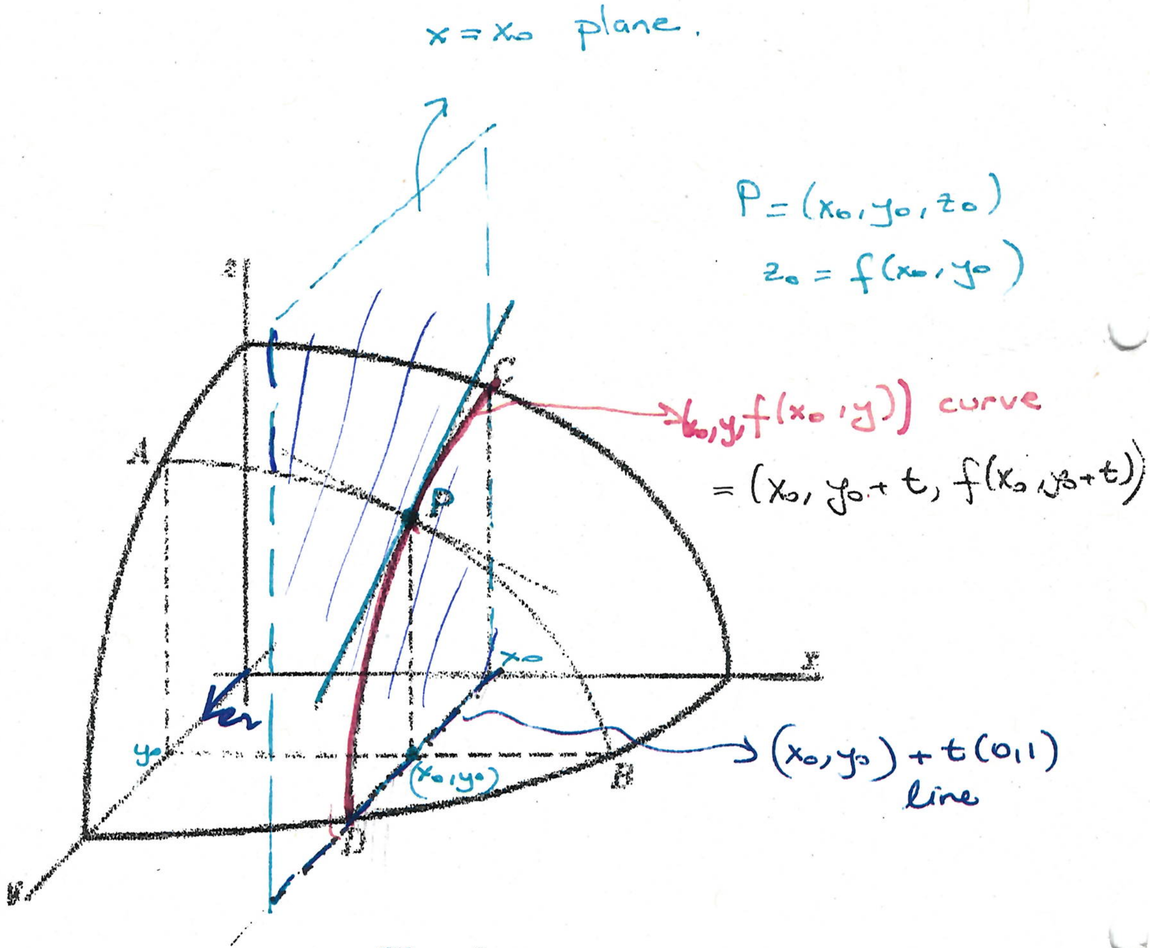
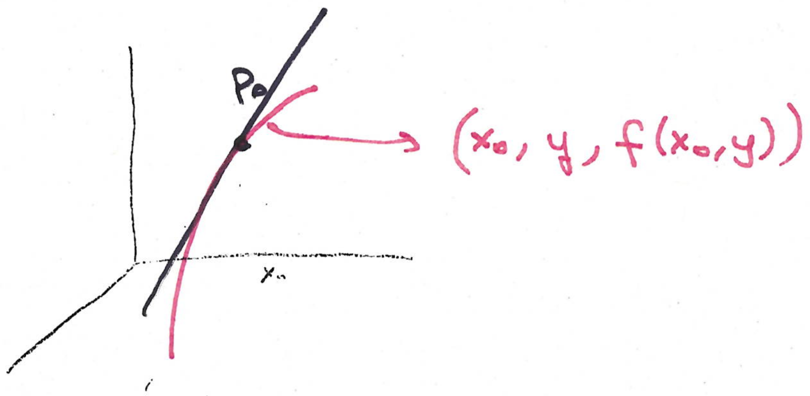


Fig. 1

$$\text{slope} = \frac{\partial f}{\partial y}(x_0, y_0) = \frac{d}{dt}\left(f(x_0, y_0 + t)\right)\Big|_{t=0}.$$



$f : \mathbb{R}^2 \to \mathbb{R}$.

$\mathrm{graph}\, f = \{(x, y, z) \mid z = f(x, y)\}$.

$z_0 = f(x_0, y_0)$

$P = (x_0, y_0, z_0)$

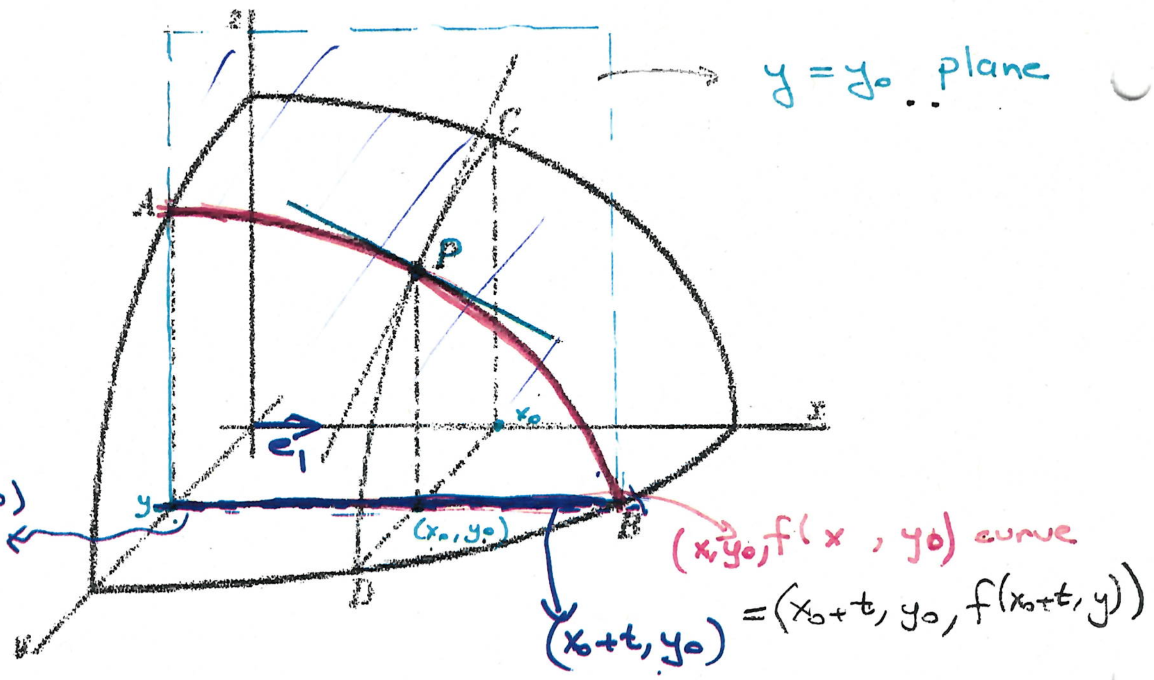
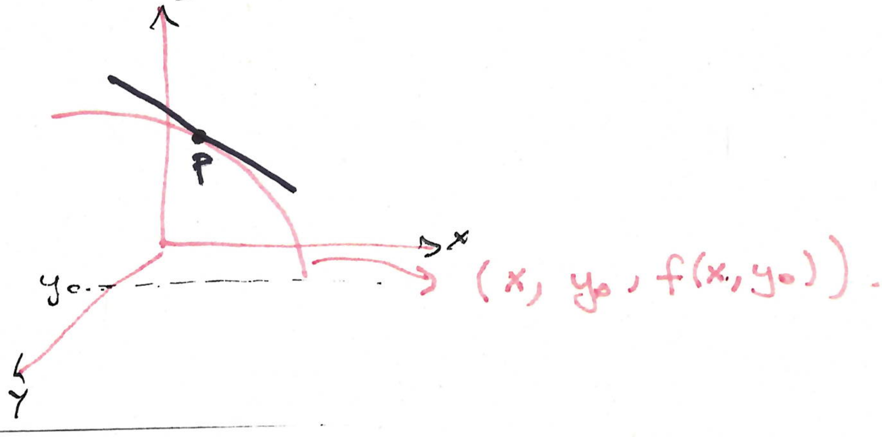


Fig. 1

$$\text{slope} = \frac{\partial f}{\partial x}(x_0, y_0) = \frac{d}{dt}\left(f(x_0 + t, y_0)\right)\Big|_{t=0}.$$



Examples. $f : \mathbb{R}^n \to \mathbb{R}$

③ $f(x_1, \dots x_n) := a_1 x_1 + \dots + a_n x_n$ for some constants $a_1 \dots a_n \in \mathbb{R}$

$$\frac{\partial f}{\partial x_j} = a_j$$

④ $f : \mathbb{R}^n \to \mathbb{R}^m$
$x \mapsto Ax = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$

A is a $m \times n$ matrix in $\mathbb{R}^{m \times n}$.

$$f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} x_1 + \dots + a_{1n} x_n \\ \vdots \\ a_{m1} x_1 + \dots + a_{mn} x_n \end{pmatrix} = Ax$$

$$\frac{\partial f}{\partial x_j}(x) = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}.$$

$f_i(x) = a_{i1} x_1 + \dots + a_{in} x_n.$

$$\left(\frac{\partial f_i}{\partial x_j}\right)_{\substack{1 \le i \le m \\ 1 \le j \le n}} = A.$$

Defn. Let $X \subset \mathbb{R}^n$ open $f : X \to \mathbb{R}^m$ with partial derivatives on X

write $f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix}$

For any $x \in X$ the matrix

$$J_f(x) = \left(\frac{\partial f_i}{\partial x_j}\right)_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

$J_f(x)$ is a matrix of m rows and n columns.

ie it is a $m \times n$ Matrix

$$f : \underline{X} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

It is called the <u>Jacobi Matrix of f</u> at <u>point x</u>

<u>Defn</u> In the special case

$$f : X \longrightarrow \mathbb{R} \qquad X \subset \mathbb{R}^n$$

the column vector

$$\begin{pmatrix} \frac{\partial f}{\partial x_1}(x_0) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x_0) \end{pmatrix}$$

is called the <u>Gradient</u> of f at x_0

It is denoted $(\nabla f)(x_0)$

Note $(\nabla f)(x_0) = \left(\frac{\partial f}{\partial x_1}(x_0), \dots \frac{\partial f}{\partial x_n}(\right.$

$$= \left(J_f(x_0)\right)^t$$

<u>Properties</u> . 1) If $f, g : X \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ X open have partial derivatives at x_0 <u>wrt</u> variable x_j in $\underline{X}$

with respect to

then so does $f+g$

$$\frac{\partial (f+g)}{\partial x_j}(x_0) = \frac{\partial f}{\partial x_j}(x_0) + \frac{\partial g}{\partial x_j}(x_0)$$

/11

2) If $f, g : \mathbb{R}^n \to \mathbb{R}$ have parh. derivatives at x_0 wrt x_j then so does fg and if $g(x) \neq 0$ then also f/g.

$$\frac{\partial(fg)}{\partial x_j} = \frac{\partial f}{\partial x_j} \cdot g + \frac{\partial g}{\partial x_j} \cdot f$$

$$\frac{\partial(f/g)}{\partial x_j} = \frac{\frac{\partial f}{\partial x_j} \cdot g - \frac{\partial g}{\partial x_j} \cdot f}{g^2}$$

<u>Rk</u>. Given $f : X \to \mathbb{R}^m$, $X \subset \mathbb{R}^n$. Assume partial derivatives of f exist everywhere. Then $(\partial_{x_i} f)(x)$ are themselves new functions

we can differentiate wrt (once again) either x_i or some other x_j

$$\partial_{x_j}(\partial_{x_i} f) = \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} f$$

$$\partial_{x_i}(\partial_{x_i} f) = \frac{\partial^2 f}{\partial x_i^2}$$

Ex: $f(x,y,z) = x^3 y z^2 + \cos x + 8$

$$\frac{\partial f}{\partial x} = 3x^2 y z^2 - \sin x$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 3x^2 z^2$$

$$\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = 6x^2 y z$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 6xyz^2 - \cos x$$

Ex: $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(0+h, 0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}.$$

$$= \lim_{h \to 0} \frac{0-0}{h} = 0.$$

$\frac{\partial f}{\partial x}(0,0)$ exists and is equal to 0.

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h \to 0} \frac{f(0, 0+h) - f(0,0)}{h}.$$

$$= 0.$$

In fact, we can look at directional derivatives in all directions.

let $v = (a,b) \neq (0,0)$

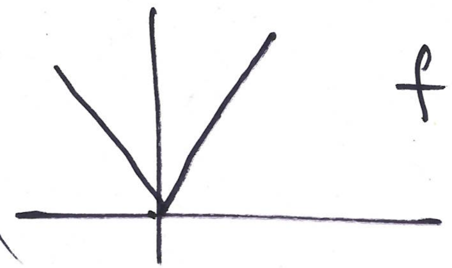
$$g(h) := f((0,0) + h\vec{v})$$
$$= f(ha, hb)$$
$$= \frac{h^2ab}{h^2(a^2+b^2)} = \frac{ab}{a^2+b^2}$$

Direc. der. of f in the direction of v

$$= g'(0) = \frac{d}{dh}\Big(\underbrace{\frac{ab}{a^2+b^2}}_{\text{constant}}\Big) = 0.$$

Recall. If $f: \mathbb{R} \to \mathbb{R}$ is differentiable at a point x_0 then f is continuous at x_0!

f cont at x_0 $\not\Rightarrow$ f is diff. at x_0.



$f(x) = |x|$ cont at 0 but not diff.

Rk: This last example has all directional derivatives at $(0,0)$ and they are all equal to 0.
Yet we know that f is not continuous at $(0,0)$.
This suggests that partial derivatives are not strong enough to take as an analog of the derivative from 1-variable case.

Let's look at the case of $f : \mathbb{R} \to \mathbb{R}$ once again.

Recall: $f : \mathbb{R} \to \mathbb{R}$ is in x_0 differentiable if it can be well approximated by the affine linear map

$$A(x) = f(x_0) + f'(x_0)(x - x_0)$$

when x is near x_0.

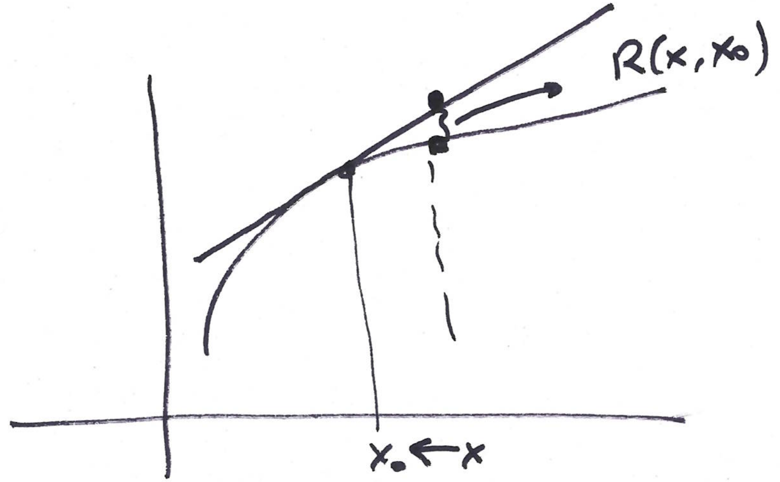
Well approximated means that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + R(x, x_0)$$

$$\iff \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{|x - x_0|} = \frac{R(x, x_0)}{|x - x_0|}$$

with $\lim\limits_{x \to x_0} \dfrac{R(x,x_0)}{|x - x_0|} = 0$.

ie $R(x,x_0)$ goes to zero faster than $|x - x_0|$ as $x \to x_0$.



The number $f'(x_0) = A$ can be thought as a 1×1 matrix A which represents the linear map

$$u : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto f'(x_0)\, x.$$

Defn. Let $X \subset \mathbb{R}^n$ open $x_0 \in X$, $f : X \longrightarrow \mathbb{R}^m$ a function. We say f is differentiable at x_0 if there exists a linear map $u : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that

$$\lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{f(x) - f(x_0) - \widetilde{u(x - x_0)}}{\|x - x_0\|} = 0$$

↳ limit here is in $\mathbb{R}^m$.

The linear map

$u : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

is called the total differential of f at x_0 and is denoted by

$df(x_0)$ or $d_{x_0} f$

Rk For $f : \mathbb{R}^n \to \mathbb{R}^m$ the total differential is **NOT** a number

It **is** a linear map!!!

Recall from lin. algebra

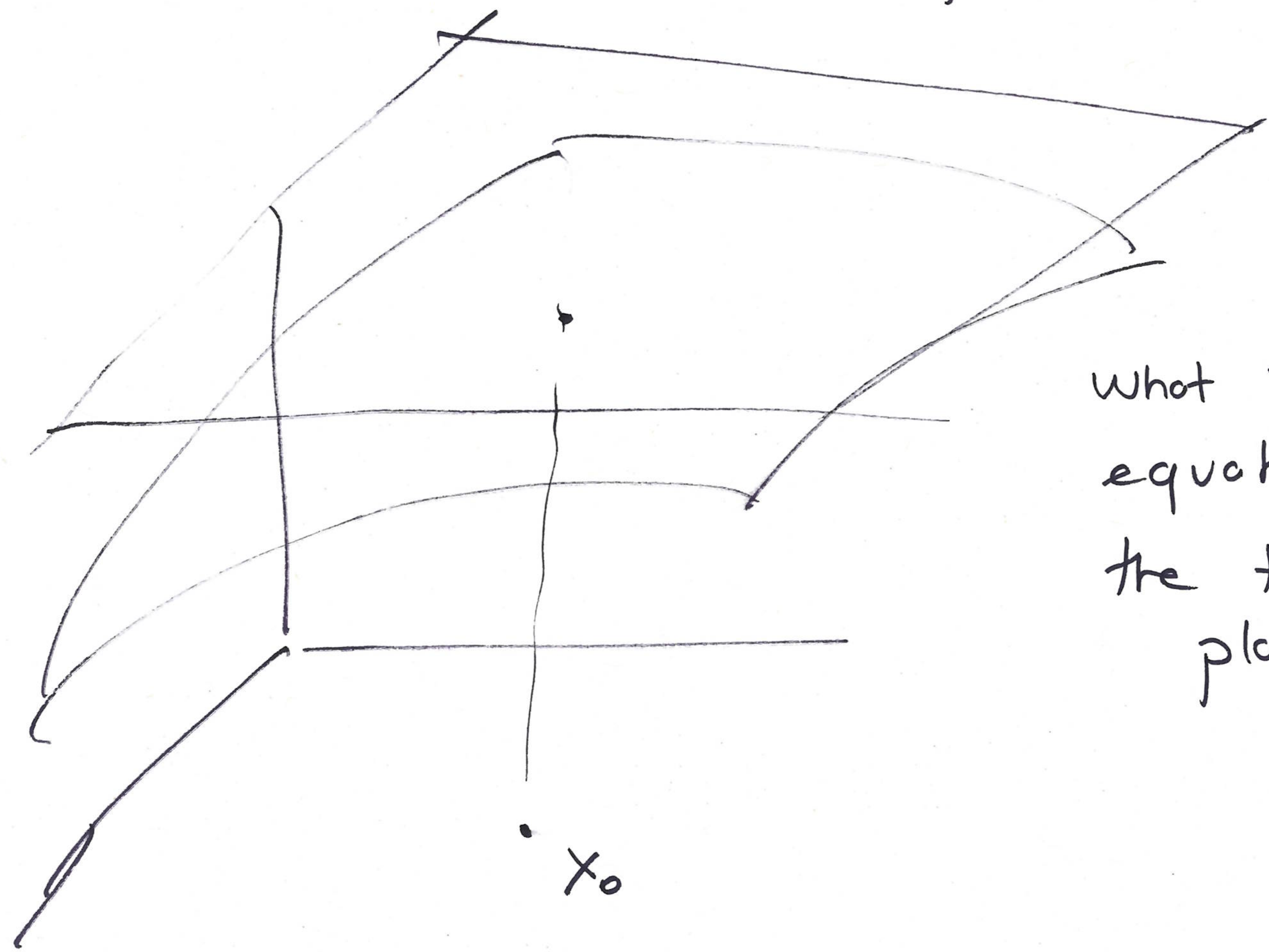
If $u : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a lin. map.

Then we can find a matrix A that represents this lin. map. A is a $m \times n$ matrix.

Question: What is the matrix representation of the differential

$df(x_0) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$



what is the equation of the tangent plane?