29-10-2020

Let $X \subset \mathbb{R}^n$ open $x_0 \in X$, $f: X \to \mathbb{R}^m$ a function. We say f is differentiable at x_0 if there exists a linear function $u: \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{f(x) - f(x_0) - u(x - x_0)}{\|x - x_0\|} = 0$$

The linear map $u: \mathbb{R}^n \to \mathbb{R}^m$ is called the differential of f at x_0 and is denoted by $df(x_0)$ or df_{x_0}. $df(x_0): \mathbb{R}^n \to \mathbb{R}^m$ is a linear map not a number!!

$f: X \to \mathbb{R}^m$ diff in x_0 if $\exists$ a lin map $u: \mathbb{R}^n \to \mathbb{R}^m$ s.t

$$f(x) = \underbrace{f(x_0) + u(x - x_0)}_{\text{affine function}} + E(f, x; x_0)$$

such that

$$\lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{E(f, x; x_0)}{\|x - x_0\|} = 0.$$

$$\boxed{\text{I.e } f \text{ is well approximated around } x_0 \text{ by the lin. map } u: \mathbb{R}^n \to \mathbb{R}^m.}$$

Rk. We have seen that existence of partial derivatives does NOT imply that f is continous.

Question) Is differentiability of f at x_0 enough to imply that f is continous at x_0?

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Question 2) If we know that f is differentiable at x_0, How do we find the linear map $df(x_0) : \mathbb{R}^n \to \mathbb{R}^m$ that approximates f well near $x = x_0$?

Thm Let $f : X \to \mathbb{R}^m$, $X \subset \mathbb{R}^n$ be diff at $x_0 \in X$ then we have

1. f is continuous at x_0.
2. f has all partial derivatives at x_0 and the matrix represents the linear map $df(x_0) : \mathbb{R}^n \to \mathbb{R}^m$, $x \mapsto Ax$ in the canonical basis

is given by the Jacobi Matrix of f at x_0.

ie. $A = J_f(x_0) = \left(\dfrac{\partial f_i}{\partial x_j} \right)$, $1 \leq i \leq m$, $1 \leq j \leq n$.

A is a $m \times n$ Matrix.

In the special case $m = 1$

ie $f : \mathbb{R}^n \to \mathbb{R}$

$df(x_0) : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto J_f(x_0)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$J_f(x_0) = \cancel{\left(\frac{\partial f}{\partial x_1}(x_0), \frac{\partial f}{\partial x_2}(x_0) \cdots \frac{\partial f}{\partial x_n}(x_0) \right.}$

$= \left(\dfrac{\partial f}{\partial x_1}(x_0) \cdots \dfrac{\partial f}{\partial x_n}(x_0) \right)$

$$\nabla f(x_0) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x_0) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x_0) \end{pmatrix} = J_f(x_0)^t$$

$$(df)(x_0) : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \left(\frac{\partial f}{\partial x_1}(x_0) \cdots \frac{\partial f}{\partial x_n}(x_0) \right) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\left[(df)(x_0)\right](x) = \frac{\partial f}{\partial x_1}(x_0) \cdot x_1 + \cdots + \frac{\partial f}{\partial x_n}(x_0)\, x_n$$

Ex: $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$(x,y,z) \longmapsto e^x y + zx$$

$\overline{x_0} = (0,1,2)$

$$J_f(x_0) = \left(\frac{\partial f}{\partial x}(x_0), \frac{\partial f}{\partial y}(x_0), \frac{\partial f}{\partial z}(x_0) \right)$$

$$\left.\frac{\partial f}{\partial x}\right|_{\overline{x_0}} = \left. e^x y + z \right|_{x_0} = 3 \qquad \frac{\partial f}{\partial y} = \left. e^x \right|_{x_0} = 1$$

$$\frac{\partial f}{\partial z} = \left. x \right|_{\overline{x_0}} = 0.$$

$$J_f(\overline{x_0}) = (3 \quad 1 \quad 0)$$

$$\nabla f(\overline{x_0}) = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

$$df(\bar{x}_0) : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

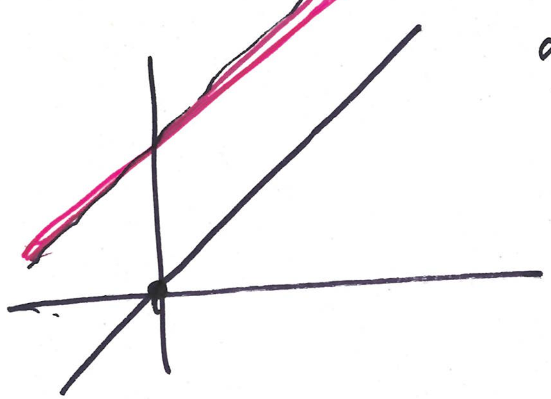
$$df(\bar{x}_0)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3x + y.$$

$$df(\bar{x}_0) : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

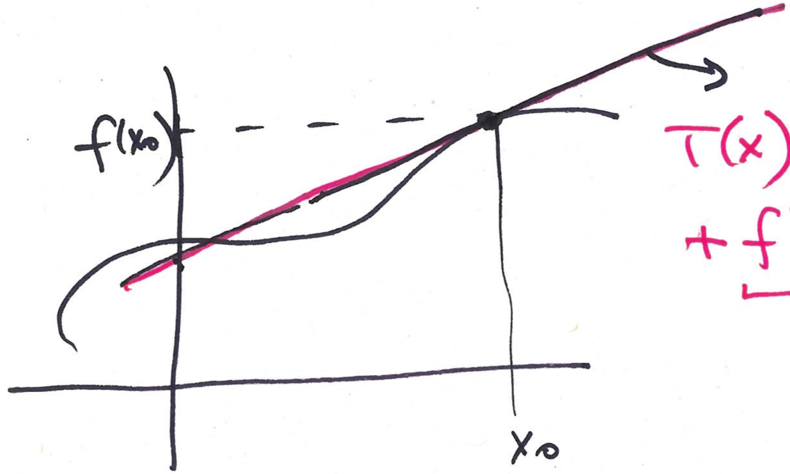
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto 3x + y$$

<u>Recall</u>. $L : \mathbb{R} \longrightarrow \mathbb{R}$ linear map has the form $L(x) = ax$ for some $a \in \mathbb{R}$

$T(x) = ax + b$



graph of L has to go thru $(0,0)$



$T(x) = f(x_0) + \underline{f'(x_0)(x - x_0)}$

$$df(x_0) : \mathbb{R} \longrightarrow \mathbb{R}.$$

$$x \longmapsto f'(x_0) \cdot x$$

Properties of the differential

3 $f, g : X \to \mathbb{R}^m$ are diff in x_0 then so is $f+g$ and

$$d(f+g)(x_0) = df(x_0) + dg(x_0).$$

sum of 2 lin. maps.

$df(x_0) : X \to \mathbb{R}^m$

$dg(x_0) : X \to \mathbb{R}^m$.

4 If $m=1$, and $f, g : \mathbb{R}^n \to \mathbb{R}$ diff in x_0 then so is $fg : \mathbb{R}^n \to \mathbb{R}$.

and if $g \neq 0$ then also f/g is diff in x_0.

Recall we've seen that

partial derivatives exist at x_0 $\not\Rightarrow$ f is diff.

$\not\Rightarrow$ f is continuous

Is there a way to use partial derivatives to claim diff. of f?

5 **Thm** If $f : X \to \mathbb{R}^m$, $X \subset \mathbb{R}^n$ has all partial derivatives $\frac{\partial f_i}{\partial x_j} : X \to \mathbb{R}^m$ and if these functions are continuous on X then f is diff. on X.

Partial deriv. exist $\not\Longrightarrow$ f diff.

+ they are continous $\Longrightarrow$ f diff.

Thm 5 says that in practice many functions are differentiable

eg all functions given by polynomials.

Ex: $f(x,y) = \begin{pmatrix} x^2+y^2 \\ 2x \\ 2y \end{pmatrix}$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$\frac{\partial f}{\partial x} = \begin{pmatrix} 2x \\ 2 \\ 0 \end{pmatrix}$ $\quad \frac{\partial f}{\partial y} = \begin{pmatrix} 2y \\ 0 \\ 2 \end{pmatrix}$

continuous everywhere,

$\Longrightarrow$ f is diff everywhere.

Thm 5

Say $x_0 = (1, 2)$

$df(x_0) : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto J_f(x_0) \begin{pmatrix} x \\ y \end{pmatrix}$

$J_f(x_0) = \begin{pmatrix} 2 & 4 \\ 2 & 0 \\ 0 & 2 \end{pmatrix}$

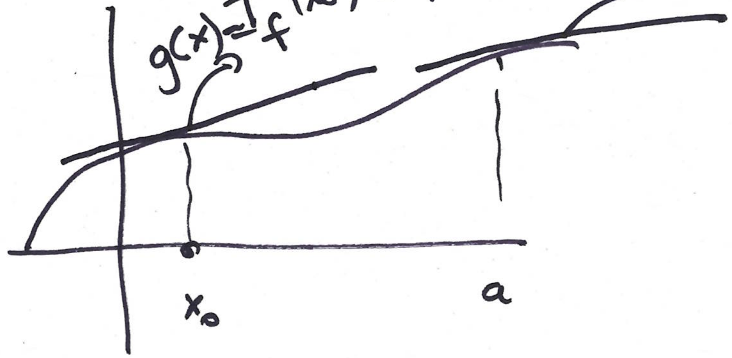
i.e. $df(x_0) : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 2 & 4 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 4y \\ 2x \\ 2y \end{pmatrix}$$

$f - \mathbb{R} \longrightarrow \mathbb{R}$

$g(x) = T_f(x_0) = f(x_0) + f'(x_0)(x - x_0)$

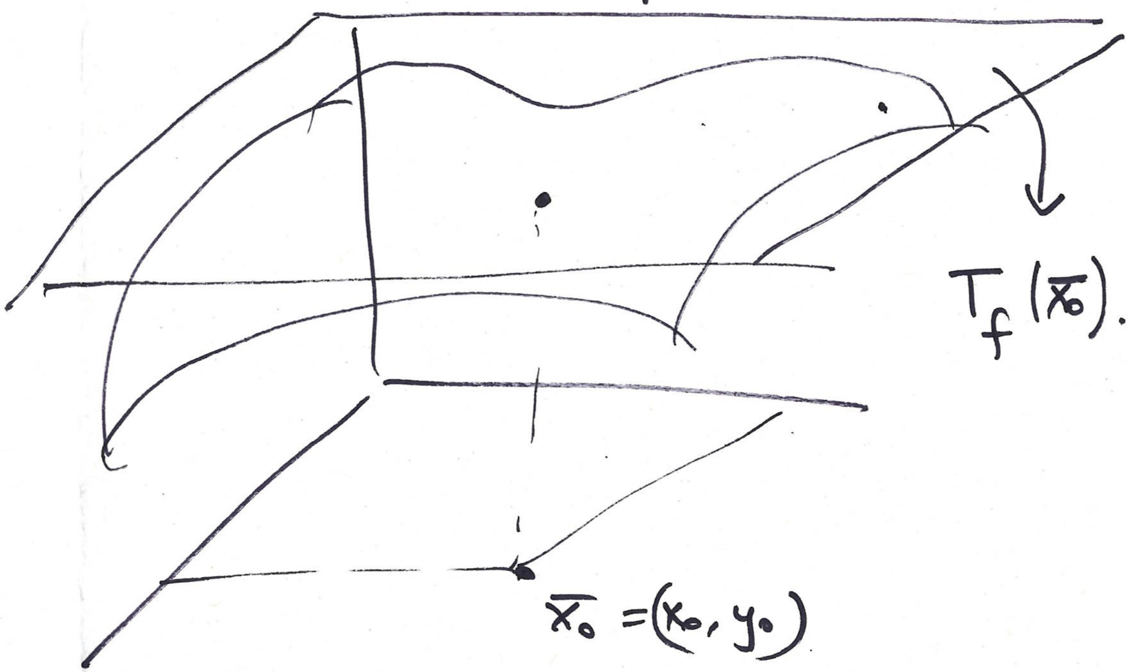
$T_f(a)$



y $a = (0,1)$

$$J_f(0,1) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$f = \mathbb{R}^2 \longrightarrow \mathbb{R}$,



$T_f(\overline{x_0})$.

$\overline{x_0} = (x_0, y_0)$

$df(a) : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 2 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ 2x \\ 2y \end{pmatrix}$$

Defn. $X \subset \mathbb{R}^n$ open

$f : X \to \mathbb{R}^m$ diff at x_0

with differential

$df(x_0) = u = \mathbb{R}^n \to \mathbb{R}^m$.

The graph of the affine linear approximation

$g : \mathbb{R}^n \to \mathbb{R}^m$

$$g(x) = f(x_0) + u(x - x_0)$$

is called the <u>Tangent space at x_0 to the graph of f</u>.

ie $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y = f(x_0) + u(x - x_0)\}$

In particular if $f : \mathbb{R}^2 \to \mathbb{R}$. diff at $\overline{x_0} = (x_0, y_0)$.

$df(x_0) = u : \mathbb{R}^2 \to \mathbb{R}$.

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \left(\frac{\partial f}{\partial x}(x_0, y_0) \;\; \frac{\partial f}{\partial y}(x_0, y_0) \right) \begin{pmatrix} x \\ y \end{pmatrix}$$

$$g(x, y) = f(x_0, y_0) + \nabla f(x_0, y_0) \cdot (\overline{x} - \overline{x_0})$$

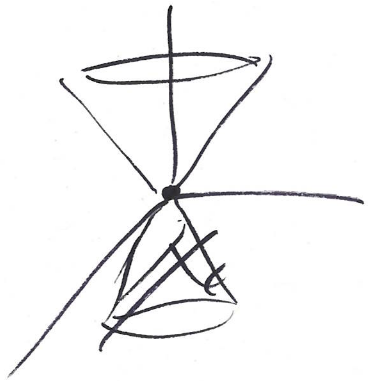
$$= f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

$\overline{x} = (x, y)$ $\qquad \overline{x_0} = (x_0, y_0)$

Ex: $f : \mathbb{R}^2 \longrightarrow \mathbb{R}.$

$$(x,y) \longmapsto \sqrt{x^2+y^2}$$

$$(x_0, y_0) = (3,4)$$



$$\nabla f(3,4) = \begin{pmatrix} ? \\ ? \end{pmatrix}$$

$$J_f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$$

$$\cancel{J_f}(3,4) = \left(\frac{3}{5}, \frac{4}{5}\right)$$

$$f(3,4) = 5$$

$$g(x,y) = f(3,4) + \begin{pmatrix} 3/5 & 4/5 \end{pmatrix}\begin{pmatrix} x-3 \\ y-4 \end{pmatrix}$$

See the picture. $\boxed{z = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4)}$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}.$

$$J_f(\bar{x}_0) = \begin{pmatrix} a & b \end{pmatrix}$$

$$\nabla f(\bar{x}_0) = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$df(x_0) : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} a & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

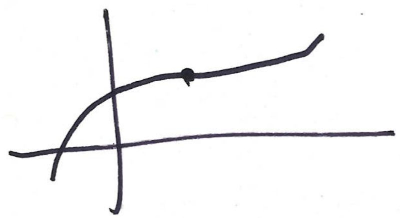
matrix multiplication

$$df(x_0)\begin{pmatrix} x \\ y \end{pmatrix} = ax + by$$

$$= \begin{pmatrix} a \\ b \end{pmatrix} \cdot (x, y)$$

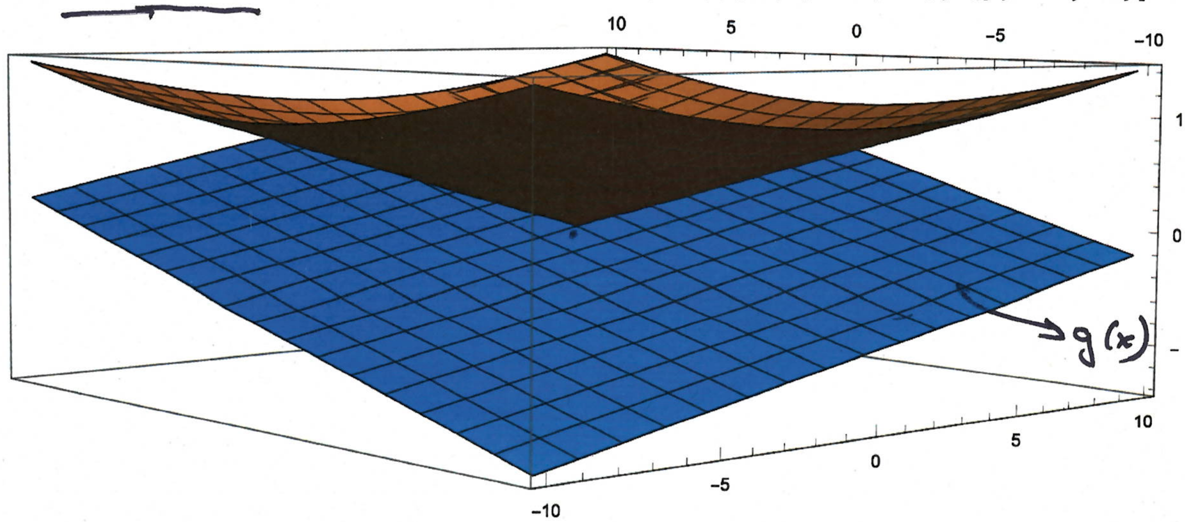
scalar product of 2 vectors.

$$= \left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle$$



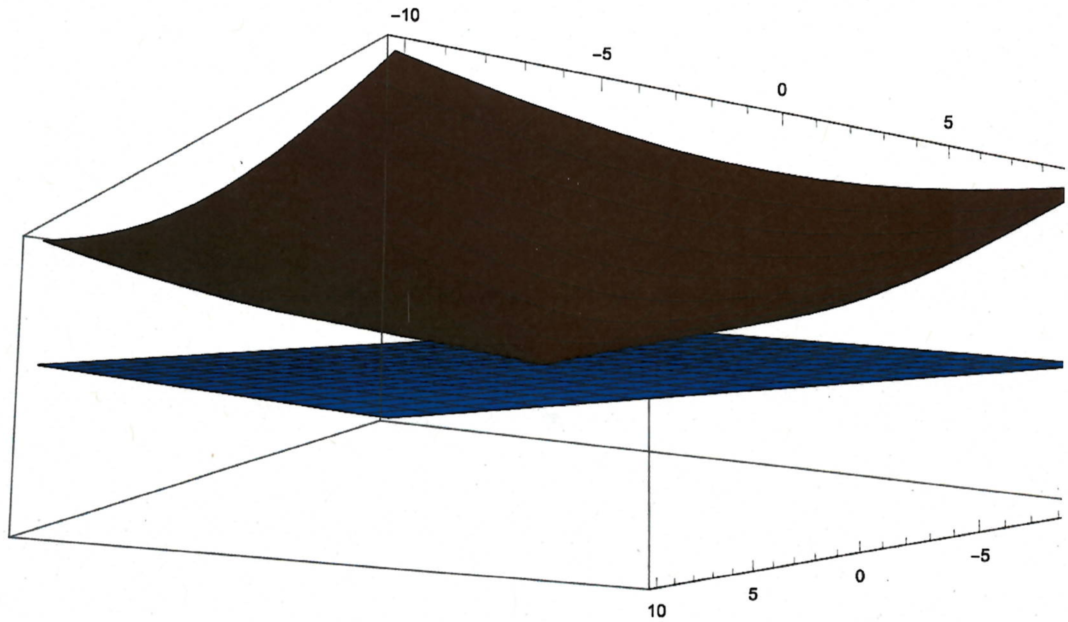
In[15]:=
```
Plot3D[{Sqrt[x^2 + y^2], 5 + (3 / 5) (x - 3) + (4 / 5) (y - 4)}, {x, -10, 10}, {y, -10, 10}]
```

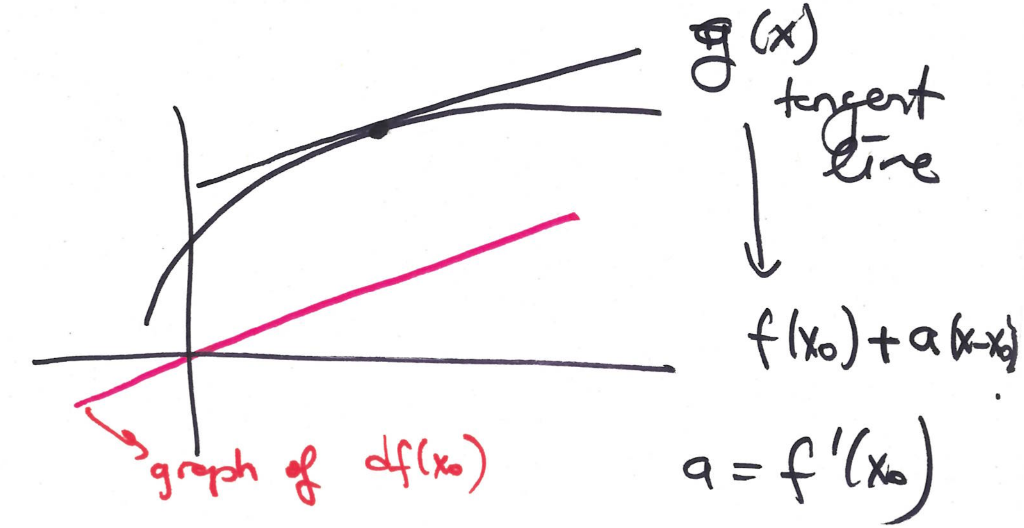
Out[15]=



In[16]:=
```
Plot3D[{Sqrt[x^2 + y^2], 5 + (3 / 5) (x - 3) + (4 / 5) (y - 4)}, {x, -10, 10}, {y, -10, 10}]
```

Out[16]=



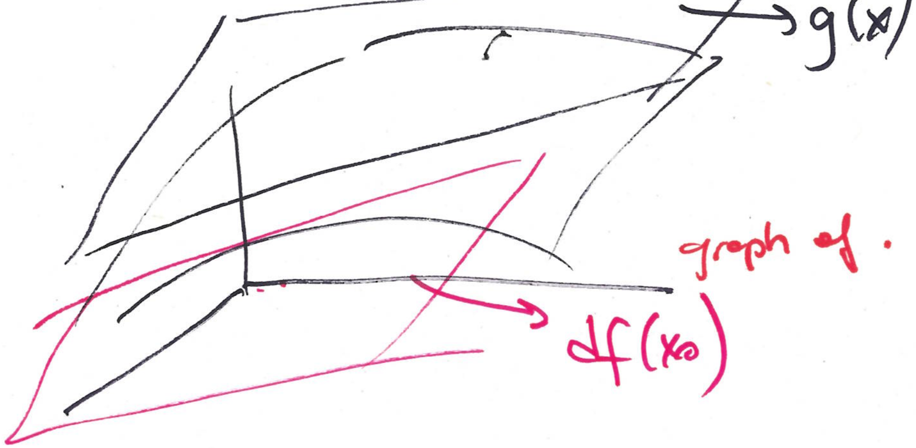


$a = f'(x_0)$

$$df(x_0) : \mathbb{R} \to \mathbb{R}$$
$$x \longmapsto ax$$

$$g(x) : \mathbb{R} \to \mathbb{R}$$
$$x \longmapsto f(x_0) + a(x-x_0)$$

$\to g(x)$



Question what is the analog of the chain rule?

$$\mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{g} \mathbb{R}.$$

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x).$$

6 **Thm** (Chain rule) let $X \subset \mathbb{R}^n$ open, $Y \subset \mathbb{R}^m$ open and $f : X \to Y$

$g : Y \to \mathbb{R}^p$ diff functions.

Then $g \circ f : X \to \mathbb{R}^p$ is differentiable in X. And for any $x_0 \in X$, its differential is given by the composition

$d(g \circ f)(x_0) : X \longrightarrow \mathbb{R}^p$

$d(g \circ f)(x_0) = dg(f(x_0)) \circ df(x_0)$

composition of maps.

In particular the Jacobi Matrix of $g \circ f$ at x_0 satisfies

$$J_{g \circ f}(x_0) = J_g(f(x_0)) \cdot J_f(x_0)$$

matrix multiplication.

$$X \xrightarrow{f} Y \xrightarrow{g} \mathbb{R}^p$$

$$x_0 \longmapsto y_0 = f(x_0)$$

$$\mathbb{R}^n \xrightarrow{df(x_0)} \mathbb{R}^m \xrightarrow{dg(y_0)} \mathbb{R}^p.$$

y_0

$$x \longmapsto Ax = y \longmapsto By.$$

$$x \longmapsto Ax \longmapsto BAx$$

$\underset{m \times n}{A} = J_f(x_0)$ $\qquad$ $\underset{p \times m}{B} = J_g(y_0)$

$J_g(f(x_0))$

$$\underbrace{B_{p \times m} A_{m \times n}}_{\displaystyle \overset{\shortparallel}{C_{p \times n}}}.$$

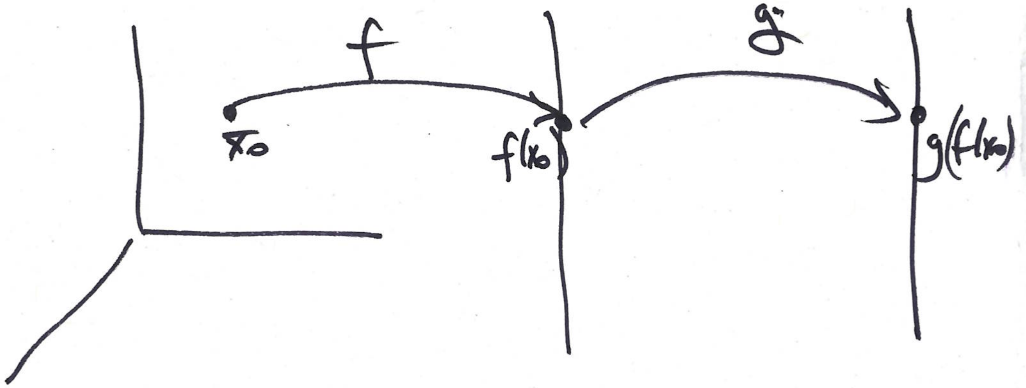
Ex (1) $m=n=p=1$

$$\mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{g} \mathbb{R}$$

$$(g\circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

the usual chain rule from Analysis I.

(2) $$\mathbb{R}^n \xrightarrow{f} \mathbb{R} \xrightarrow{g} \mathbb{R}$$



$$d(g\circ f)(x_0) = g'(f(x_0)) \circ (df)(x_0)$$

e.g. $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto e^{xy}$$

We can think of h as a composition of 2 maps. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto xy$$

$$g : \mathbb{R} \longrightarrow \mathbb{R}$$
$$t \longrightarrow e^t$$

$$h = g\circ f$$

We can do this in 2 different ways (a) directly (b) Using chain rule.

(a) $\frac{\partial h}{\partial x} = y e^{xy}$

$\frac{\partial h}{\partial y} = x e^{xy}$

cont. everywhere

$\Rightarrow h$ is diff. everywhere

$J_h(x,y) = \begin{pmatrix} y e^{xy} & x e^{xy} \end{pmatrix}$

(b). $f : \mathbb{R}^2 \to \mathbb{R}$
$(x,y) \mapsto xy$.

$g : \mathbb{R} \to \mathbb{R}$
$t \to e^t$

$J_f(x,y) = \begin{pmatrix} y & x \end{pmatrix}$

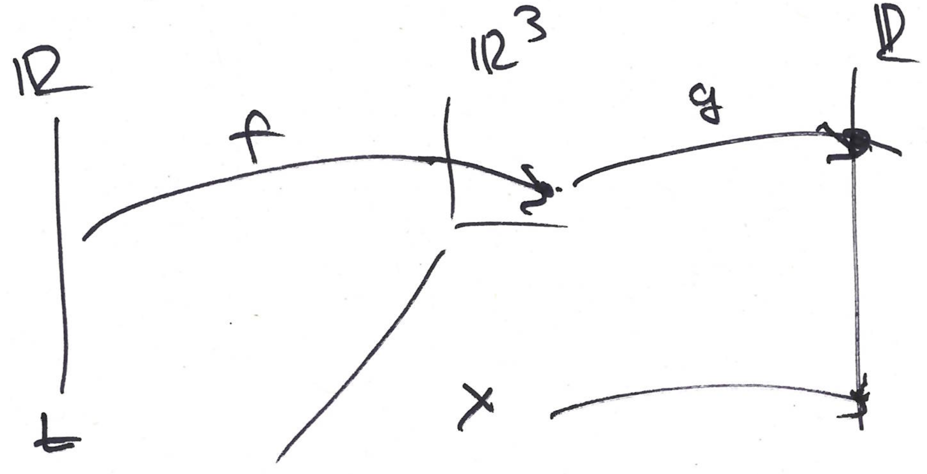
$J_g(t) = e^t$

$J_h(x,y) = \underbrace{J_g(\underline{f(x,y)})}_{\underbrace{J_g(xy)}_{e^{xy}}} \cdot \underbrace{J_f(x,y)}_{(y,\ x)}$

$e^{xy} \cdot (y, x) = \begin{pmatrix} e^{xy} y, & x e^{xy} \end{pmatrix}$

(3). $f : \mathbb{R} \longrightarrow \mathbb{R}^n$
$t_0 \longmapsto \begin{pmatrix} f_1(t_0) \\ f_2(t_0) \\ \vdots \\ f_n(t_0) \end{pmatrix}$

$g : \mathbb{R}^n \longrightarrow \mathbb{R}$.



$g \circ f : \mathbb{R} \longrightarrow \mathbb{R}$.

$d(g \circ f)'(t_0) = \underline{dg}(f(t_0)) \cdot f'(t_0)$

$= \begin{pmatrix} \frac{\partial g}{\partial x_1}(f(t_0)), & \frac{\partial g}{\partial x_2}(f(t_0)) \ldots & \frac{\partial g}{\partial x_n}(f(t_0)) \end{pmatrix} \begin{pmatrix} f_1'(t_0) \\ \vdots \\ f_n'(t_0) \end{pmatrix}$

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$$= \left\langle \nabla g(f(t_0)), \begin{pmatrix} f_1'(t_0) \\ \vdots \\ f_n'(t_0) \end{pmatrix} \right\rangle$$

We can apply this to

$$M : \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad J_M = (y \;\; x)$$
$$(x, y) \longmapsto (xy)$$

and

$$f : \mathbb{R} \longrightarrow \mathbb{R}^2$$
$$t \longmapsto \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}.$$

$$\underbrace{d(M \circ f)(t)}_{\downarrow} = \underbrace{(dM)(f(t))}_{(f_2(t) \;\; f_1(t))} \circ df(t) \begin{pmatrix} f_1'(t) \\ f_2'(t) \end{pmatrix}$$

$$(f_1(t) f_2(t))' = f_2(t) f_1'(t) + f_1(t) f_2'(t)$$

which reproves the product rule of differentiation from Analysis I.

(4) $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}$

$$x \longrightarrow y = y_1 \ldots y_m$$

$$g \circ f : \mathbb{R}^n \longrightarrow \mathbb{R}.$$

$$J_{g \circ f}(x_0) = J_g(\overbrace{f(x_0)}^{y_0}) \cdot J_f(x_0)$$

$$= \begin{pmatrix} \frac{\partial g}{\partial y_1}(y_0), \ldots \frac{\partial g}{\partial y_m}(x_0) \end{pmatrix}_{1 \times m} \cdot J_f(x_0)$$

$$J_f(x_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}_{m \times n}$$

This gives for example that

$$\frac{\partial (g\circ f)}{\partial x_1} = \frac{\partial g}{\partial y_1}(y_0)\frac{\partial f_1}{\partial x_1}(x_0) + \frac{\partial g}{\partial y_2}(y_0)\frac{\partial f_2}{\partial x_1}(x_0) + \cdots + \frac{\partial g}{\partial y_m}(y_0)\frac{\partial f_m}{\partial x_1}(x_0).$$

$$= \sum_{j=1}^{m} \frac{\partial g}{\partial y_j}(y_0)\frac{\partial f_j}{\partial x_1}(x_0)$$

<u>Ex</u>

$$\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2 \xrightarrow{g} \mathbb{R}^3$$

$$(x,y) \longmapsto \begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} g_1(u,v) \\ g_2(u,v) \\ g_3(u,v) \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$$

$$J_f(x,y) = \begin{pmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{pmatrix}$$

$$J_g(u,v) = \begin{pmatrix} \partial_u g_1 & \partial_v g_1 \\ \partial_u g_2 & \partial_v g_2 \\ \partial_u g_3 & \partial_v g_3 \end{pmatrix}$$

$$J_{g\circ f}(x,y) = J_g(u,v) \cdot J_f(x,y)$$

$f: \mathbb{R}^n \to \mathbb{R}$

If f is ~~diff~~ at x_0 then all partial derivatives of f at x_0 exist.

Recall, the partial deriv. of f can be thought as the directional derivatives in the direction of unit vectors $e_1, e_2, \dots$

In general given $\vec{v} \in \mathbb{R}^n$ we define the directional derivative of f at x_0 in the direction of $\vec{v}$ as the derivative at 0 of

$$g(t) := f(x_0 + t\vec{v})$$

7 **Thm** $f: X \to \mathbb{R}^m$, $X \subset \mathbb{R}^n$, $x_0 \in X$, $v \in \mathbb{R}^n$, $v \neq 0$. If f is diff at x_0 then the direc. der. of f at x_0 in the direc. of v exists and is given by $df(x_0)(v)$.

$$\frac{d}{dt} f(x_0 + t\vec{v}) \Big|_{t=0} = df(x_0)(v)$$

$$= J_f(x_0) \cdot v$$

matrix vector product

Ex: $f(x,y) = \begin{pmatrix} x^2 + y^2 \\ 2x \\ 2y \end{pmatrix}$

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$$(df)(1,2) \equiv \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto J_f(1,2)\begin{pmatrix} x \\ y \end{pmatrix}$$

$$J_f(1,2) = \begin{pmatrix} 2 & 4 \\ 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$(df)(1,2) : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \underbrace{\begin{pmatrix} 2 & 4 \\ 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}_{\begin{pmatrix} 2x+4y \\ 2x \\ 2y \end{pmatrix}}$$

dir. der. of f at $(1,2)$ in the dir of $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, is given by

$$(df(1,2))\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix} \rightarrow$$ dir. der of $(1,2)$ in dir $(1,1)$ ev

$$\frac{\partial f}{\partial x} \cancel{(1,2)} \begin{pmatrix} 2x \\ 2 \\ 0 \end{pmatrix} \qquad \frac{\partial f}{\partial y} = \begin{pmatrix} 2y \\ 0 \\ 2 \end{pmatrix}$$

$$\frac{\partial f}{\partial x}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \qquad \frac{\partial f}{\partial y}(1,2) = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$$

dir. der of f at $(1,2)$ in the dir. of $e_1 = (1,0)$

dir. der at $(1,2)$ in dir. of $e_2 = (0,1)$