- $f : X \to \mathbb{R}$, $X \subset \mathbb{R}^n$, $f \in C^2(X : \mathbb{R})$

$$\mathrm{Hess}_f(x_0) := \left(\frac{\partial^2 f (x_0)}{\partial x_i \, \partial x_j} \right)_{\substack{1 \le i \le n \\ 1 \le j \le n}}$$

Hessian of f at $x_0 \in X$

- $f : X \to \mathbb{R}$, $f \in C^2$

$$f(x) = \underbrace{\boxed{f(x_0) + \nabla f(x_0) \cdot (x - x_0) + \frac{1}{2} (x - x_0)^t \, \mathrm{Hess}_f(x_0) (x - x_0)}}^{T_2 f(x - x_0 ; x_0)} + E_2(f, x ; x_0)$$

- For $f \in C^k$,

$$\boxed{T_k f(y ; x_0) = \sum_{\substack{m \\ |m| \le k}} \frac{1}{m!} \partial_x^m f(x_0) \, y^m}$$

Taylor polynomial of f of order k at x_0.

where
$$\sum_{\substack{m = (m_1, \dots, m_n) \\ |m| = m_1 + \dots + m_n = \ell}} \frac{1}{m!} \partial_x^m f(x_0) y^m = \sum_{m_1 + \dots + m_n = \ell} \frac{1}{m_1! \cdots m_n!} \frac{\partial^\ell f}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} (x_0) \cdot y_1^{m_1} \cdots y_n^{m_n}.$$

Thm $f(x) = T_k f(x - x_0 ; x_0) + E_k(f, x ; x_0)$ where

$$\frac{E_k(f, x ; x_0)}{\| x - x_0 \|^k} \xrightarrow[x \to x_0]{} 0$$

$f : \overline{X} \longrightarrow \mathbb{R}, \quad \overline{X} \subseteq \mathbb{R}^n$

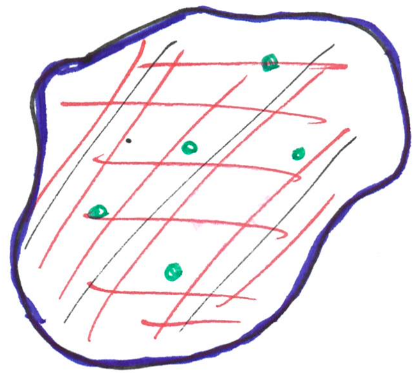
Thm Let $X \subset \mathbb{R}^n$ be open and f differentiable. If $x_0 \in X$ is a local extrema (i.e a local min or max) then

$\nabla f(x_0) = 0.$

Defn If $\nabla f(x_0) = 0$ for some $x_0 \in X$ then x_0 is called a critical point.

A critical point of f which is not a local min or max is called a saddle point.

Thm If $f : \overline{X} \longrightarrow \mathbb{R}$ diff. on the interior of $\overline{X}$ where $\overline{X}$ is closed and bounded, then global extrema of f on $\overline{X}$ exists and it is either at a critical point of f or on the boundary of $\overline{X}$.



$\overline{X} = \mathrm{int}(X) \cup$ boundry($\overline{X}$)

• critical points

$\underline{n=1}$ $f : [a,b] \longrightarrow \mathbb{R}.$

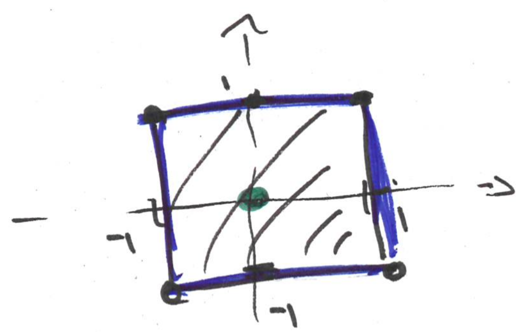
then f attains its min, max on $[a,b]$.

max, min of f occurs either at a critical pt or $x=b,a$

2)

Ex. $f(x,y) = x^2 + y^2$

on X = square $[1,1] \times [-1,1]$



Critical points: $\nabla f = \begin{pmatrix} 2x \\ 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow x = 0, \; y = 0.$

We need to check f on the boundary of the square

on top boundary line $y=1$, $-1 \le x \le 1$

$f(x,y) = x^2 + 1 \to$ maximum when $x = \pm 1$
min at $x = 0$.

Since both $f(x,y)$ is sym in x,y and the boundy is sym, we have that the max can occur at $(\pm 1, \pm 1)$, also $(0, \pm 1)$, $(\pm 1, 0)$

$f(0,0) = 0 \longrightarrow$ global min.
$f(\pm 1, \pm 1) = 2 \longrightarrow$ global max
$f(0, \pm 1) = 1$
$f(\pm 1, 0) = 1.$

Question: How do we determine if a critical point is a local min or max?

Defn. A non-degenerate critical point x_0, of $f \in C^2$ is a critical point for which $\det \mathrm{Hess}_f(x_0) \neq 0$.

Recall. $f : \mathbb{R} \to \mathbb{R}$

x_0 is a critical pt ($f'(x_0)=0$)

then the nature of the pt x_0, is determined by

$f''(x_0) > 0$	$f''(x_0) = 0$,	$f''(x_0) < 0$
min	saddle	max

In several variables the role of $f''(x_0)$ is played by the $\mathrm{Hess}_f(x_0)$.

Recall Lin. Alg.

A sym matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ $\det A \neq 0$ is

(a) Positive definite iff $A > 0$

$x A x^t > 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}$.

e.values > 0

(b) negative def. iff $x A x^t < 0$ $\forall x \neq 0$.

$A < 0$. (e.values < 0)

(c) indefinite otherwise.

It has positive and negat. eigenvalues.

Criteria. A sym. matrix is positive def $\Longleftrightarrow$ for $1 \leq j \leq n$ $\det A_j > 0$. where $A_j = (a_{k\ell})_{\substack{1 \leq k \leq j \\ 1 \leq \ell \leq j}}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \vdots \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix}$$

A_1, A_2

if $A = 2\times 2$ matrix
then $\left(A > 0 \iff a_{11} > 0,\ \det A > 0\right)$.

Thm. $f : X \to \mathbb{R}^n$, $f \in C^2(X, \mathbb{R})$.
$X \subseteq \mathbb{R}^n$. Let x_0 be a non-deg. critical pt of f.
$\left(\nabla f(x_0) = 0,\ \det \mathrm{Hess}_f(x_0) \neq 0\right)$.

Then 1) If $\mathrm{Hess}_f(x_0) > 0$ then x_0 is a loc. min.

2) If $\mathrm{Hess} f(x_0) < 0$ then x_0 is a loc. max

3) If $\mathrm{Hess}_f(x_0)$ is indef then x_0 is a saddle

Ex ④ $f_1(x,y) := x^2 + y^2$ has a loc. min at $(0,0)$

$f_2(x,y) = -x^2 - y^2$ loc. max at $(0,0)$

$f_3(x,y) = xy$. saddle pt at $(0,0)$

$\nabla f_1 = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$ $\quad \nabla f_2 = \begin{pmatrix} -2x \\ -2y \end{pmatrix}$

$\nabla f_3 = \begin{pmatrix} y \\ x \end{pmatrix} \Rightarrow x_0 = (0,0)$ is a critical pt of f_1, f_2, f_3.

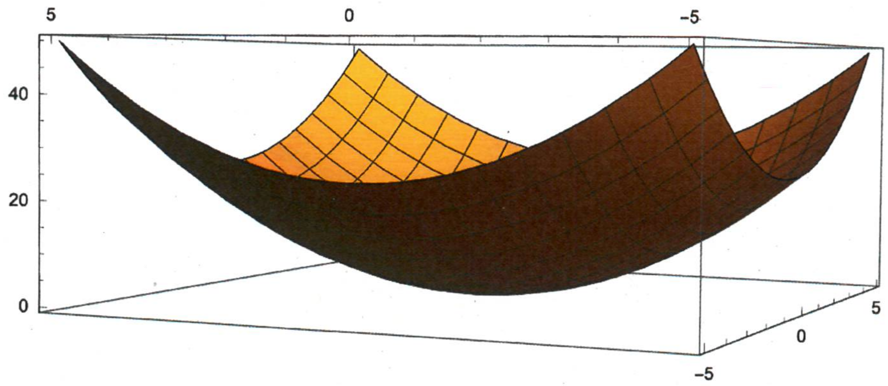
$\mathrm{Hess}_{f_1}(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} > 0$

$\mathrm{Hess}_{f_2}(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} < 0$.

$\mathrm{Hess}_{f_3}(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ indefinite.

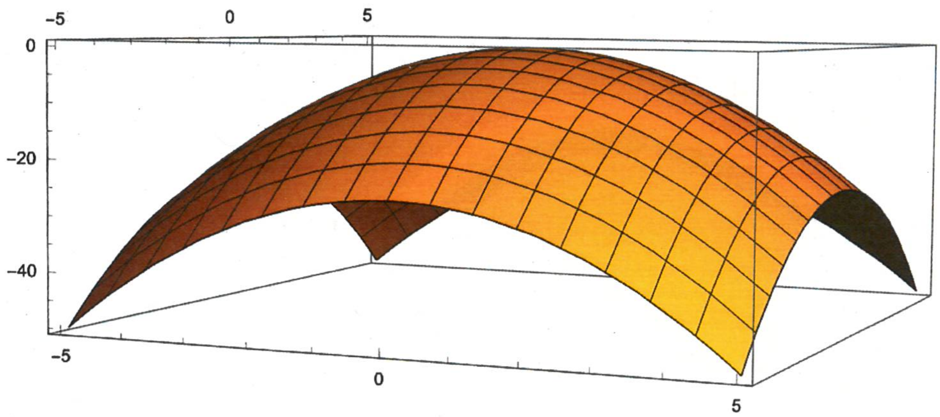
```
In[1]:= Plot3D[x^2 + y^2, {x, -5, 5}, {y, -5, 5}]
```

Out[1]=



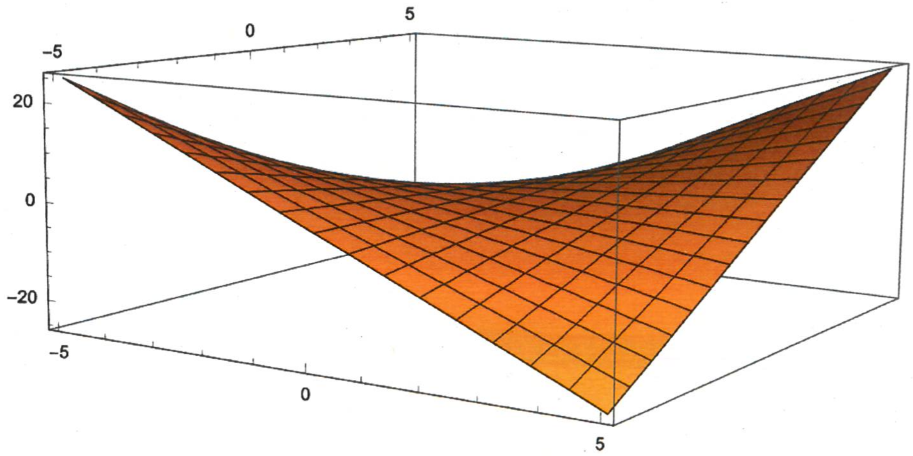

```
In[2]:= Plot3D[-x^2 - y^2, {x, -5, 5}, {y, -5, 5}]
```

Out[2]=




```
In[4]:= Plot3D[x * y, {x, -5, 5}, {y, -5, 5}]
```

Out[4]=



5.1

Ex ②. $f(x,y) = x\sin y$

$$\nabla f = \begin{pmatrix} \sin y \\ -x\cos y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

$\Rightarrow \sin y = 0 \Longrightarrow y = \pi k \quad k \in \mathbb{Z}.$

$x\cos y = 0. \Longrightarrow x = 0$ or $y = \frac{\pi}{2}k$

k odd integer.

But both partial derivatives are zero only if $x_0 = (0, \pi k)$

$k \in \mathbb{Z}$.

$f_{xx} = 0 \quad f_{xy} = \cos y.$

$f_{yy} = -x\sin y.$

$$\mathrm{Hess}_f(x) = \begin{pmatrix} 0 & \cos y \\ \cos y & -x\sin y \end{pmatrix}$$

$$\mathrm{Hess}_f(0, 2\pi) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$2\pi k$

$$\mathrm{Hess}_f(0, \pi) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

πk

k odd.

$\mathrm{Hess}_f(x_0)$ is indefinite.

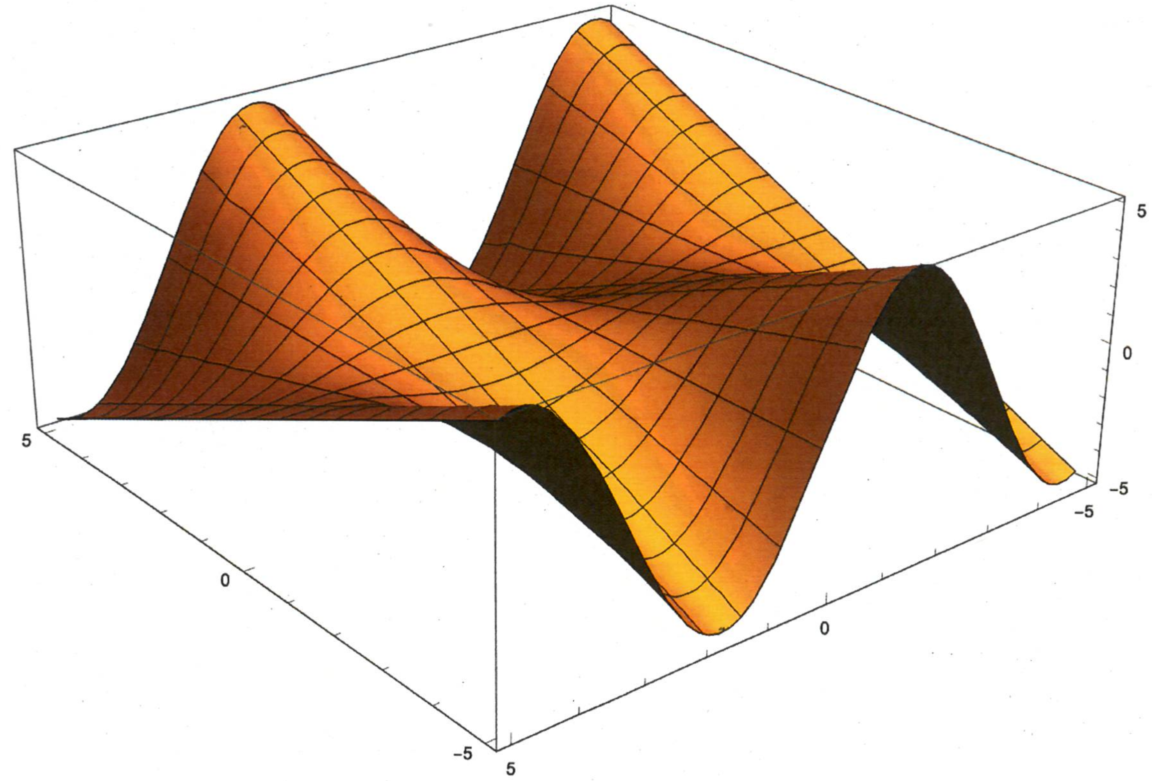
$x_0 = (0, \pi k)$ are all non-deg.

$\det \mathrm{Hess} \neq 0$.

They're all saddle points.

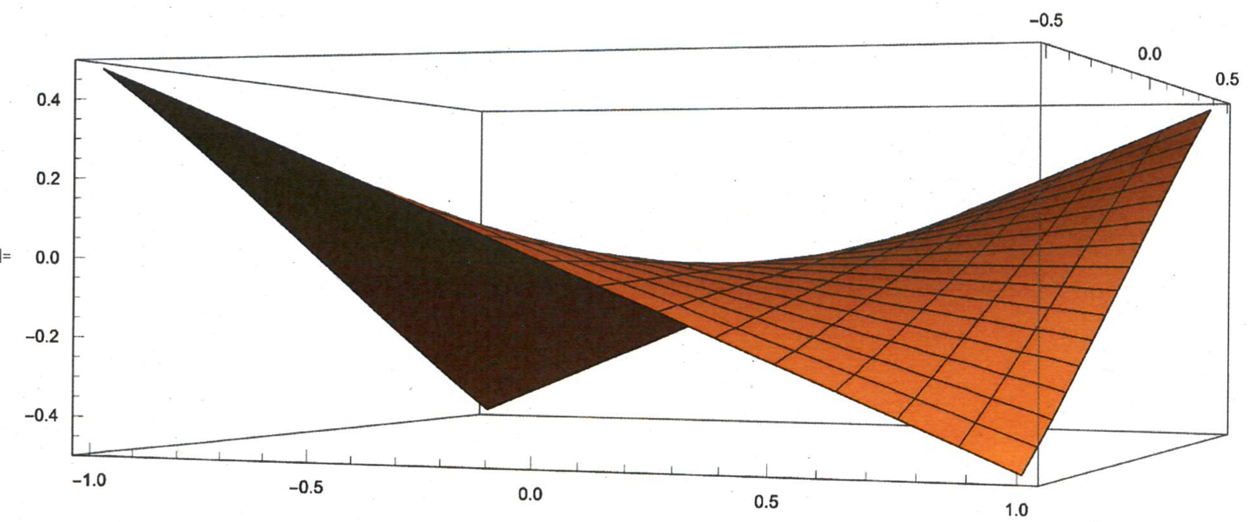
In[6]:= `Plot3D[x * Sin[y], {x, -5, 5}, {y, -5, 5}]`

Out[6]=



In[10]:= `Plot3D[x * Sin[y], {x, -1, 1}, {y, -0.5, 0.5}]`

Out[10]=



17

Rk: What happens at a degenerate critical point x_0

i.e. if $\det \mathrm{Hess}_f(x_0) = 0$?

- We cannot use the thm to decide on the nature of x_0. We have to decide case by case.

$f_1(x,y) = x^4 + y^4 \rightarrow$ has a local min

$f_2(x,y) = -x^4 - y^4 \rightarrow$ loc max

$f_3(x,y) = x^4 - y^4 \rightarrow$ saddle pt.

$\nabla f_i(0,0) = 0 \qquad i = 1,2,3$

$$\mathrm{Hess}_f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Clicker question

Let $f : \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = \sin x + 2y^2 + xy$.

What is the eqn of the tangent plane at $(0,1,f(0,1))$?

$\frac{\partial f}{\partial x} = \cos x + y \qquad \frac{\partial f}{\partial y} = 4y + x$

at the pt $(0,1)$, $\frac{\partial f}{\partial x}(0,1) = 2 \qquad \frac{\partial f}{\partial y}(0,1) =$

eqn. of tangent plane:

$$z = f(0,1) + \frac{\partial f}{\partial x}(0,1)(x-0) + \frac{\partial f}{\partial y}(0,1)(y-1)$$

$$z = 2 + 2x + 4(y-1)$$

$$\Rightarrow \boxed{2x + 4y - z = 2}.$$

§ 4. Integration in $\mathbb{R}^n$.

Recall: if $f : \mathbb{R} \to \mathbb{R}^n$

$$t \mapsto \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

$f_i(t) : \mathbb{R} \to \mathbb{R}$.

The derivative of f is simply

is simply $\begin{pmatrix} f_1'(t) \\ \vdots \\ f_n'(t) \end{pmatrix}$

Similarly the integral of f from a to b is

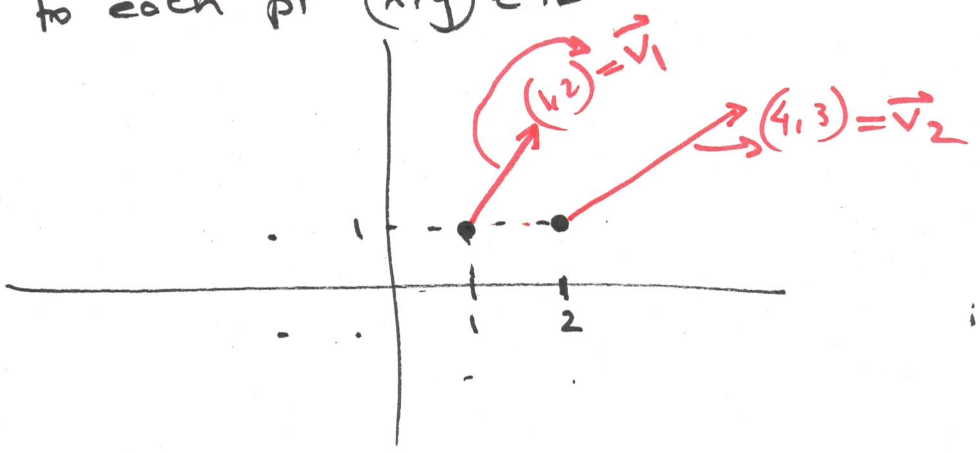
$$\int_a^b f(t)\,dt := \begin{pmatrix} \int_a^b f_1(t)\,dt \\ \vdots \\ \int_a^b f_n(t)\,dt \end{pmatrix}$$

Today: $f : \mathbb{R}^n \to \mathbb{R}^n$

vector fields.

One way to visualize is : $\underline{n=2}$

to assign a vector $f(x,y) \in \mathbb{R}^2$ to each pt $(x,y) \in \mathbb{R}^2$



$f(x,y) = (x^2, x+y)$

$f(1,1) = (1, 2) = \vec{v}_1$

$f(2,1) = (4, 3) = \vec{v}_2$

We start ~~recalling~~ definition of a curve in $\mathbb{R}^n$.

This is simply the image of a function $\gamma : [a, b] \longrightarrow \mathbb{R}^n$

where the function γ, is continuous, piecewise C^1.

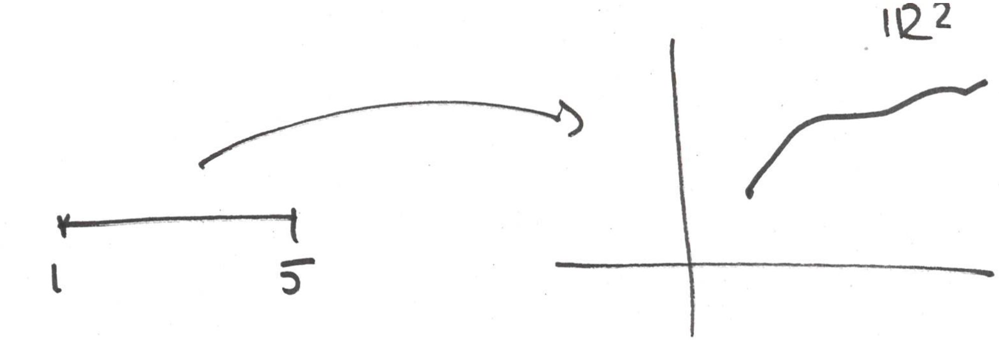
ie $\exists\, k > 1$ and a partition

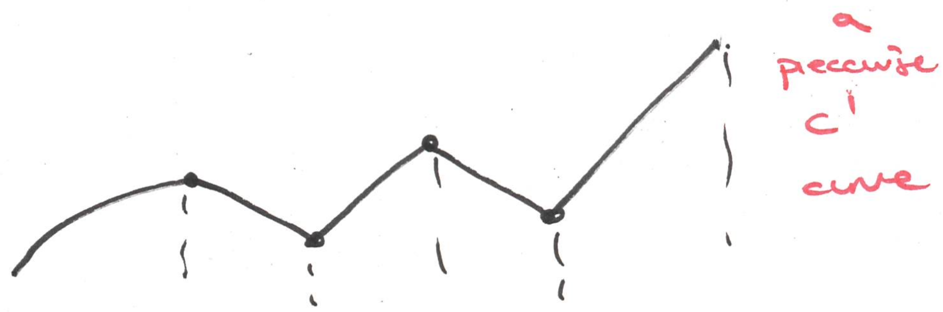
$a = t_0 < t_1 \ldots < t_k = b$

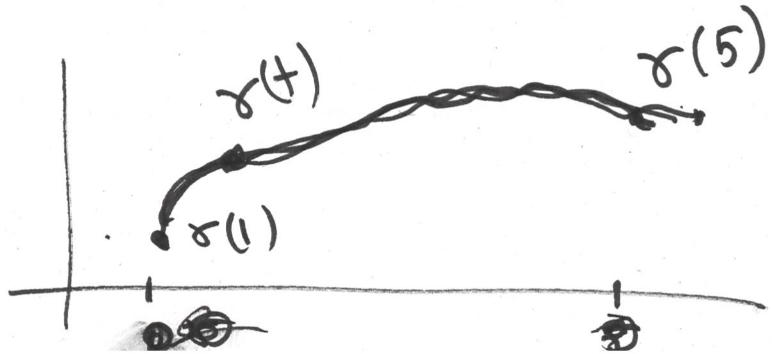
such that $\gamma\big|_{(t_j, t_{j+1})} \in C^1$

γ is called a parametrization of the curve

e.g. $\gamma : [1, 5] \longrightarrow \mathbb{R}^2$

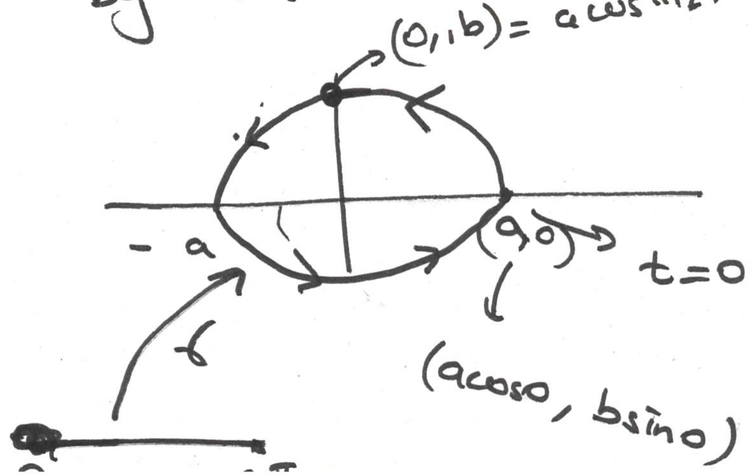




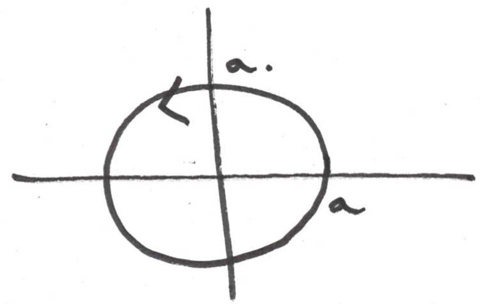


example $\gamma : [0, 2\pi] \longrightarrow (a\cos t, b\sin t)$

is a curve which is given by an ellipse.

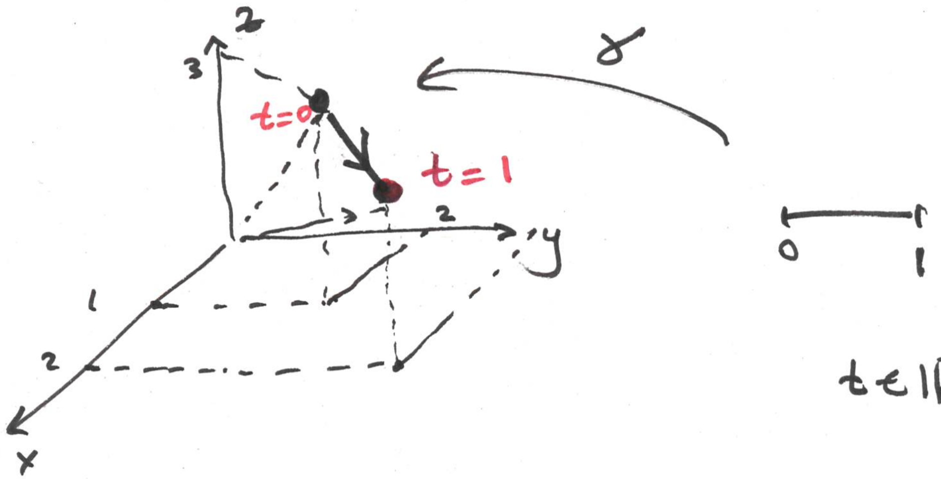


If $a = b$ then we get a parametrization of the circle of radius a.



② $\gamma : [0,1] \longrightarrow \mathbb{R}^3$

$t \longrightarrow (1+t,\ 2+t,\ 3-t)$



$t \in \mathbb{R}$

In general if $a = (a_1, a_2, a_3)$

and $b = (b_1, b_2, b_3)$

$\gamma : [0,1] \longrightarrow \mathbb{R}^3$

$t \longmapsto (a_1 + b_1 t,\ a_2 + b_2 t,\ a_3 + b_3 t)$

is the parametrization of the line segment between the vector $\vec{a}$, and $\vec{a} + \vec{b}$.

we get a line segment

$r(t) = \vec{a} + \vec{b}t \qquad t \in [0,1]$.

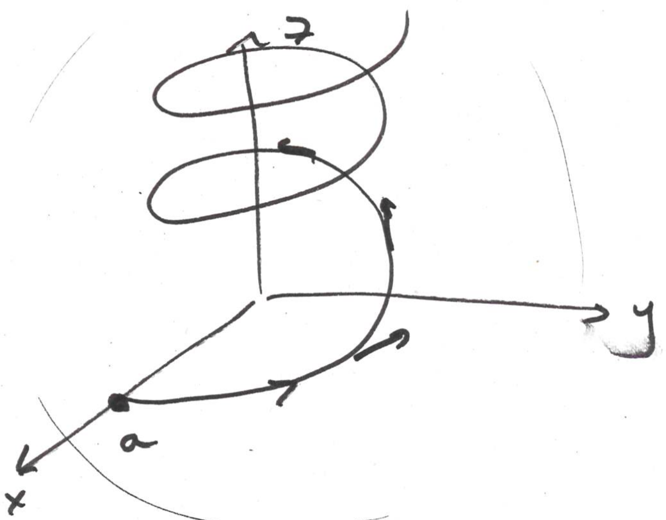
③ $f : [a,b] \longrightarrow \mathbb{R}$. $\in C^1$

Then its graph is a curve in $\mathbb{R}^2$.

A parametrization

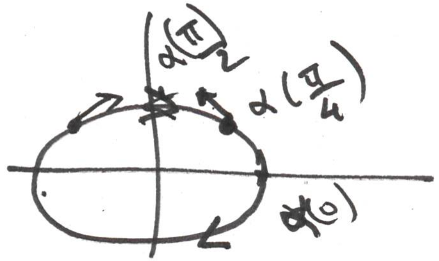
$$\gamma : [a,b] \longrightarrow \mathbb{R}^2$$
$$t \longmapsto (t, f(t))$$

4. $\gamma : \mathbb{R} \to \mathbb{R}^3$
$$t \to (a\cos t,\ a\sin t,\ t)$$



5. $\alpha : [0, 2\pi] \longmapsto (a\cos(2\pi - t),\ b\sin(2\pi - t))$

traces the same ellipse but in the opposite direction.



$\gamma'(t)$ gives the tangent vector at t, to the curve.

Defn. Let $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ be a parametrization of a curve. Let $X \subset \mathbb{R}^n$ be a sset which contains the image of γ. Let $f : X \longrightarrow \mathbb{R}^n$ a continuous function. The integral

$$\int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt \in \mathbb{R}$$

is called the line integral of f

It is denoted by

$$\int_\gamma f(s).\,d\vec{s}\,.$$

<u>Rk</u> if $f = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}$ $\quad f : \mathbb{R}^n \to \mathbb{R}^n$

$f_i : \mathbb{R}^n \to \mathbb{R}$.

Another notation that is used for the line integral is

$$\int f(s)\,d\vec{s} = \int f_1(x)\,dx_1 + \dots + f_n(x)\,dx_n$$

<u>eg.</u> $\gamma : I \to \mathbb{R}^2$

$t \to (\gamma_1(t), \gamma_2(t)) = (x, y)$

$x = \gamma_1(t) \qquad y = \gamma_2(t)$

$dx = \gamma_1'(t)dt \qquad dy = \gamma_2'(t)$

$$\int_\gamma f\,ds = \int_a^b \begin{pmatrix} f_1(\gamma(t)) \\ f_2(\gamma(t)) \end{pmatrix} \cdot \begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \end{pmatrix} dt$$

$$= \int_a^b \big(f_1(\gamma(t)) \cdot \gamma_1'(t) + f_2(\gamma(t)) \cdot \gamma_2'(t)\big)\,dt$$

$$\int_a^b f_1(x,y)\,dx + f_2(x,y)\,dy\,.$$

$f : \mathbb{R}^2 \to \mathbb{R}^2$

$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$

<u>Ex</u>: ① $f(x,y) = (-y, x)$

$\gamma(t) = (\cos t, \sin t) \qquad t \in [0, 2\pi]$

$$\int_\gamma f(s)\,ds = \int_0^{2\pi} \underbrace{f(\gamma(t))} \cdot \underbrace{\gamma'(t)}\,dt$$

$$= \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\,dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = \int_0^{2\pi} 1\,dt = 2\pi$$

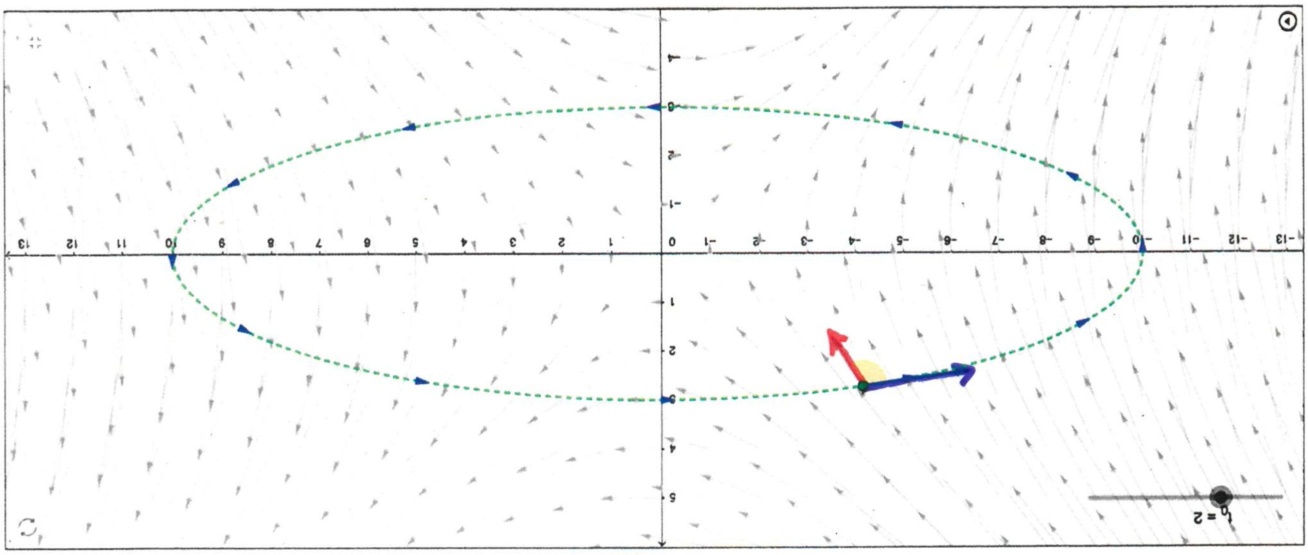
(2) If we take the same curve but in the opposite direction $\tilde{\gamma}(t) = (\cos t, -\sin t)$

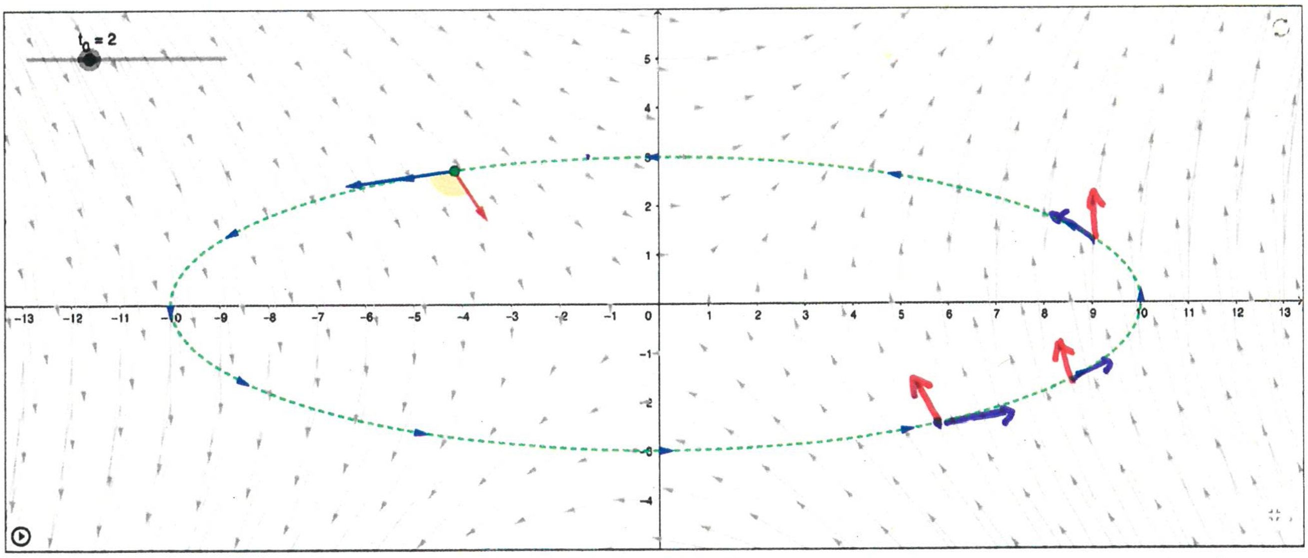
$$\int_{\tilde{\gamma}} f \cdot ds = \int_0^{2\pi} (\sin t, \cos t) \cdot (-\sin t, -\cos t)\, dt$$

$$= \int_0^{2\pi} -\sin^2 t - \cos^2 t \; dt$$

$$= -\int_0^{2\pi} dt = -2\pi$$

When we reverse the curve the line integral changes sign.





scalar product

$$\int_a^b f(s)\,ds := \int_a^b \underbrace{f(\gamma(t))}_{\text{red vector}} \cdot \underbrace{\gamma'(t)}_{\text{blue vector}}\,dt$$

Geogebra: Line Integral of a vector field in 2-space

Author: Kristen Beck

https://www.geogebra.org/m/YZWyyM47