Integration in $\mathbb{R}^n$

- If $f : \mathbb{R} \to \mathbb{R}^n$

$$t \mapsto \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

then

$$\int_a^b f(t)\,dt = \begin{pmatrix} \int_a^b f_1(t)\,dt \\ \vdots \\ \int_a^b f_n(t)\,dt \end{pmatrix}$$

- If $f : \mathbb{R}^n \to \mathbb{R}^n$ then f is called a vector field.

- Let $\gamma : [a,b] \to \mathbb{R}^n$ be a curve whose image is contained in $X \subset \mathbb{R}^n$

- Let $f : X \to \mathbb{R}^n$ a continuous vector field.

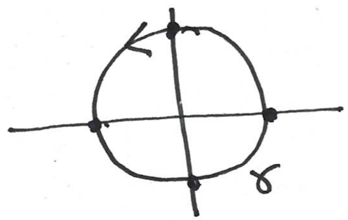
The line integral of f along γ is

$$\boxed{\int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt}$$

It is denoted by $\int_\gamma f\,d\vec{s}$

- Ex: $\gamma : [0, 2\pi] \to \mathbb{R}^2$

$$t \mapsto (\cos t, \sin t)$$



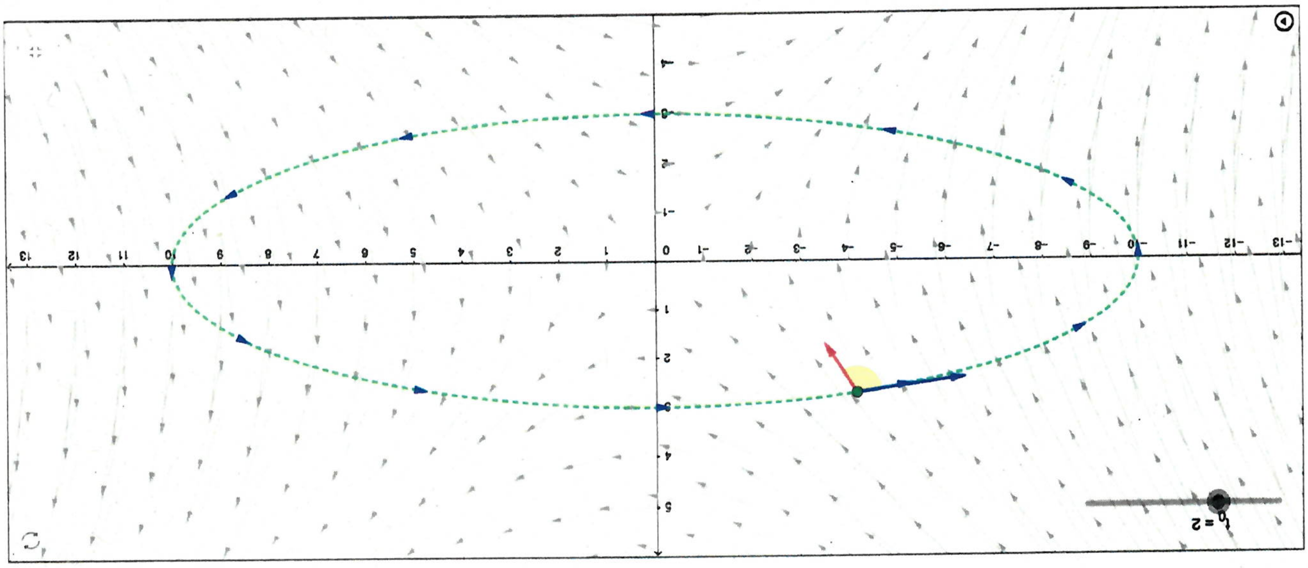
Let $f(x,y) = (-y, x)$.

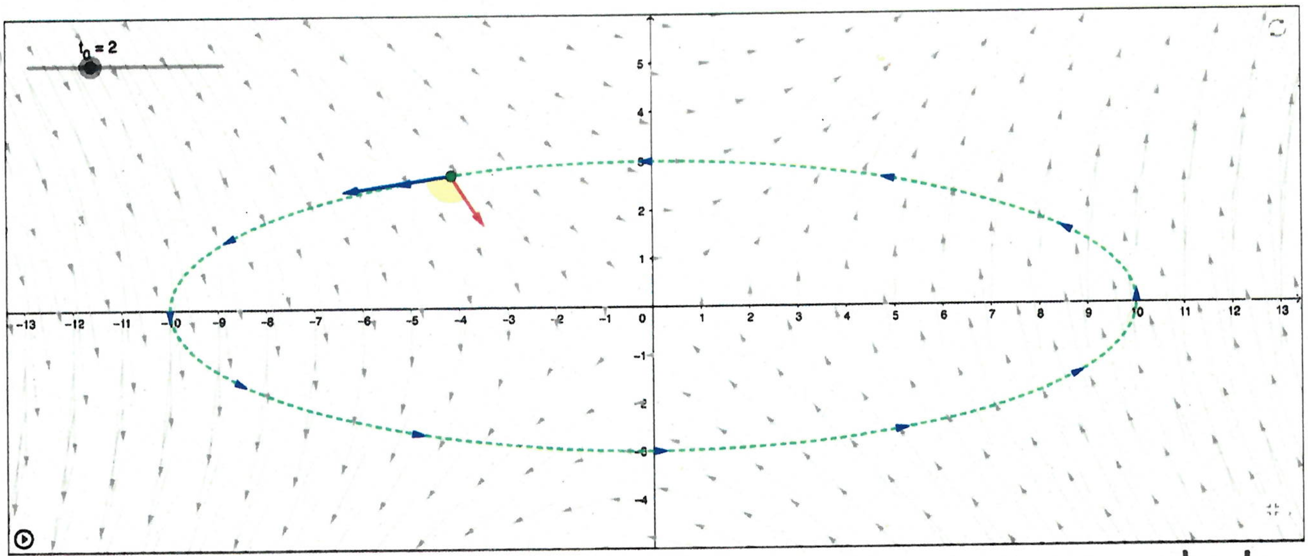
Then $\int_\gamma f\,ds =$

$$\int_0^{2\pi} f(\gamma(t)) \cdot \gamma'(t)\,dt = \int_0^{2\pi} \underbrace{(-\sin t, \cos t)}_{f(\gamma(t))} \cdot \overbrace{(-\sin t, \cos t)}^{\gamma'(t)}\,dt$$

$$= 2\pi$$

$f : \mathbb{R}^2 \to \mathbb{R}^2$





scalar product.

$$\int_a^b f\,ds = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt$$

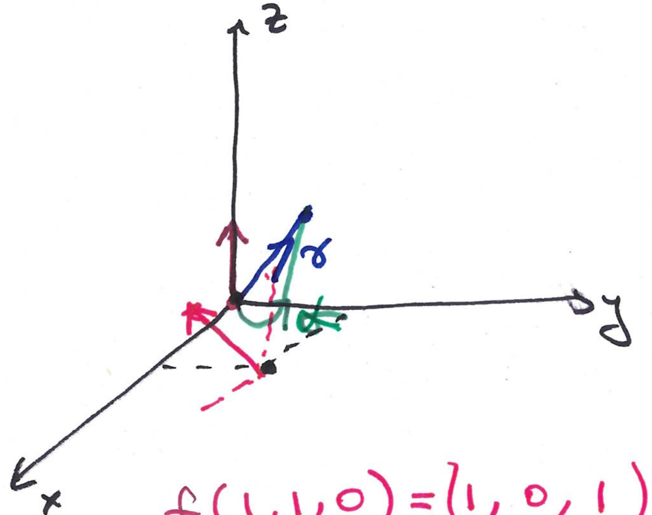
red vector

blue vector

2)

Ex $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$(x,y,z) \longmapsto (y^2, xz, 1).$$



$f(1,1,0) = (1,0,1)$

(a)

$$\gamma : [0,1] \longrightarrow \mathbb{R}^3$$
$$t \longmapsto \begin{pmatrix} t \\ t \\ t \end{pmatrix}$$

$$\int_\gamma f\,ds = \int_0^1 \underbrace{f(\gamma(t))}\cdot \gamma'(t)\,dt$$

$$= \int_0^1 f(t,t,t)\cdot(1,1,1)\,dt$$

$$= \int_0^1 (t^2, t^2, 1)\cdot(1,1,1)\,dt$$

$$= \int_0^1 (t^2 + t^2 + 1)\,dt = \int_0^1 (2t^2+1)\,dt$$

$$= \frac{2t^3}{3} + t \Big|_0^1 = \frac{2}{3} + 1 = 5/3.$$

(b)

$$\alpha : [0,1] \longrightarrow \mathbb{R}^3$$
$$t \longmapsto \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$$

$$\int_\alpha f\,ds = \int_0^1 f(\alpha(t))\cdot \alpha'(t)\,dt$$

$$= \int_0^1 \underbrace{f(t, t^2, t^3)}\cdot(1, 2t, 3t^2)\,dt$$

$$= \int_0^1 (t^4, t^4, 1)\cdot(1, 2t, 3t^2)\,dt$$

$$= \int_0^1 t^4 + 2t^5 + 3t^2\,dt = \cdots\ 23/15\ /3$$

Properties of the line integral.

1. It is independent of orientation preserving reparametrization of the curve.

i.e. if $\gamma : [a,b] \to \mathbb{R}^n$ a C^1 curve and let

$\tilde{\gamma} : [c,d] \to \mathbb{R}^n$ such that

$\tilde{\gamma} = \gamma \circ \phi$ where $\phi : [c,d] \to [a,b]$

ϕ is C^1 such that $\phi(c) = a$, $\phi(d) = b$

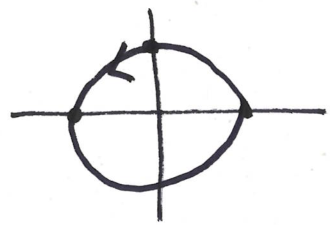
with $\phi' > 0$ $\forall t \in [c,d]$, Then

$$\int_\gamma f \, ds = \int_{\tilde{\gamma}} f \, ds$$

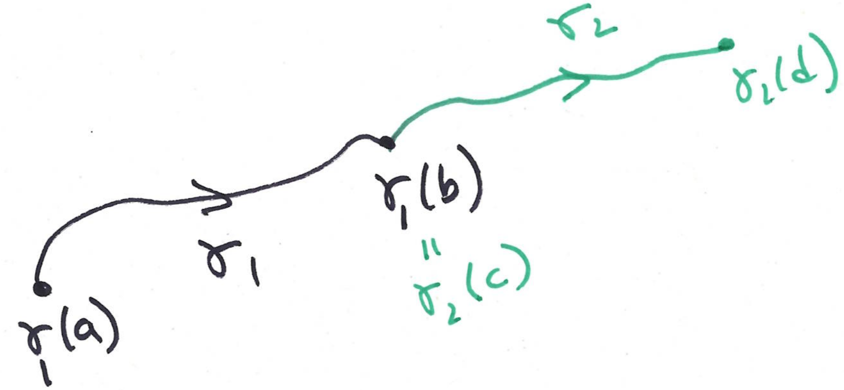
eg: $\gamma : [0, 2\pi] \to \mathbb{R}^2$
$t \to \cos t, \sin t$

$\phi : [0, \pi] \to [0, 2\pi]$.
$t \to 2t$, $\phi' = 2 > 0$

$\tilde{\gamma} = \gamma \circ \phi : [0, \pi] \to \mathbb{R}^2$
$t \to (\cos 2t, \sin 2t)$



2. let $\gamma_1 : [a,b] \to \mathbb{R}^n$
$\gamma_2 : [c,d] \to \mathbb{R}^n$ 2 path
with, $\gamma_1(b) = \gamma_2(c)$



we define $\gamma_1+\gamma_2$ the path formed by juxtaposition (concatenation) of the 2 curves

$$\gamma_1+\gamma_2 := \begin{cases} \gamma_1(t) & t\in[a,b] \\ \gamma_2(t-b+c) & t\in[b, d+b-c] \end{cases}$$

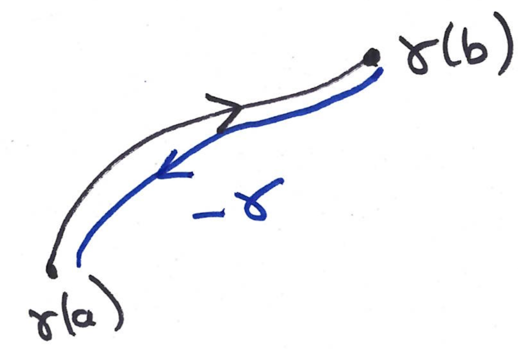
then

$$\int_{\gamma_1+\gamma_2} f\,ds = \int_{\gamma_1} f\,ds + \int_{\gamma_2} f\,ds.$$

③ If $\gamma:[a,b]\to\mathbb{R}^n$ a path

let $-\gamma:[a,b]\to\mathbb{R}^n$ same path traced in the opposite direction.

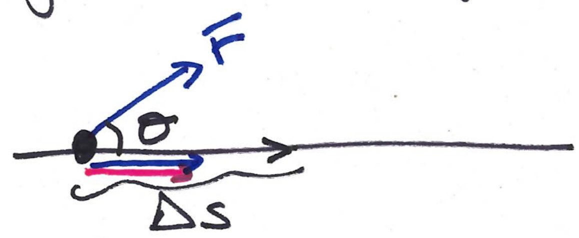
ie $(-\gamma)(t) := \gamma(a+b-t)$



$$\int_{-\gamma} f\,ds = -\int_{\gamma} f\,ds$$

Why do we define such an integral?

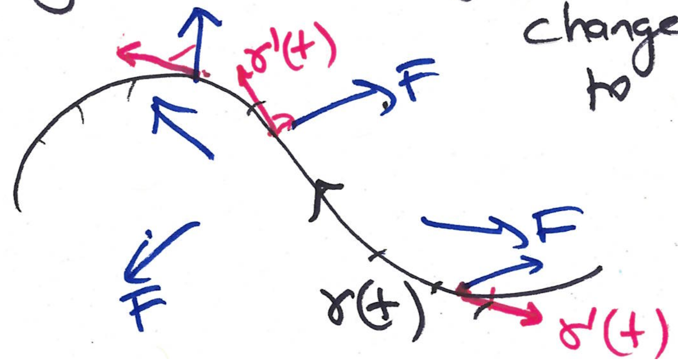
A motivating example comes from physics. Assume a point mass moves under the influence of a constant force field $F:\mathbb{R}^2\to\mathbb{R}^2$



If it is moved by a distance Δs, then the amount of work done is given

$$w = \vec{F} \cdot \vec{ds} = \|F\| \|s\| \cos\theta$$
$$= \underbrace{\|s\| \|F\| \cos\theta}_{\substack{\text{amount of work done} \\ \text{in the direct. of the} \\ \text{movement.}}}$$

Suppose the particle is moved along a curve under ~~under~~ the influence of a Force field which changes from pt to pt.



$$\Delta w_i = \vec{F_i} \cdot \vec{\Delta \gamma_i}$$

$$\text{Total work} = \sum_{i=1}^{n} \Delta w_i$$

$$= \sum_{i=1}^{n} F(\gamma_i(t)) \frac{\Delta \gamma_i}{\Delta t_i} \cdot \Delta t_i$$

$n \to \infty$

$$\downarrow$$

$$\int F(\gamma(t))\, \gamma'(t)\, dt$$

Important example

Suppose $f : X \to \mathbb{R}^n$ $X \subset \mathbb{R}^n$ is a vector field such that

$\exists\, g : X \to \mathbb{R}$, $g \in C^1$

such that $\nabla g = f$

let $\gamma : [a,b] \to \mathbb{R}^n$ a curve such that $\gamma([a,b]) \subset X$

Then

$$\int_\gamma f\, ds \overset{\text{defn}}{=} \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt$$

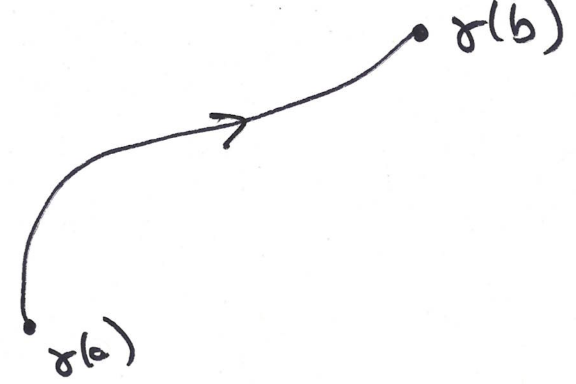
$$\overset{\text{assumption that } \nabla g = f}{=} \int_a^b \underbrace{\nabla g(\gamma(t)) \cdot \gamma'(t)}_{\frac{d}{dt}(g \circ \gamma) \text{ by chain rule}}\, dt$$

$$= \int_a^b \frac{d}{dt}(g\circ\gamma)\, dt$$

$$= (g\circ\gamma)(b) - (g\circ\gamma)(a)$$

Fund. thm of Integral calculus from Analysis I.

$$= g(\gamma(b)) - g(\gamma(a))$$



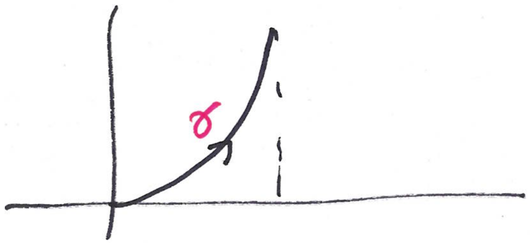
$\int_\gamma f\, ds$ only depends on the end points of the curve.

Recall: $F : \mathbb{R} \to \mathbb{R}$. $\int_a^b F'(t)\, dt = F(b) - F(a)$

Defn. A differentiable scalar field $g : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $\nabla g = f$, $f : X \to \mathbb{R}^n$ is called a potential for f

Clicker question

$f(x,y) = (-y, xy)$



$\gamma(t) = (t, t^2)$

$\gamma : [0,1] \to \mathbb{R}^2$

$t \to (t, t^2)$

$$\int_\gamma f\,ds = \int_0^1 f(\gamma(t)) \cdot \gamma'(t)\,dt$$

$$= \int_0^1 f(t, t^2) \cdot (1, 2t)\,dt$$

$$= \int_0^1 (-t^2, t^3) \cdot (1, 2t)\,dt$$

$$= \int_0^1 (-t^2 + 2t^4)\,dt = -\frac{t^3}{3} + \frac{2t^5}{5}\Big|_0^1 = \frac{-1}{3} + \frac{2}{5} = \frac{-5+6}{15} = \frac{1}{15}.$$

Rk ① If $n=1$, a potential is same as primitive of f. , g is a primitive of f if $g' = f$.

② If $f : \mathbb{R} \to \mathbb{R}$ is continuous then f always has a primitive namely $g(x) := \int_a^x f(t)\,dt$

$(g'(x) = f(x))$

Question: Is this also the case for $n \geq 2$?

Ex: $f = (2xy^2, 2x)$

Suppose $\exists g : \mathbb{R}^2 \to \mathbb{R}$ s.t. $\nabla g = f$

We can show that in fact such a g cannot exist! ie this f does not have a potential g.

If g were a potential then $\nabla g = f$, $\frac{\partial g}{\partial x} = 2xy^2$

$$\frac{\partial g}{\partial y} = 2x$$

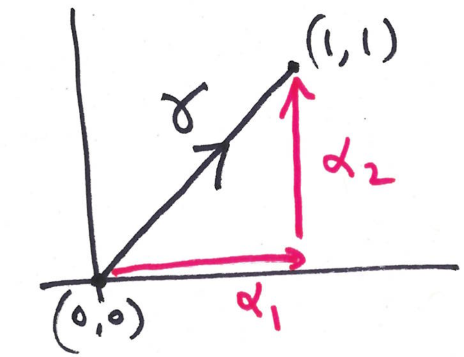
$$\frac{\partial g}{\partial x} = 2xy^2 \Rightarrow g(x,y) = \int 2xy^2\,dx$$

$$= x^2y^2 + h(y).$$

$$\frac{\partial g}{\partial y} = 2x^2y + h'(y) = 2x$$

cannot have a solution

Ex.: Let $f = (2xy^2, 2yx^2)$



$$\gamma : [0,1] \to \mathbb{R}^2$$
$$t \to (t,t)$$

$$\alpha_1 : [0,1] \to \mathbb{R}^2$$
$$t \to (t,0)$$

$$\alpha_2 : [0,1] \to \mathbb{R}^2$$
$$t \to (1,t)$$

$$\int_\gamma f\,ds = \int_0^1 f(t,t)\cdot(1,1)\,dt$$
$$= \int_0^1 (2t^3, 2t^3)\cdot(1,1)\,dt$$
$$= \int_0^1 4t^3\,dt = t^4 \Big|_0^1 = 1$$

$$\int_{\alpha_1+\alpha_2} f\,ds = \int_{\alpha_1} f\,ds + \int_{\alpha_2} f\,ds = \int_0^1 f(\alpha_1(t))\cdot(1,0)\,dt + \int_0^1 f(\alpha_2(t))\cdot(0,1)\,dt$$
$$= \cdots = 1.$$

In this case there is a potential for f, namely

$$g(x,y) = x^2y^2$$
$$\nabla g = f.$$

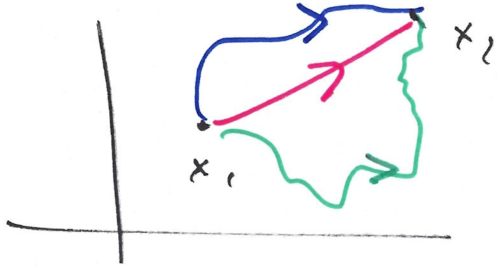
Line integ of f is indep of path of integration, only depends on the end pts of the path.

Defn Let $X \subset \mathbb{R}^n$, $f : X \to \mathbb{R}^n$ be a continuous vector field.

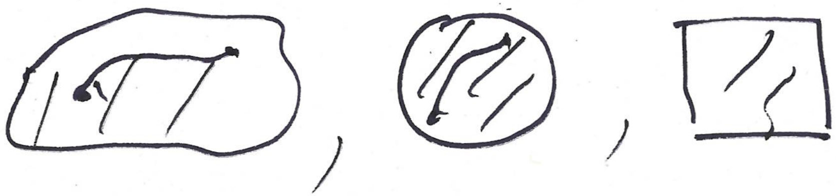
If for any $x_1, x_2 \in X$ the line integral $\int_\gamma f\,ds$ is indep. of the curve in X from x_1 to x_2.

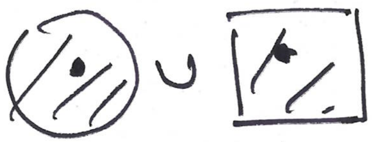
Then we say the vector field f is **conservative**,

How do we decide if a vector field f is conservative?



Defn. Let $X \subset \mathbb{R}^n$ open. X is said to be **path connected** if for every pair of pts $x, y \in X$, $\exists$ a C^1-path $\gamma : [0,1] \to X$ with $\gamma(0) = x$, $\gamma(1) = y$.





not path connected.

Thm. Let f be a cont. vector field on an open path connected set $X \subset \mathbb{R}^n$. Then TFAE.

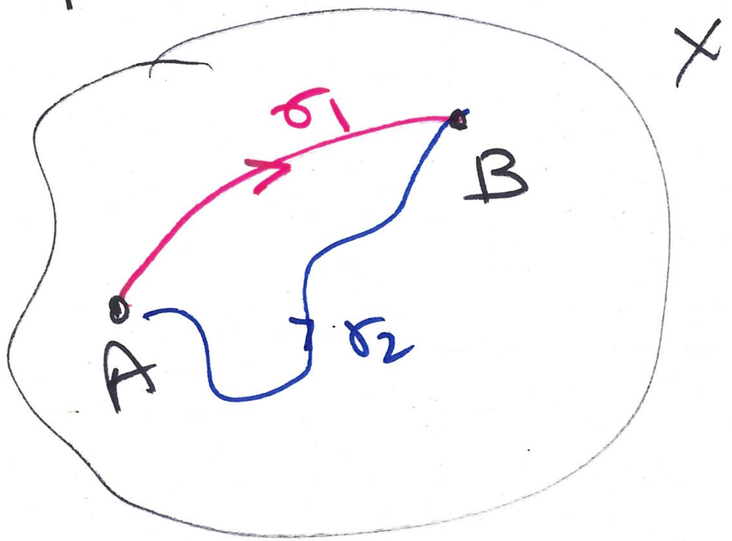
(1) f is the gradient of a func $g : X \to \mathbb{R}$ ie $f = \nabla g$, g is a potential for f

(2) The line integ. of f ~~betw~~ is indep of path between 2 pts.

ie. if $\gamma_1 : [a,b] \to X$
$\gamma_2 : [c,d] \to X$
are 2 curves

$\gamma_1(a) = \gamma_2(c) = A$

$\gamma_1(b) = \gamma_2(d) = B$.



$$\int_{\gamma_1} f\,ds = \int_{\gamma_2} f\,ds.$$

(3) The line integral of f around any closed curve is 0.

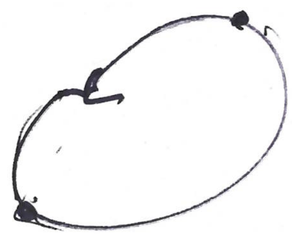
Remark: We've seen (1) $\Rightarrow$ (2)

(2) $\Leftrightarrow$ (3) clear.

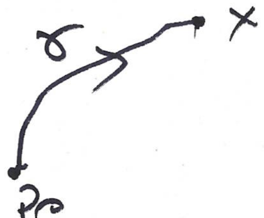
(3) $\Rightarrow$ (1)

$$\int_{\gamma_1 - \gamma_2} f\,ds = 0 = \int_{\gamma_1} f\,ds - \int_{\gamma_2} f\,ds$$

$$\Rightarrow \int_{\gamma_1} f\,ds = \int_{\gamma_2} f\,ds.$$



(2) $\Rightarrow$ (1) Idea is similar to 1-variable case. For each $x \in X$, let $\gamma : [0,1] \to X$ be a path $\gamma(0) = p_0$, $\gamma(1) = x$



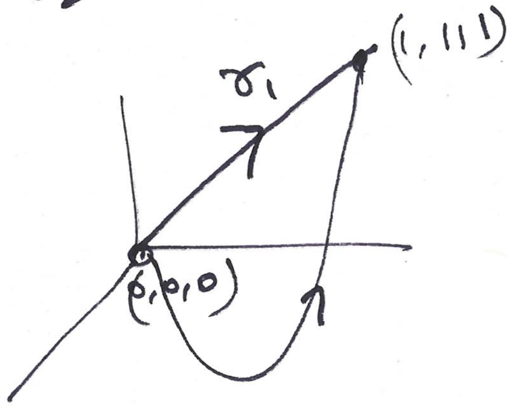
Define $g(x) := \int_{p_0}^{x} f\,ds$

Since it is indep of path we write $\int_{p_0}^{x}$ instead of $\int f\,ds$.

Ex: $f = (y^2, xz, 1)$ is not conservative

Because $\int_{\gamma_1} f\,ds = 5/3$.

$$\int_{\gamma_2} f\,ds = 23/15$$



$$\gamma_2 = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$$

we can also check

if there were g s.t $\nabla g = f$

we get into a contradiction

For **example** $\frac{\partial g}{\partial x} = y^2$ $\quad \frac{\partial g}{\partial y} = xz$

$$\frac{\partial g}{\partial z} = 1$$

$\Rightarrow \frac{\partial g}{\partial x} = y^2 \Rightarrow g(x,y,z) = y^2 x + h(y,z)$

$$\frac{\partial g}{\partial y} = 2xy + \frac{\partial h(y,z)}{\partial y} = xz$$

$$= - - -$$

Rk - If we know that a vector field is conservative the above method ~~also~~ can also be used to find a potential g.

eg: $f = (2xy^2, 2yx^2)$

$\frac{\partial g}{\partial x} = 2xy^2$, $\quad \frac{\partial g}{\partial y} = 2yx^2$

$\Downarrow$

$g(x,y) = x^2y^2 + h(y) \Rightarrow \frac{\partial g}{\partial y} = 2yx^2 + h'(y)$

$= 2yx^2$

$\Rightarrow h'(y) = 0 \Rightarrow h(y) = \text{const.}$

$\Rightarrow \boxed{g(x,y) = x^2y^2 + c}$.

Is there an easy criteria to check if a vector field is conservative or not?

We have the following <u>necessary</u> condition

<u>Thm</u> Let $X \subset \mathbb{R}^n$ open

$f : X \to \mathbb{R}^n$ a C^1 vectorfield. $f = (f_1 \dots f_n)$

If f is conservative then

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$$

<u>Pf:</u> If $f = \nabla g$, $g \in C^2$

Then $\dfrac{\partial^2 g}{\partial x_i \partial x_j} = \dfrac{\partial^2 g}{\partial x_j \partial x_i}$ $\quad \dfrac{\partial}{\partial x_j}\underbrace{\left(\dfrac{\partial g}{\partial x_i}\right)}_{f_i}$

$$\Rightarrow \frac{\partial}{\partial x_i}\left(\frac{\partial g}{\partial x_j}\right) = \frac{\partial}{\partial x_i} f_j$$

<u>eg.</u> $f = (\underbrace{2xy^2}_{f_1}, \underbrace{2x}_{f_2})$

$$\frac{\partial f_2}{\partial x} = 2 \neq \frac{\partial f_1}{\partial y} = 2xy$$

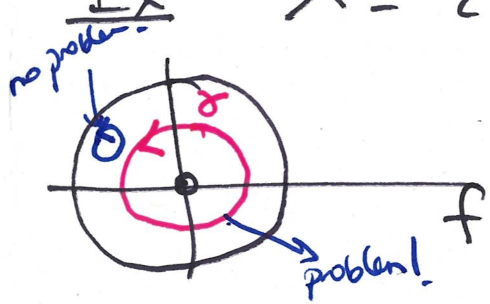
$\Rightarrow f$ is not conservative.

<u>Warning</u>. $\dfrac{\partial f_i}{\partial x_j} = \dfrac{\partial f_j}{\partial x_i} \not\Rightarrow f$ is conser.

$\not\Rightarrow \int f$ is indep. of path of integration

<u>See</u> Ex 10.3.

<u>Ex</u> $X = \{(x,y) \mid 0 < x^2+y^2 < 2\}$.



Take $\gamma(t) = (\cos t, \sin t)$ $t \in [0, 2\pi]$

$$f(x,y) = \begin{pmatrix} -y/x^2+y^2 \\ x/x^2+y^2 \end{pmatrix}$$

Then check.

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$$

But $\int_\gamma f\,ds \neq 0$.

$$= 2\pi.$$

ie f is not conservative

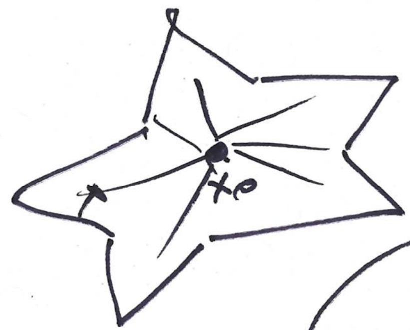
yet $\dfrac{\partial f_1}{\partial y} = \dfrac{\partial f_2}{\partial x}$.

Defn. A subset $X \subset \mathbb{R}^n$ is called **star shaped** if $\exists\, x_0 \in X$ such that $\forall x \in X$, the line segment joining x_0 to x is contained in X.

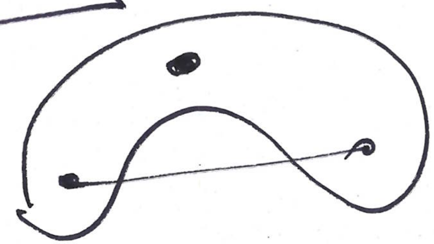
Rk Convex $\Rightarrow$ star shaped

"

any 2 pts can be joined by a line in the set.

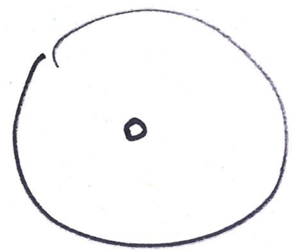


star-shaped



not star shaped.

not convex



It is not star-shaped.

Thm Let X be a star-shaped open sset of $\mathbb{R}^n$. $f \in C^1$ vector field s.t $\boxed{\dfrac{\partial f_i}{\partial x_j} = \dfrac{\partial f_j}{\partial x_i} \text{ on } X}$ $\forall i, j$ then f is conservative.

$f : X \to \mathbb{R}^n$.

$X = \mathbb{R}^n$, Ball, all star-shaped.

<u>$n=2$</u> These conditions are

$$\partial_x f_2 = \partial_y f_1$$

<u>$n=3$</u> 3 conditions

$$\partial_y f_3 - \partial_z f_2 = 0$$

$$\partial_z f_1 - \partial_x f_3 = 0$$

$$\partial_x f_2 - \partial_y f_1 = 0.$$

<u>Defn.</u> Let $X \subset \mathbb{R}^3$ open, $f : X \to \mathbb{R}^3$ C^1 vector field. Then the <u>curl of f</u> is the vector field on X defined by

$$\boxed{\mathrm{curl}(f) := \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_3 \end{pmatrix}.}$$

where $f : X \to \mathbb{R}^3$

$$\bar{x} \to \begin{matrix} f_1(\bar{x}) \\ f_2(\bar{x}) \\ f_3(\bar{x}) \end{matrix}$$

$\bar{x} = (x, y, z)$

<u>Rk</u> This can be remembered more easily by the formal determinant

$$\begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{vmatrix}$$