$\gamma : [a,b] \to \mathbb{R}^n$ a curve

$f : \mathbb{R}^n \to \mathbb{R}^n$ continuous vector field.

Line Integral of f along γ is

$$\int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt = \int_\gamma f(s)\, ds$$

Properties of Line Integrals.

(1) It is independent of orientation preserving reparametrization of the curve. ie. if

$\tilde{\gamma} : [c,d] \to \mathbb{R}^n$ is such that

$\tilde{\gamma} = \gamma \circ \phi$ with $\phi : [c,d] \to [a,b]$

a C^1 function such that $\phi(c) = a$

$\phi(d) = b$ and $\phi'(t) > 0 \ \forall\, t \in [c,d]$

Then $\int_\gamma f\, ds = \int_{\tilde{\gamma}} f\, ds.$

(2) $\int_{\gamma_1 + \gamma_2} f\, ds = \int_{\gamma_1} f\, ds + \int_{\gamma_2} f\, ds$

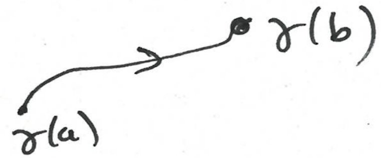
(3) $\int_{-\gamma} f\, ds = -\int_\gamma f\, ds$

(4) If $\exists\, g : X \to \mathbb{R}$, $g \in C^1$ such that $\nabla g = f$. Then

For any $\gamma : [a,b] \to \mathbb{R}$ with $\gamma([a,b]) \subset X$, we have

$$\int_\gamma f\, ds = (g \circ \gamma)(b) - (g \circ \gamma)(a)$$

ie $\int_\gamma f\, ds$ only depends on the endpoints of the curve



Defn

- A differentiable function $g : X \to \mathbb{R}$, $X \subset \mathbb{R}^n$ such that $\nabla g = f$ (where $f : X \to \mathbb{R}^n$ a vector field) is called a potential for f.

If $n=1$, a potential g for f is just a primitive of f.

Defn Let $X \subset \mathbb{R}^n$, $f : X \to \mathbb{R}^n$ a continuous vector field. Let $x_1, x_2 \in X$. If the line integral of f along any curve between x_1, x_2 has the same value, ie $\int_\gamma f \, ds$ is independent of the path between the end points x_1, x_2, then f is called conservative.

Thm f cont. vector field $f : X \to \mathbb{R}^n$, X open, path connected. Then TFAE

1. $\exists g : X \to \mathbb{R}$ such that $f = \nabla g$
2. f is conservative
3. If γ is any closed path, [i.e. $\gamma : [a,b] \to X \subset \mathbb{R}^n$ such that $\gamma(a) = \gamma(b)$] then $\int_\gamma f \, ds = 0$.

Thm (Necessary condition for f to be conservative)

$f : X \to \mathbb{R}^n$, X open, f is C^1.
If f is conservative then $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ $\forall\, i = 1, \dots, n$, $j = 1, \dots, n$, $i \neq j$

Thm If X is star-shaped, open in $\mathbb{R}^n$ then $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ $\forall i, j$ is also sufficient.

Let $f : X \to \mathbb{R}^n$ vector field

X open, star shaped.

If $\dfrac{\partial f_i}{\partial x_j} = \dfrac{\partial f_j}{\partial x_i}$ on X $\forall i, j$

then f is conservative.

e.g. • $n=2$ $\quad f : X \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$$

X, starshaped (for example if $X = \mathbb{R}^2$, $B_r^\circ(a)$ - a Ball)

then need to check one condition : $\partial_x f_2 = \partial_y f_1$

• $n=3$ Need to check 3 conditions

$$f : X \to \mathbb{R}^3$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \\ f_3(x,y,z) \end{pmatrix}$$

$$\partial_y f_3 - \partial_z f_2 = 0$$

$$\partial_z f_1 - \partial_x f_3 = 0$$

$$\partial_x f_2 - \partial_y f_1 = 0.$$

Defn $f : X \to \mathbb{R}^3$ C^1 vector field

$\operatorname{curl} f : X \to \mathbb{R}^3$, the curl of f is the vector field defined by

$$\operatorname{curl} f := \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix}$$

$$\begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$f : \mathbb{R}^3 \to \mathbb{R}^3$

Ex: $f(x,y,z) = (e^x \cos y + yz,\ xz - e^x \sin y,\ xy + z)$

Is f conservative?

$\mathbb{R}^3$ is star-shaped so it is enough to check if $\operatorname{curl}(f) = \underline{0}$?

$$\operatorname{curl} f = \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} x - x \\ y - y \end{pmatrix}$$

$(z - e^x \sin y) - (-e^x \sin y + z)$

$$\operatorname{curl} f = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \checkmark \quad f \text{ is conservative}$$

Hence $\exists\ g : \mathbb{R}^3 \to \mathbb{R}$ st. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto g(x,y,z)$

s.t $\boxed{\nabla g = f}$

To find $g(x,y,z)$ such that $\nabla g = f$, we use

$$\frac{\partial g}{\partial x} = e^x \cos y + yz \quad (1)$$

$$\frac{\partial g}{\partial y} = xz - e^x \sin y \quad (2)$$

$$\frac{\partial g}{\partial z} = xy + z \quad (3)$$

$(1) \Rightarrow g(x,y,z) = \int (e^x \cos y + yz)\,dx$

$= e^x \cos y + yzx + h(y,z)$

diff wrt y.

$$\frac{\partial g}{\partial y} = -e^x \sin y + zx + \frac{\partial h}{\partial y}(y,z)$$

$= (2) = xz - e^x \sin y$

$$\Rightarrow \frac{\partial h}{\partial y}(y, z) = 0$$

$$\Rightarrow h(y, z) = h(z)$$

$$\Rightarrow g(x, y, z) = e^x \cos y + yzx + h(z)$$

$$\frac{\partial g}{\partial z} = xy + h'(z) = (3) = xy + z$$

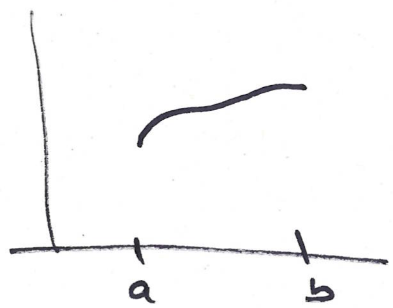
$$\Rightarrow h'(z) = z$$

$$\Rightarrow h(z) = \frac{z^2}{2} + c$$

$$\Rightarrow g(x, y, z) = e^x \cos y + yzx + \frac{z^2}{2} + c$$

Check that indeed

$$\nabla g = f$$



$$\int_{[a,b]=\gamma} f(\underbrace{\gamma(t)}_{x}) \cdot \underbrace{\gamma'(t)\,dt}_{dx} = \int_a^b f(x)\,dx.$$

$$f : \mathbb{R} \to \mathbb{R}$$

$f : \mathbb{R}^n \to \mathbb{R}$

scalar field

$\int f \, dx$

where?

$f : \mathbb{R}^n \to \mathbb{R}^n$

line integral of f along curves.

f is a vector field.

/5

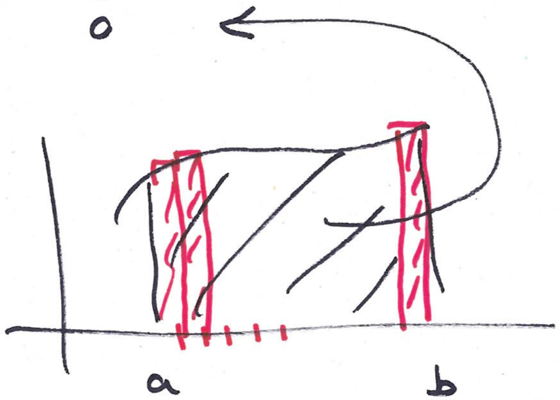
§ 4.2. Riemann Integral in $\mathbb{R}^n$.

$f : \mathbb{R}^2 \to \mathbb{R}$

$n = 1$

$f : [a, b] \to \mathbb{R}$

$\int_a^b f(x)\,dx$, say $f \geq 0$



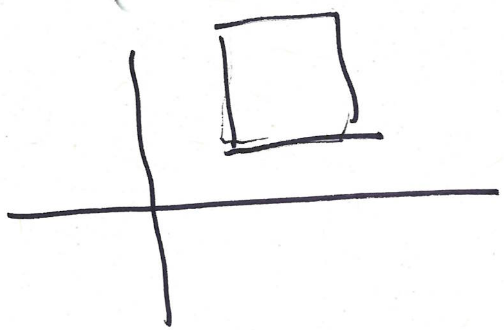
$\Sigma(f, P, \xi_i)$

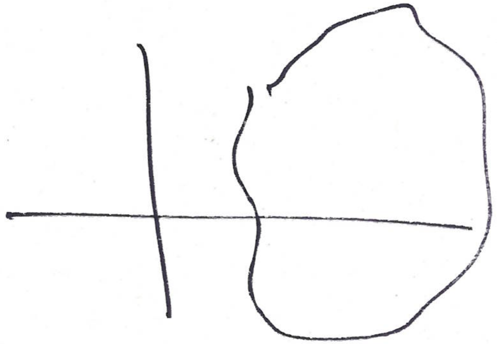
$\delta(P) = \min_i (x_i - x_{i-1})$

$P = \{a = x_0 < x_1 \cdots < x_n = b\}$.

$$\int_a^b f(x)\,dx = \lim_{\delta(P) \to 0} \Sigma(f, P, \xi_i)$$

We can construct multiple integrals of functions $f : \mathbb{R}^n \to \mathbb{R}$ over regions in $\mathbb{R}^n$.

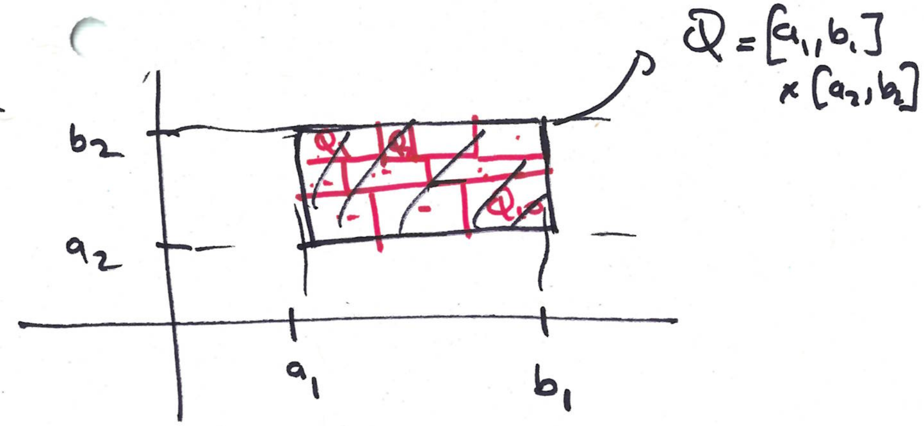




Analog of a closed interval in $\mathbb{R}^n$ is a closed rectangle $Q = I_1 \times \dots \times I_n$ where $I_k = [a_k, b_k]$

$$Q = \{ (x_1 \dots x_n) \in \mathbb{R}^n \mid x_k \in I_k, \ 1 \le k \le n \}$$

$n = 2$



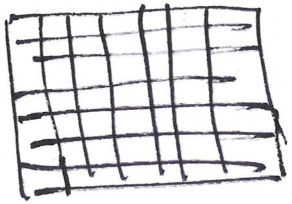
$$\text{vol}(Q) = \prod_{i=1}^{n} (b_i - a_i) = \mu(Q)$$

A partition P of Q is a subcollection of rectangular boxes $Q_1, \dots Q_k \subset Q$ such that

(1) $Q = \bigcup_{j=1}^{k} Q_j$

(2) $\text{Int}\, Q_i \cap \text{int}\, Q_j = \phi$, $i \neq j$

Norm (P) $= \delta_P := \max(\operatorname{diam} Q_j)$



For each Q_j, we choose $\xi_j \in Q_j$

Riemann sum of f, for partition P, intermediate point $\{\xi_i\}$ is

the sum

$$R(f, P, \xi) := \sum_{j=1}^{k} f(\xi_i)\,\mathrm{vol}(Q_j)$$

Lower-R-sum $:= \sum_{j=1}^{k} \left(\inf_{x\in Q_j} f(x)\right) \mathrm{vol}(Q_j) =: L_f(P)$

Upper R-sum $:= \sum_{j=1}^{k} \left(\sup_{x\in Q_j} f\right)(\mathrm{vol}\, Q_j) =: U_f(P)$

Lower R-integ

$$\underline{I}(f) = \underline{\int_Q} f\,dx := \sup\{L_f(P) \mid P \text{ runs all partitions of } Q\}.$$

Upper R-integr

$$\overline{I}(f) = \overline{\int_Q} f\,dx := \operatorname{Inf}\{U_f(P) \mid P \text{ a partition of } Q\}.$$

f is called integrable if $\underline{I}(f) = \overline{I}(f)$ and in that case we write $\int_Q f(x)\,dx$

See pictures (8-1), (8-2)

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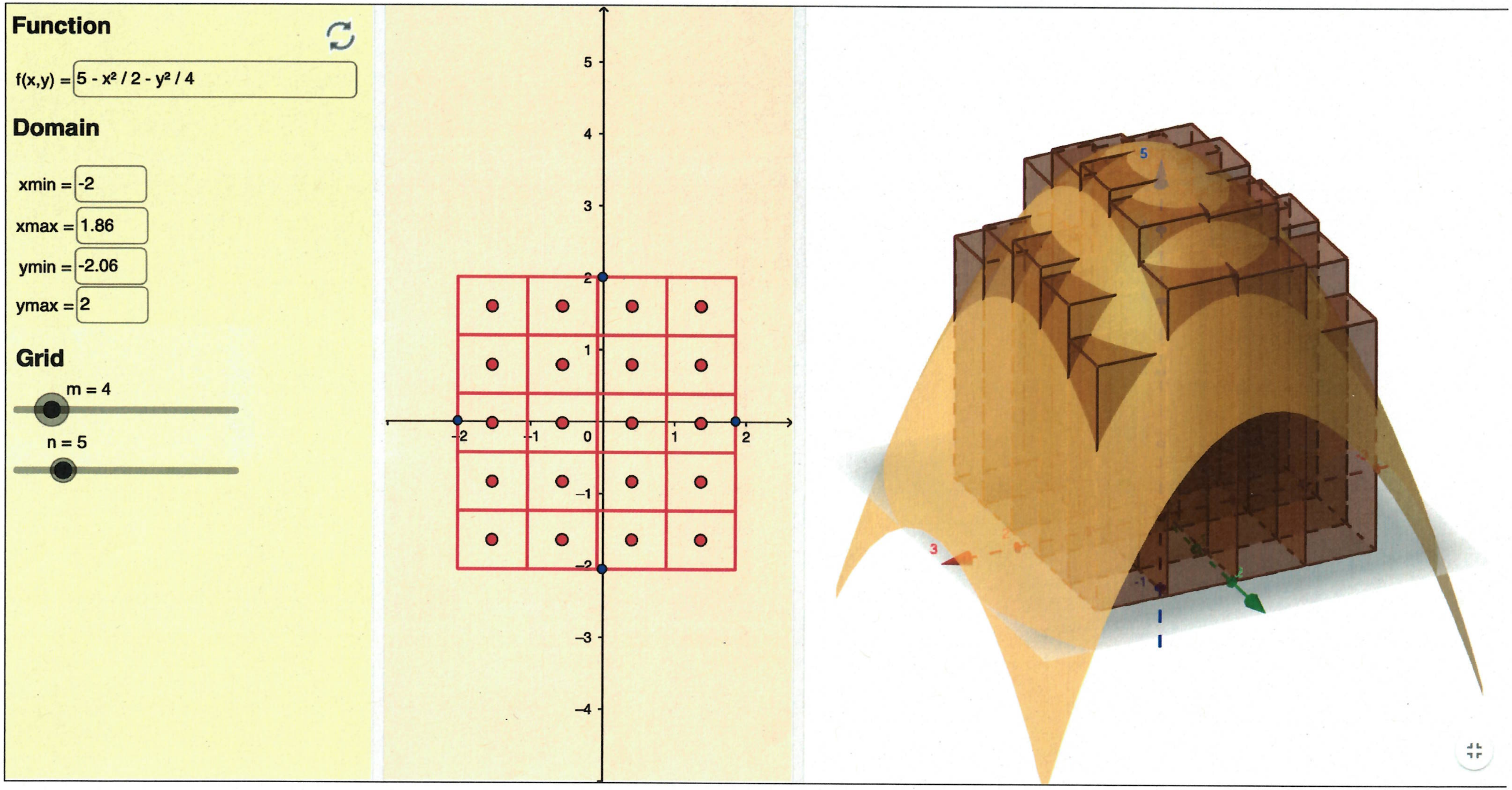
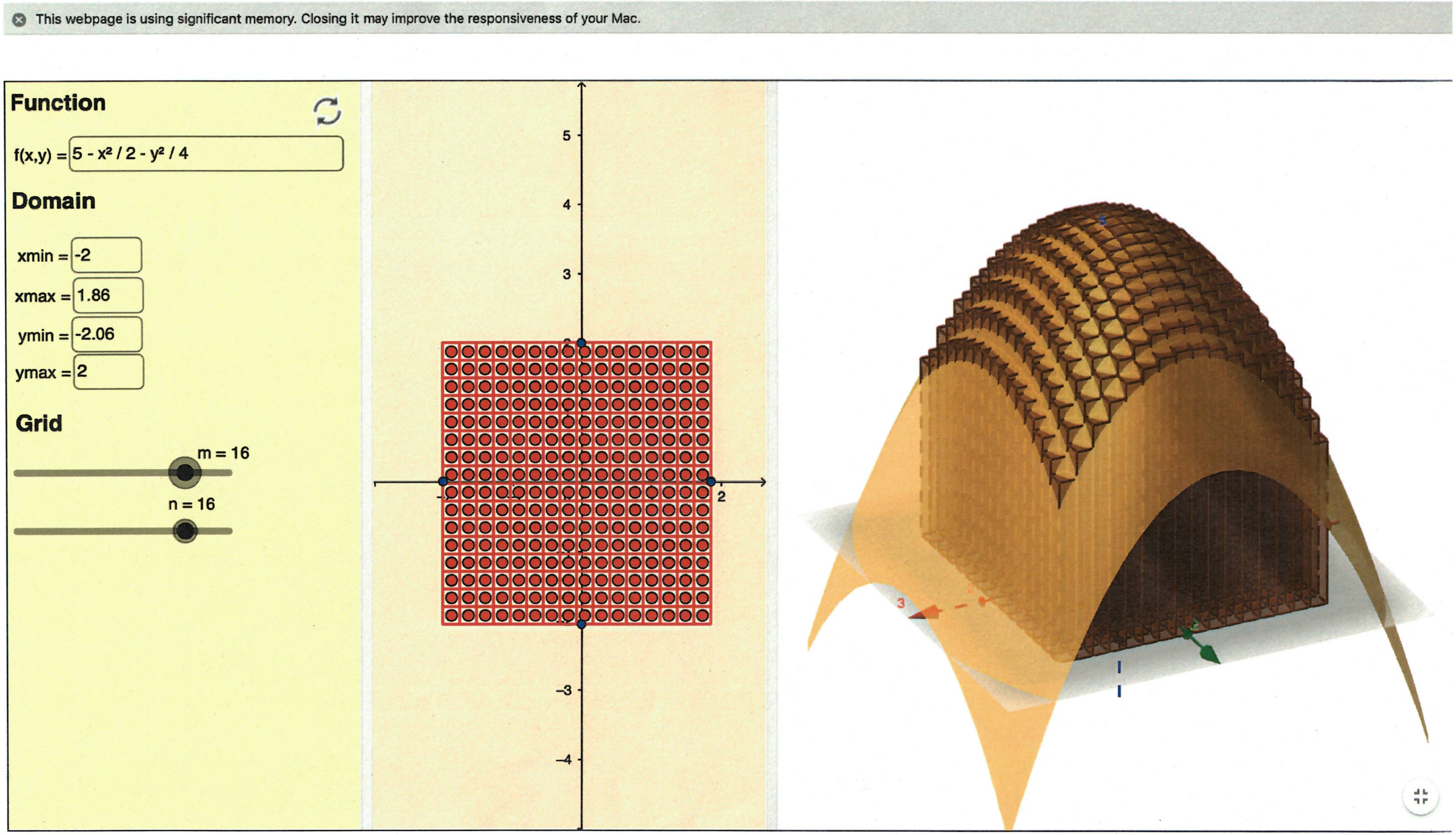




This webpage is using significant memory. Closing it may improve the responsiveness of your Mac.
Function
f(x,y) = 5 - x² / 2 - y² / 4
Domain
xmin = -2
xmax = 1.86
ymin = -2.06
ymax = 2
Grid
m = 16
n = 16
5
4
3
2
−3
−4

8.2

$$\int_Q f(x_1, \ldots x_n)\, dx_1 dx_2 \ldots dx_n .$$

$f : \mathbb{R}^n \to \mathbb{R}$, Q rectangular box in $\mathbb{R}^n$

Thm If f is continuous on Q then f is integrable.

Thm $f, g : Q \subset \mathbb{R}^n \to \mathbb{R}$ integrable, [illegible] $\alpha, \beta \in \mathbb{R}$.

then

(1) $\alpha f + \beta g : Q \to \mathbb{R}$ is integ. and

$$\int_Q (\alpha f + \beta g)\, dx = \alpha \int_Q f\, dx + \beta \int_Q g\, dx$$

(2) If $f(x) \leq g(x)$ $\forall x \in Q$

then $\int_Q f(x)\, dx \leq \int_Q g(x)\, dx$.

(3) If $f(x) \geq 0$ then $\int_Q f\, dx \geq 0$.

(4) $\left| \int_Q f\, dx \right| < \int_Q |f|\, dx$

$\leq \left(\sup_Q |f|\right) \text{vol } Q$

(5) Fubini's theorem.

$Q = I_1 \times \ldots \times I_n$, f continuous on Q

then $\int_Q f(x_1 \ldots x_n)\, dx_1 \ldots dx_n = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \ldots \int_{a_n}^{b_n} f(x)\, dx_n\, dx_{n-1} \ldots dx_1$

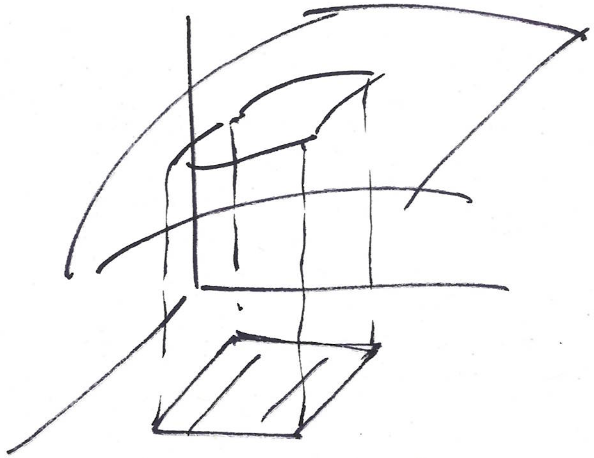
/9.

6) If $f = 1$ then

$$\int_Q 1 \, dx = \mathrm{vol}(Q)$$

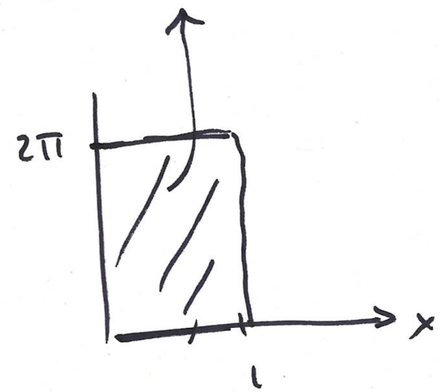
Rk If $f \geq 0$ the integral of f is the volume of the set $\{(x,y) \in Q \times \mathbb{R} \mid 0 \leq y \leq f(x)\}$

ie the volume under the graph of f above Q.



Ex: $f = e^x \sin y$

$Q = [0,1] \times [0, 2\pi]$.



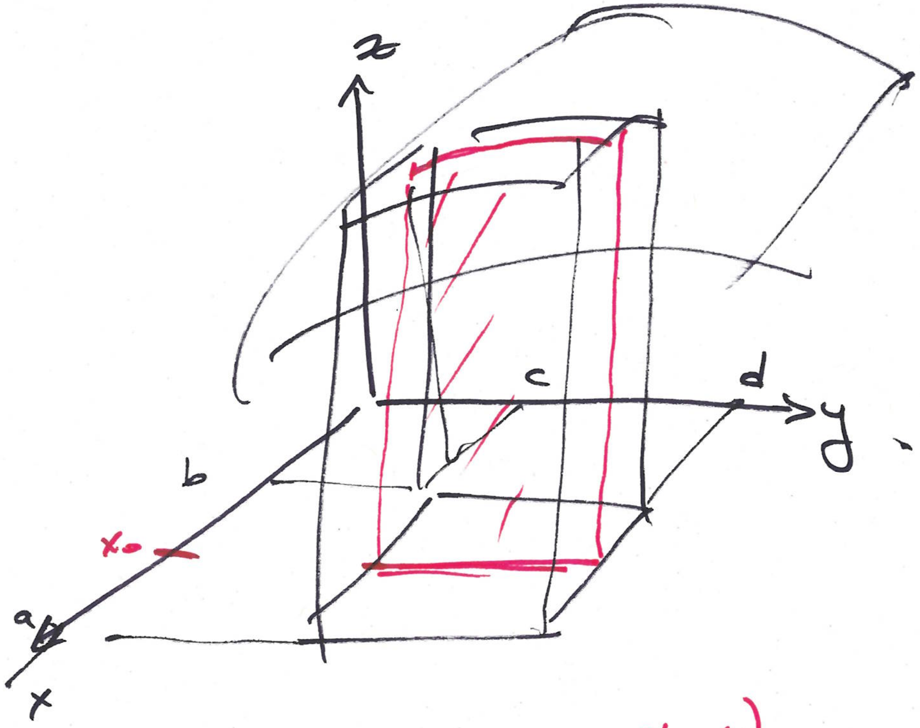
$$\int_Q f \, dx\,dy = \int_0^1 \left(\int_0^{2\pi} e^x \sin y \, dy \right) dx$$

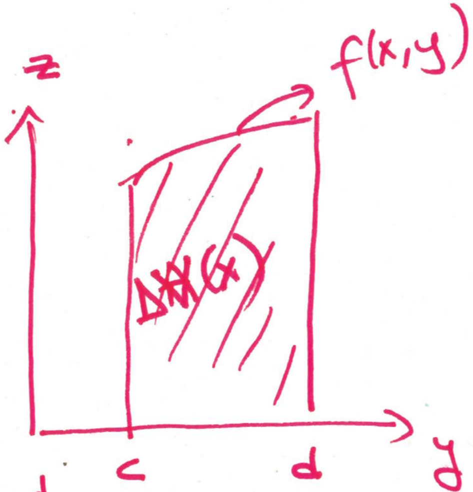
$$= \int_0^{2\pi} \left(\int_0^1 e^x \sin y \, dx \right) dy.$$

$$\int_0^1 e^x \underbrace{\left(\int_0^{2\pi} \sin y \, dy \right)}_{-\cos y \big|_0^{2\pi}} dx = \int_0^1 e^x \cdot 0 \, dx$$

$$= 0.$$

Geometric interpretation





$$\Delta V(x) = \int_c^d f(x,y)\,dy$$

(see geogebra pitches 11.1 11.2)

$$\int_a^b \Delta V(x)\,dx = \text{total volume.}$$

But in $\mathbb{R}^n$, we have more complicated regions than rectangular boxes.

es in $\mathbb{R}^2$, what about integrals over other regions.

Suppose $D \subset \mathbb{R}^2$ a region is of the following form

Order of integration:

☐ $dA = dy\,dx$

☑ $dA = dx\,dy$

Region of integration:

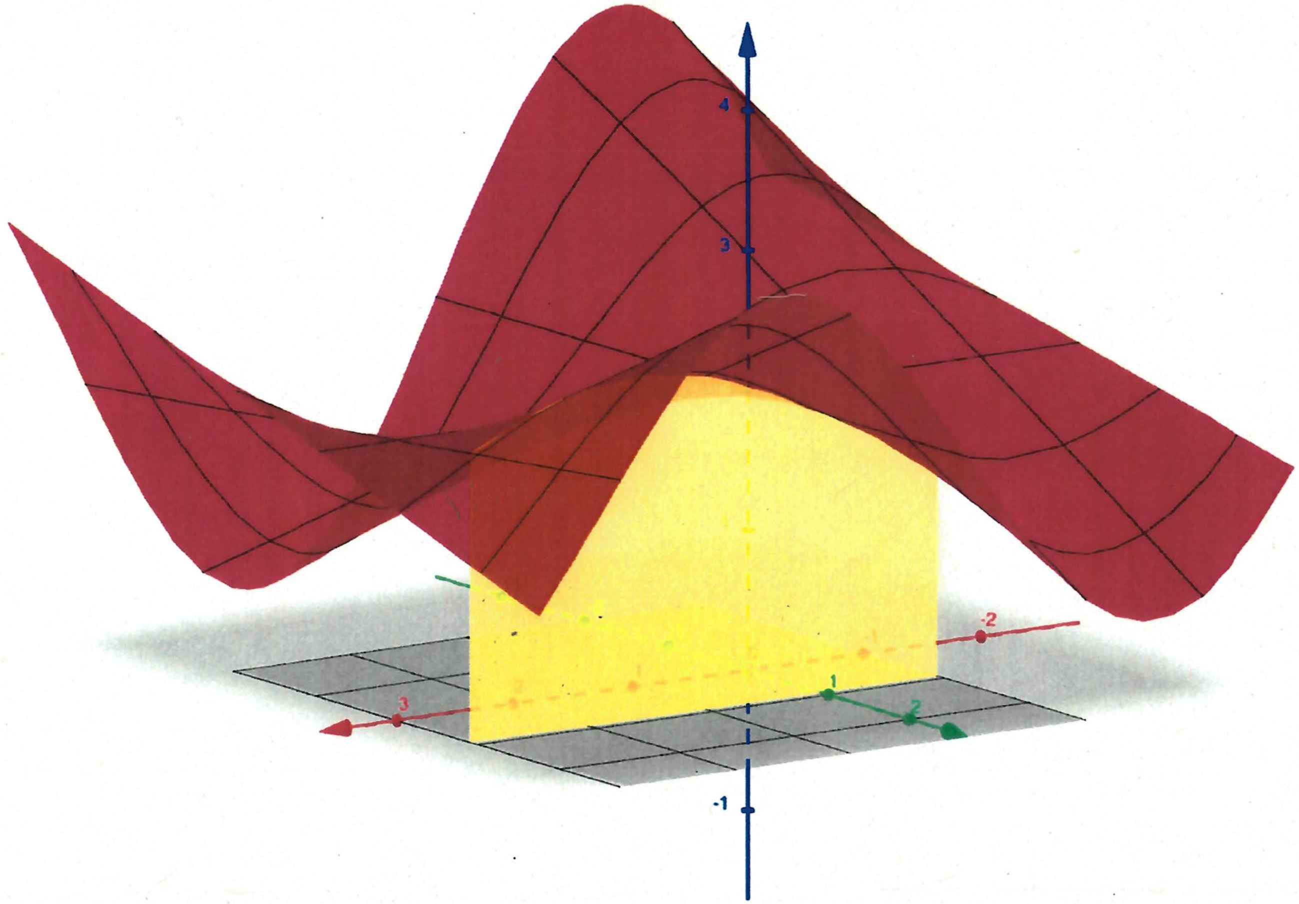
$$\left.\begin{array}{l} -1 \le x \le 3 \\ -2 \le y \le 3 \end{array}\right\} \mathcal{D}$$

$y_0 = 0.9$

$$\iint_{\mathcal{D}} f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

$$= \int_c^d A(y)\,dy$$

$A(y_0)$ = area of yellow region



11.1

Order of integration:

- [ ] $dA = dy\,dx$
- [x] $dA = dx\,dy$

Region of integration:

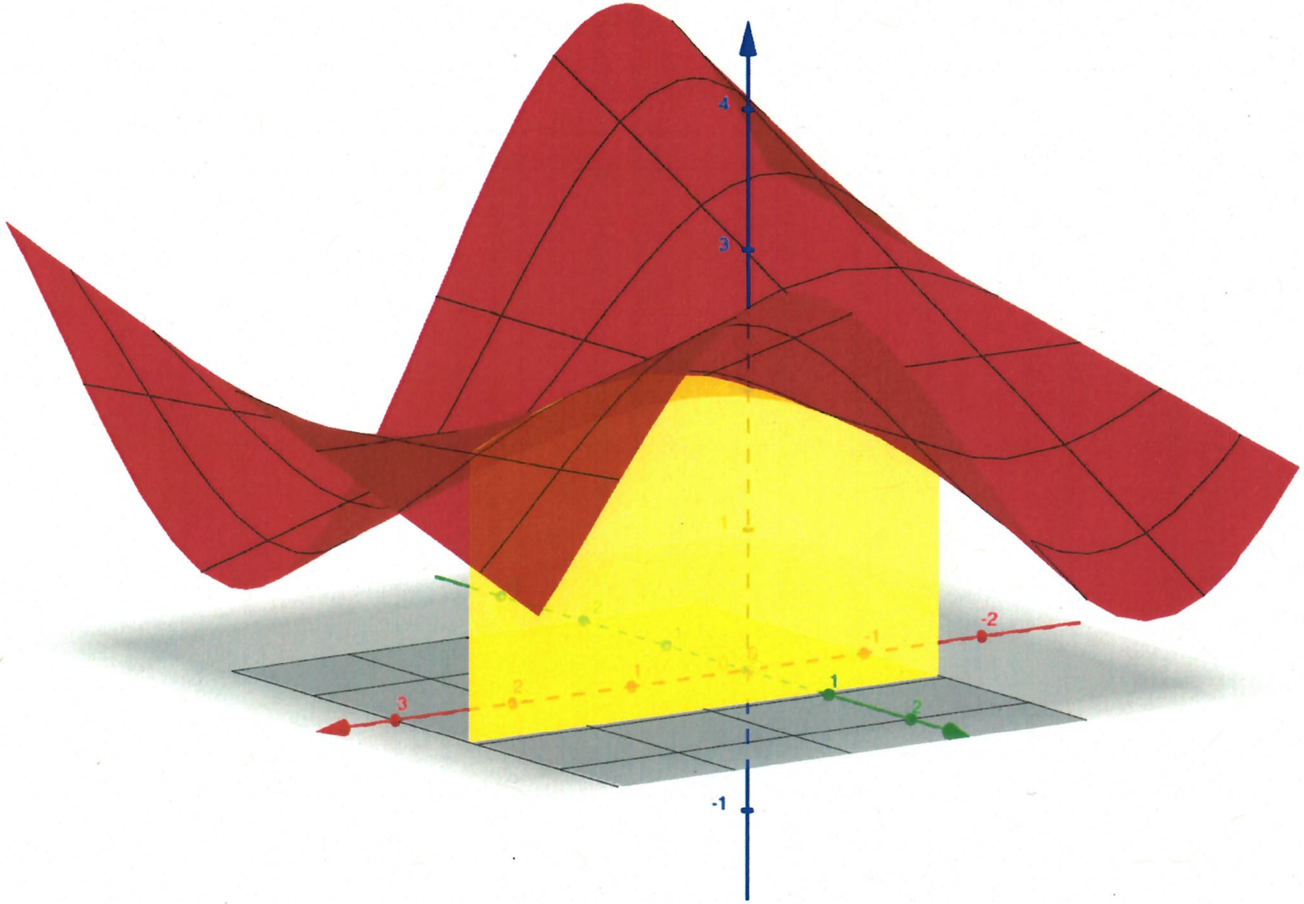
$$\left.\begin{array}{l} -1 \le x \le 3 \\ -2 \le y \le 3 \end{array}\right\}\mathcal{D}$$

$y_0 = 0.9$

$$\iint_{\mathcal{D}} f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

$$= \int_c^d A(y)\,dy$$

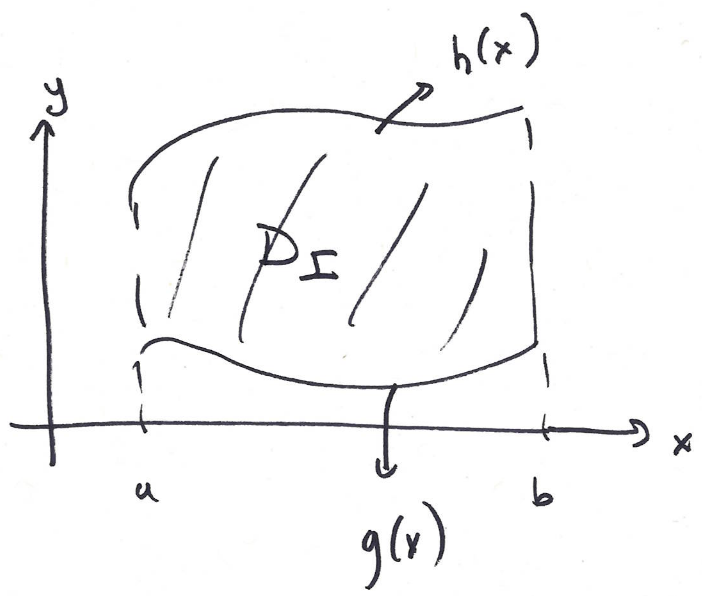
$A(y_0)$ = area of yellow region

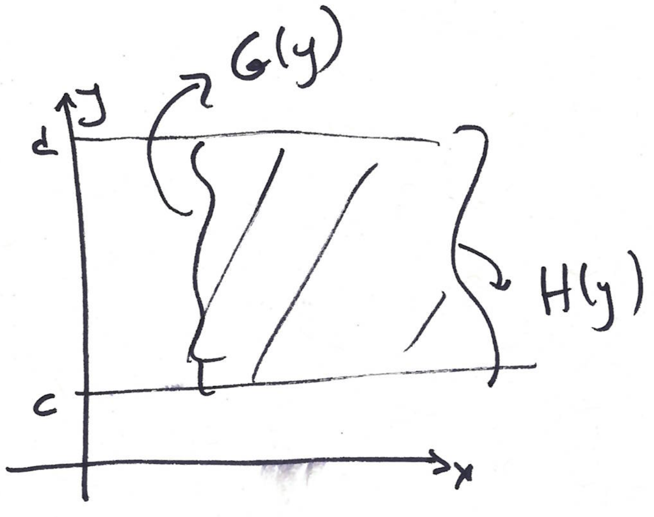


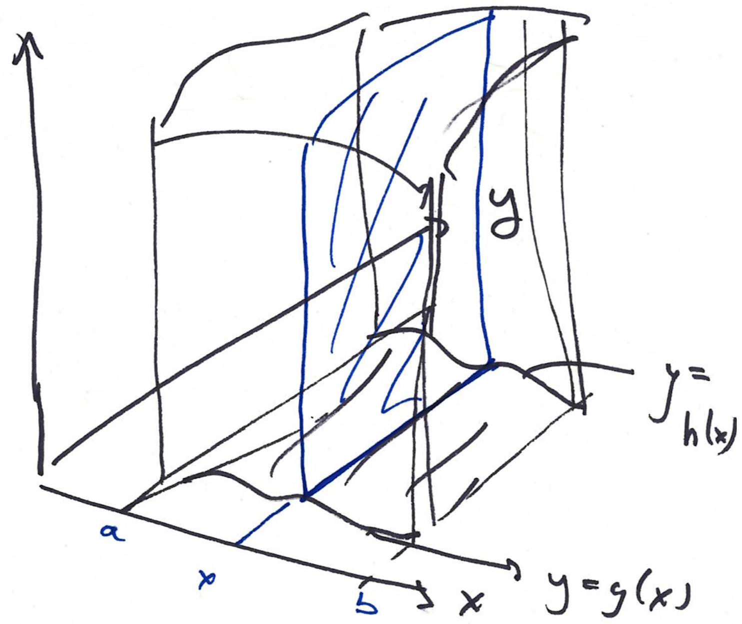
$$D_I := \{(x,y) \mid a \leq x \leq b,\ g(x) < y < h(x)\}$$

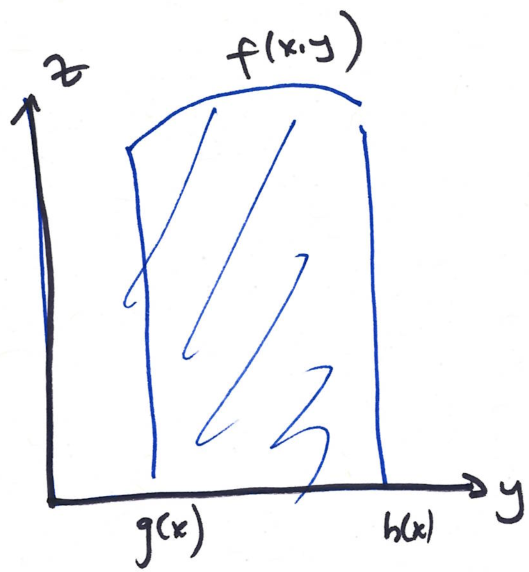
or

$$D_{II} := \{(x,y) \mid c \leq y \leq d,\ G(y) < x < H(y)\}$$

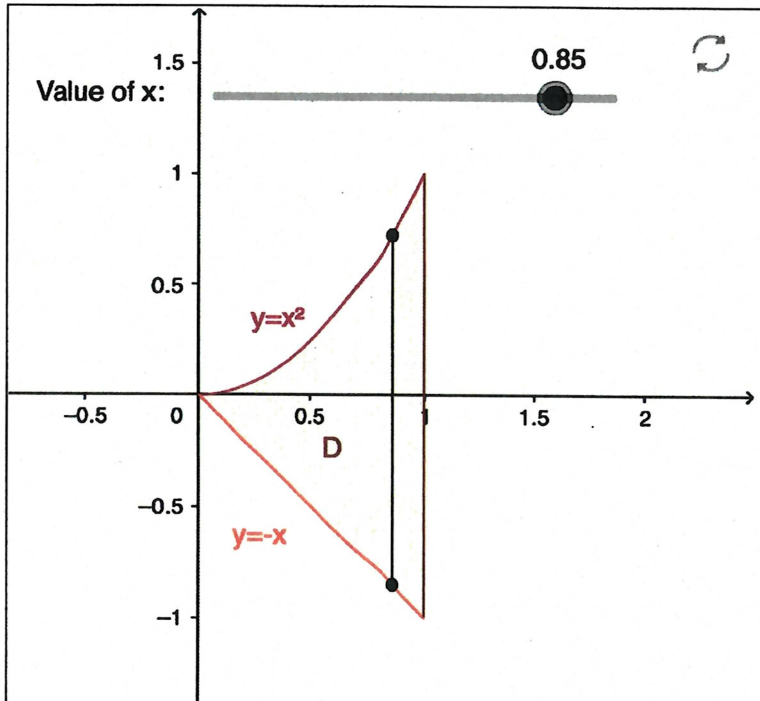








see geogebra picture (12.1)



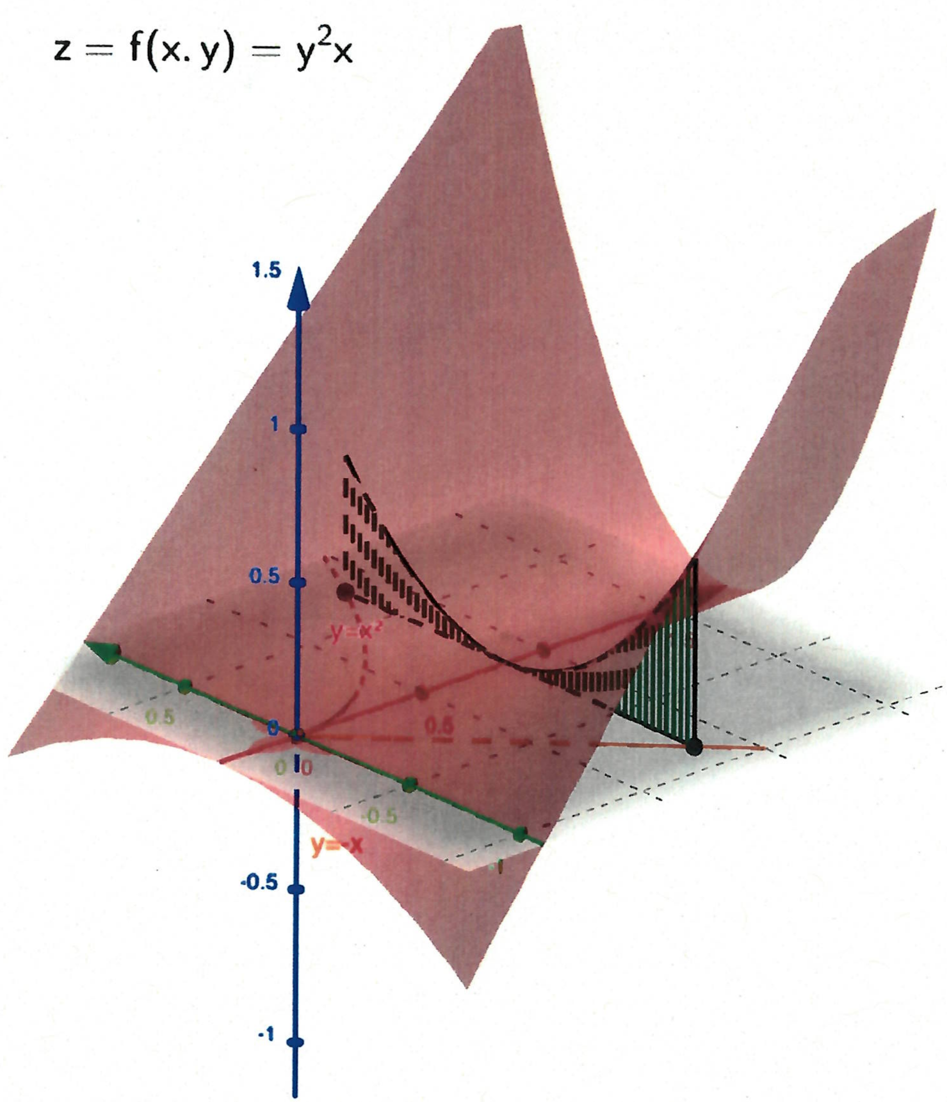
Green cross-sectional area

$$A(x) = \int_{-x}^{x^2} y^2x \, dy$$

Volume under the surface over D

$$\iint_D y^2x \, dx = \int_0^1 A(x) \, dx$$

$$= \int_0^1 \left(\int_{-x}^{x^2} y^2x \, dy \right) dx$$



$$V = \int_a^b \Delta V(x)\, dx$$

$$= \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\, dy \right) dx$$

Thm (Fubini) ① If D is of type I then

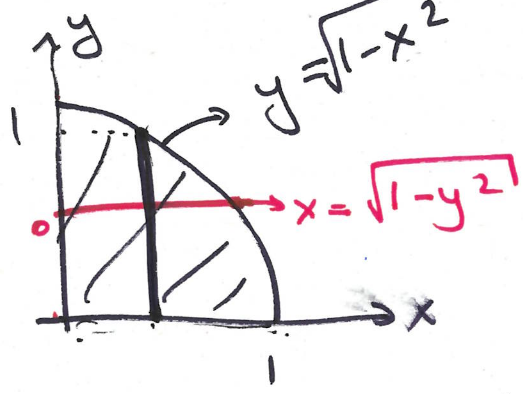
$$\int_D f\, dx\, dy = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\, dy\, dx$$

② If D is of type II

$$\int_D f\, dx\, dy = \int_c^d \left(\int_{G(y)}^{H(y)} f(x,y)\, dx \right) dy$$

Examples ① $f(x,y) = x+y$

D = region given by a quarter of a unit disc



$$\int_D f\, dx\, dy = \int_0^1 \left(\int_{0=y}^{\sqrt{1-x^2}=y} (x+y)\, dy \right) dx$$

$$\int_0^1 \left(\int_0^{\sqrt{1-y^2}} x+y\, dx \right) dy$$

$$\int_0^1 \left(\int_0^{\sqrt{1-y^2}} (x+y)\, dx \right) dy .$$

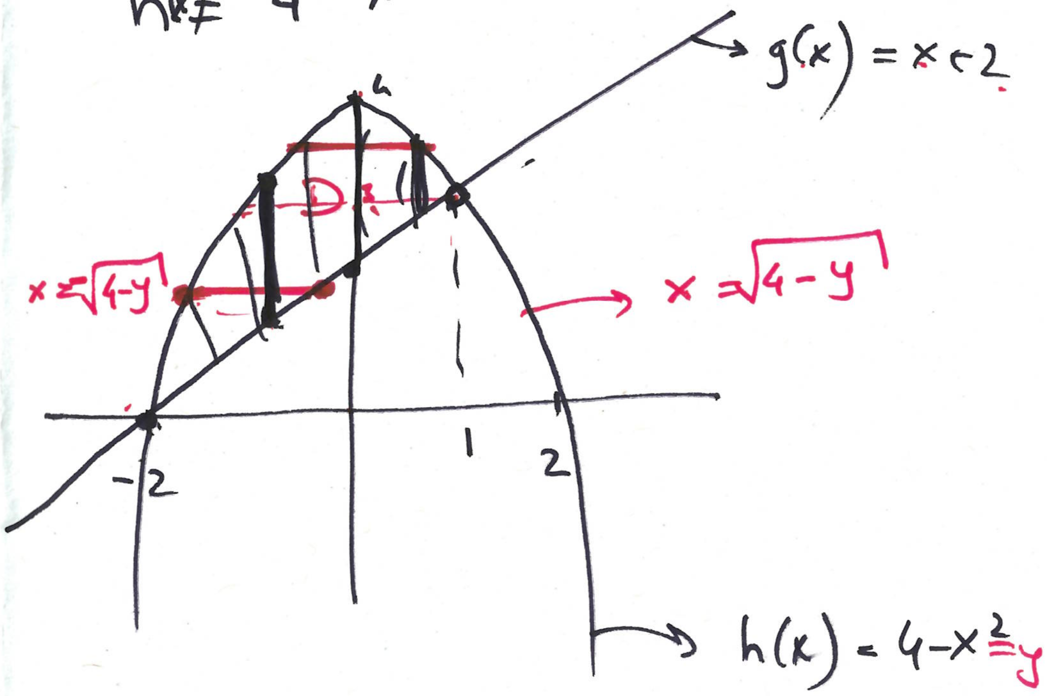
$$\int_0^1 \left(\frac{x^2}{2} + xy \Big|_0^{\sqrt{1-y^2}} \right) dy .$$

$$\int_0^1 \left(\frac{\left(\sqrt{1-y^2}\right)^2}{2} + y\sqrt{1-y^2} - 0 \right) dy .$$

$$= \int_0^1 \left(\frac{1-y^2}{2} + y\sqrt{1-y^2} \right) dy = \cdots 2/3 ,$$

② $f(x,y) = x$

D region bdd by the line $g(x) = x+2$ and the parabola $h(x) = 4 - x^2$



pt. of intersect. $4 - x^2 = x + 2$

$x^2 + x - 2 = 0 \rightarrow x = -2$, $x = 1$

$$\int_{-2}^{1} \left(\int_{y=x+2}^{y=4-x^2} x \, dy \right) dx = \int_D f \, dy\, dx$$

$$= \int_{-2}^{1} \left(xy \Big|_{y=x+2}^{y=4-x^2} \right) dx$$

$$= \int_{-2}^{1} \left[x(4-x^2) - x(x+2) \right] dx$$

$$= \int_{-2}^{1} 4x - x^3 - x^2 + 2x \, dx = \cdots$$

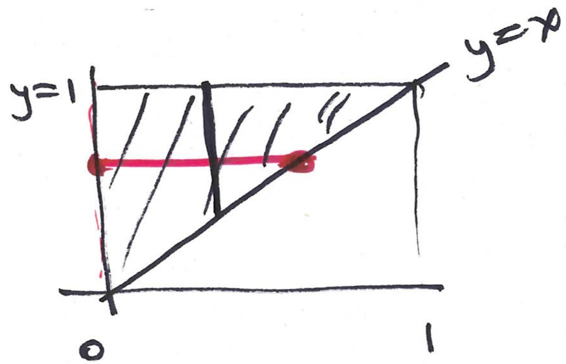
$$\int_D f \, dx\, dy = \int_{0}^{3} \int_{-\sqrt{4-y}}^{y-2} x \, dx \, dy + \int_{3}^{4} \int_{-\sqrt{4-y}}^{\sqrt{4-y}} x \, dx \, dy =$$

③ $\int_0^1 \int_x^1 e^{y^2} \, dy \, dx$

$\int e^{y^2} dy$ does not have an explicit primitive.

We can try to change the order of integration

First we need to understand the region of integration.



$$\int_0^1 \int_x^1 e^{y^2}\,dy\,dx$$

$$\int_0^1 \left(\int_{x=0}^{x=y} e^{y^2}\,dx \right) dy$$

$$= \int_0^1 \left(e^{y^2} \cdot x \Big|_{x=0}^{x=y} \right) dy$$

$$= \int_0^1 \left(y e^{y^2} - 0 \right) dy.$$

$$= \int_0^1 y e^{y^2}\,dy = \frac{e^{y^2}}{2} \Big|_0^1 = \frac{1}{2}(e-1)$$

Rk This example shows sometimes we <u>have</u> to change order of integration to be able to evaluate the integral.