3.12.20

Riemann Integral in $\mathbb{R}^n$.

$f: \mathbb{R}^n \longrightarrow \mathbb{R}$

$Q = [a_1, b_1] \times \cdots \times [a_n \times b_n]$

$\mathrm{vol}\, Q = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$

P: a partition of $Q = \{Q_1, \dots, Q_k\}$

i.e. • $Q = \bigcup_{j=1}^{k} Q_j$

• $(\mathrm{Int}\, Q_i) \cap (\mathrm{Int}\, Q_j) = \emptyset$ for $i \neq j$

$$L_f(P) = \sum_{j=1}^{k} \left(\inf_{Q_j} f\right) \mathrm{vol}(Q_j)$$

$$U_f(P) = \sum_{j=1}^{k} \left(\sup_{Q_j} f\right) \mathrm{vol}(Q_j)$$

$$\underline{I}(f) = \underline{\int_Q} f\, dx = \sup \{ L_f(P) \mid P \text{ partition of } Q \}$$

$$\overline{I}(f) = \overline{\int_Q} f\, dx = \inf \{ U_f(P) \mid P \text{ partitions of } Q \}$$

f is called integrable if

$$\underline{I}(f) = \overline{I}(f).$$

In that case we write

$\int_Q f(x)\, dx$ for the integral of f Q.

$$\int_Q f(x_1, \dots x_n)\, dx_1 dx_2 \cdots dx_n.$$

Thm 1) If f is continuous then f is integrable

2) if f, g are integrable, $\alpha, \beta \in \mathbb{R}$ then $\alpha f + \beta g$ is integrable

and $\int_Q (\alpha f + \beta g)\, dx = \alpha \int_Q f\, dx + \beta \int_Q g\, dx$

(Linearity)

Thm 3) **Monotonicity**

if $f(x) < g(x)$ then

$$\int_Q f(x)\,dx \le \int_Q g(x)\,dx$$

4) if $f(x) \ge 0$ then $\int_Q f(x)\,dx \ge 0$

5) $\left|\int_Q f\,dx\right| \le \int_Q |f(x)|\,dx$

$\le \left(\sup_Q |f|\right)(\mathrm{vol}\ Q)$

6) **Fubini's theorem**

$Q = [a_1, b_1] \times \dots \times [a_n, b_n]$

f continous on Q.

Then

$$\int_Q f(x)\,dx = \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2} \cdots \left(\int_{a_n}^{b_n} f(x)\,dx_n\right)\cdots\right)dx_1$$

7) $\int_Q 1\,dx = \mathrm{vol}\ Q$

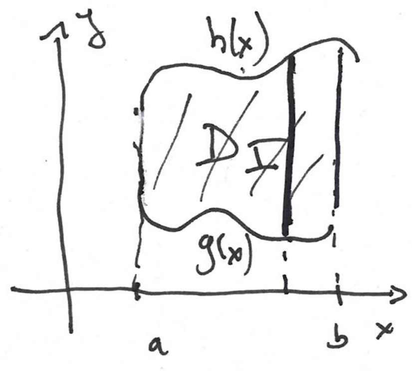
n = 2 **Thm** **Fubini**

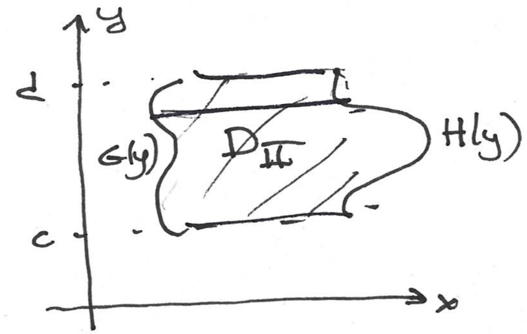
If $D_I = \{(x,y) \mid a \le x \le b,\ g(x) < y < h(x)\}$

then $$\int_{D_I} f(x,y)\,dx\,dy = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy\right)dx.$$

If $D_{II} = \{(x,y) \mid c \le y \le d,\ G(y) < x < H(y)\}$

then $$\int_{D_{II}} f(x,y)\,dx\,dy = \int_c^d \left(\int_{G(y)}^{H(y)} f(x,y)\,dx\right)dy$$





Then 3) Monotonicity (

If $f(x) \le g(x)$ then

$$\int_\Omega f(x)\,dx \le \int_\Omega g(x)\,dx$$

4) If $f(x) \ge 0$ then $\int_\Omega f(x)\,dx \ge 0$

5) $\left|\int_\Omega f\,dx\right| \le \int_\Omega |f(x)|\,dx$

$\le (\sup_\Omega f)(\mathrm{vol}\,\Omega)$

6) Fubini's theorem

$\Omega = [a_1, b_1] \times \cdots \times [a_n, b_n]$

f continuous on Ω.

Then

$$\int_\Omega f(x)\,dx = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f\,dx_n \cdots dx_1$$

7) $\int_\Omega 1\,dx = \mathrm{vol}\,\Omega$

<u>Def.</u> <u>Type I</u> <u>Type II</u>

If $D^I = \{(x,y) \mid a \le x \le b,\ g(x) \le y \le h(x)\}$

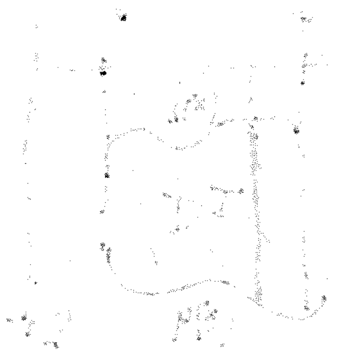
then

$$\iint_{D^I} f(x,y)\,dx\,dy = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy\right) dx$$

If $D^{II} = \{(x,y) \mid c \le y \le d,\ \varphi(y) \le x \le \psi(y)\}$

then

$$\iint_{D^{II}} f(x,y)\,dx\,dy = \int_c^d \left(\int_{\varphi(y)}^{\psi(y)} f(x,y)\,dx\right) dy$$



$f : X \to \mathbb{R}$, continuous

$X \subset \mathbb{R}^n$ bounded, closed.

Integral of f over X,

$\int_X f\,dx$ satisfies

- Linearity: if $f, g : X \to \mathbb{R}$ continuous, $\alpha, \beta \in \mathbb{R}$ then

$$\int_X (\alpha f + \beta g)\,dx = \alpha \int_X f\,dx + \beta \int_X g\,dx$$

- Positivity: if $f \leq g$ then

$$\int_X f\,dx \leq \int_X g\,dx$$

In particular if $f \geq 0$ then

$$\int_X f\,dx \geq 0.$$

- Upper bound and triangle inequality

$$\left| \int_X f\,dx \right| \leq \int_X |f|\,dx$$

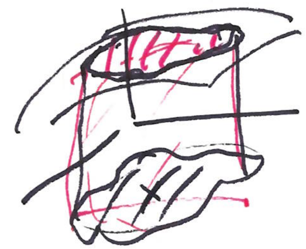
$$\left| \int_X (f(x) + g(x))\,dx \right| \leq \int_X |f|\,dx + \int_X |g|\,dx.$$

- Volume: if $f = 1$

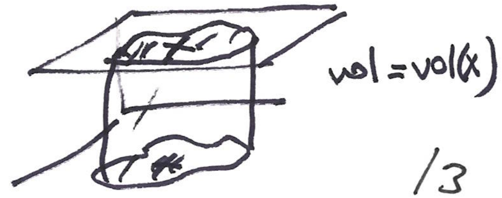
$$\int_X 1\,dx = \mathrm{vol}(X)$$

If $f \geq 0$, then

$$\int_X f\,dx = \mathrm{vol}\,\underbrace{\{(x, y) \in X \times \mathbb{R} \mid 0 \leq y \leq f(x)\}}_{\subseteq \mathbb{R}^{n+1}}$$



$f : \mathbb{R}^2 \to \mathbb{R}$



How does Fubini's thm look in general?

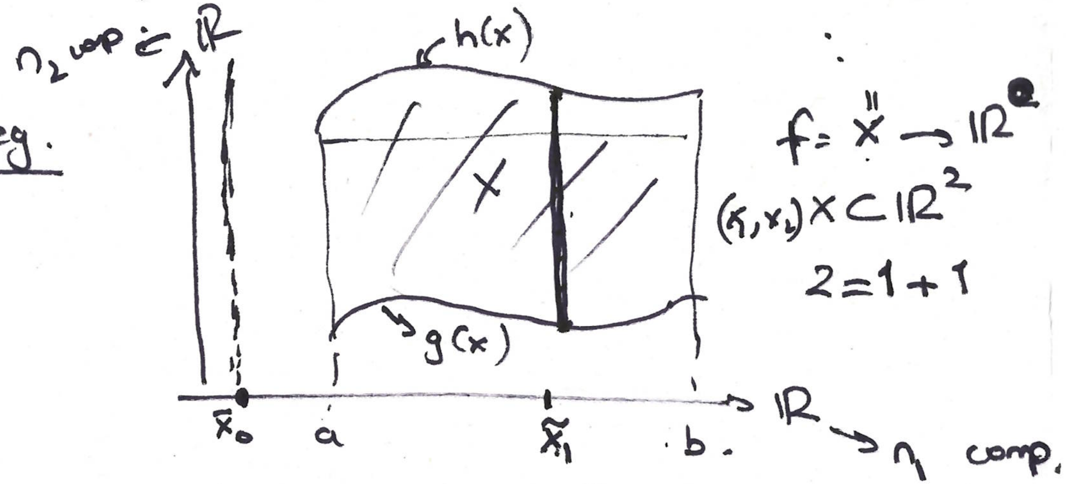
$f: X \to \mathbb{R}$, $X \subset \mathbb{R}^n$.

$n = n_1 + n_2$, $n_i \geq 1$ $x \in \mathbb{R}^n$, $x = (x_1, x_2)$, $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$

let $x_1 \in \mathbb{R}^{n_1}$, let

$$X_{x_1} = \{ x_2 \in \mathbb{R}^{n_2} \mid (x_1, x_2) \in X \}.$$

eg.



$f: X \to \mathbb{R}$

$(x_1, x_2)\, X \subset \mathbb{R}^2$

$2 = 1 + 1$

$$X_{\tilde{x}_1} = [g(\tilde{x}_1), h(\tilde{x}_1)] \in \mathbb{R}$$

$$X_{\tilde{x}_0} = \emptyset$$

Fubini

let X_1 be the set of $x_1 \in \mathbb{R}^{n_1}$ such that

$$X_{x_1} \neq \emptyset$$

for example in our example.

$$X_1 = [a, b] \subset \mathbb{R}^{n_1}$$

Then in general we have that X_1 is compact in $\mathbb{R}^{n_1}$, and X_{x_1} is compact in $\mathbb{R}^{n_2}$ $\forall\, x_1 \in X_1$

If $g(x_1) := \int_{X_{x_1}} f(x_1, x_2)\, dx_2$ on X_1 continuous then

$$\int_X f(x_1, x_2)\, dx = \int_{\mathbb{X}_1} g(x_1)\, dx_1$$

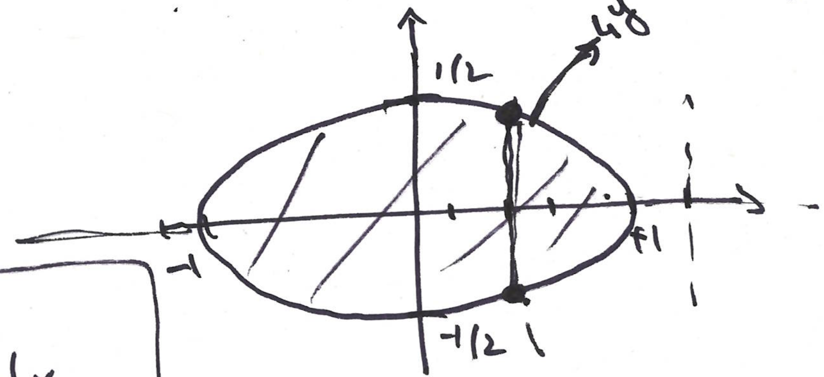
$$\boxed{\int_X f\, dx = \int_{\mathbb{X}_1} \left(\int_{\mathbb{X}_{x_1}} f(x_1, x_2)\, dx_2 \right) dx_1}$$

Similarly, exchanging roles of x_1, x_2 we have

$$\boxed{\int_X f(x_1, x_2)\, dx = \int_{\mathbb{X}_2} \int_{\mathbb{X}_{x_2}} f(x_1, x_2)\, dx_1\, dx_2}$$

$n = 3$ $\quad 1+2$, $2+1$

$n = 4$ $\quad 1+3$, $3+1$, $2+2$



$4y^2 = 1 - x^2$

$y = \pm \frac{\sqrt{1-x^2}}{2}$

$\mathbb{X} = \{ x_1^2 + 4x_2^2 \leq 1 \} \subset \mathbb{R}^2$

$2 = 1 + 1$

$\mathbb{X} = \{ (x_1, x_2) \mid x_1^2 + 4x_2^2 \leq 1 \}$

$\mathbb{X}_{x_1} = \phi$ if $x_1 > 1$ or $x_1 < -1$

$$\mathbb{X}_1 = \{ x_1 \in \mathbb{R} \mid \mathbb{X}_{x_1} \neq \emptyset \}$$

$$= [-1, 1].$$

if $x_1 \in \mathbb{X}_1$ then

$$\mathbb{X}_{x_1} = \{ x_2 \in \mathbb{R} \mid (x_1, x_2) \in \mathbb{X} \}$$

$$= \left[-\sqrt{\frac{1-x_1^2}{2}}, \sqrt{\frac{1-x_1^2}{2}} \right]$$

$$\int_{\mathbb{X}} f(x_1,x_2)\,dx = \int_{\mathbb{X}_1} \left(\int_{\mathbb{X}_{x_1}} f(x_1,x_2)\,dx_2 \right) dx_1$$

$$= \int_{-1}^{1} \int_{-\sqrt{\frac{1-x_1^2}{2}}}^{\sqrt{\frac{1-x_1^2}{2}}} f(x_1,x_2)\,dx_2\,dx_1 .$$

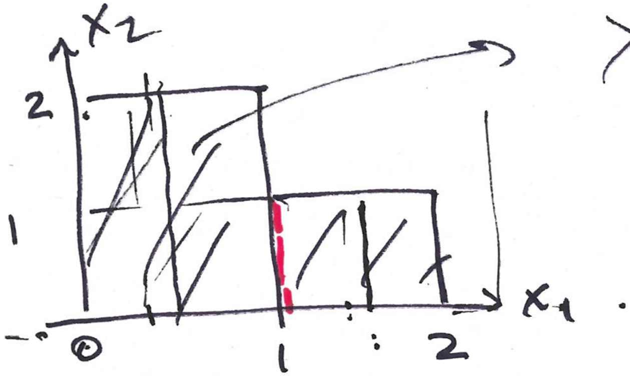
In Fubini's thm we have the assumption

$$g(x_1) := = \int_{\mathbb{X}_{x_1}} f(x_1,x_2)\,dx_2$$

is continuous.

It might not be continuous.

eg. $n=2$, $f=1$



$\mathbb{X}$, closed bdd,

$n = 2 = 1+1$

$$\mathbb{X}_1 = [0, 2] = \{ x_1 \mid \mathbb{X}_{x_1} \neq \emptyset \}.$$

$$\mathbb{X}_{x_1} = \begin{cases} [0,2] & 0 < x_1 < 1 \\ [0,1] & 1 < x_1 < 2 \end{cases}$$

$$g(x_1) = \int_{\mathbb{X}_{x_1}} 1 \, dx_2 = \begin{cases} 2 & \text{if } x_1 \leq 1 \\ 1 & \text{if } 1 < x_1 \leq 2 \end{cases}$$

$g(x_1)$ is not continuous.

But not really a problem because we can divide $\mathbb{X}$ into smaller sets for which the function $g(x_1)$ is continuous.

We are integrating twice over the red piece but these are in 2 dim. negligeable. In the same way that a point in an interval is negligeable.

$$f : [a,b] \to \mathbb{R}$$

$$\int_a^b \qquad \int_a^c f \, dx + \int_c^b f \, dx$$

In $\mathbb{R}^2$ any curve is negligeable, any finite fin. of pts are negligeable

in $\mathbb{R}^3$, any surface, any curve or points are negligeable

In $\mathbb{R}^2$ a curve has no area.

In $\mathbb{R}^3$ a surface or a curve has no volume.

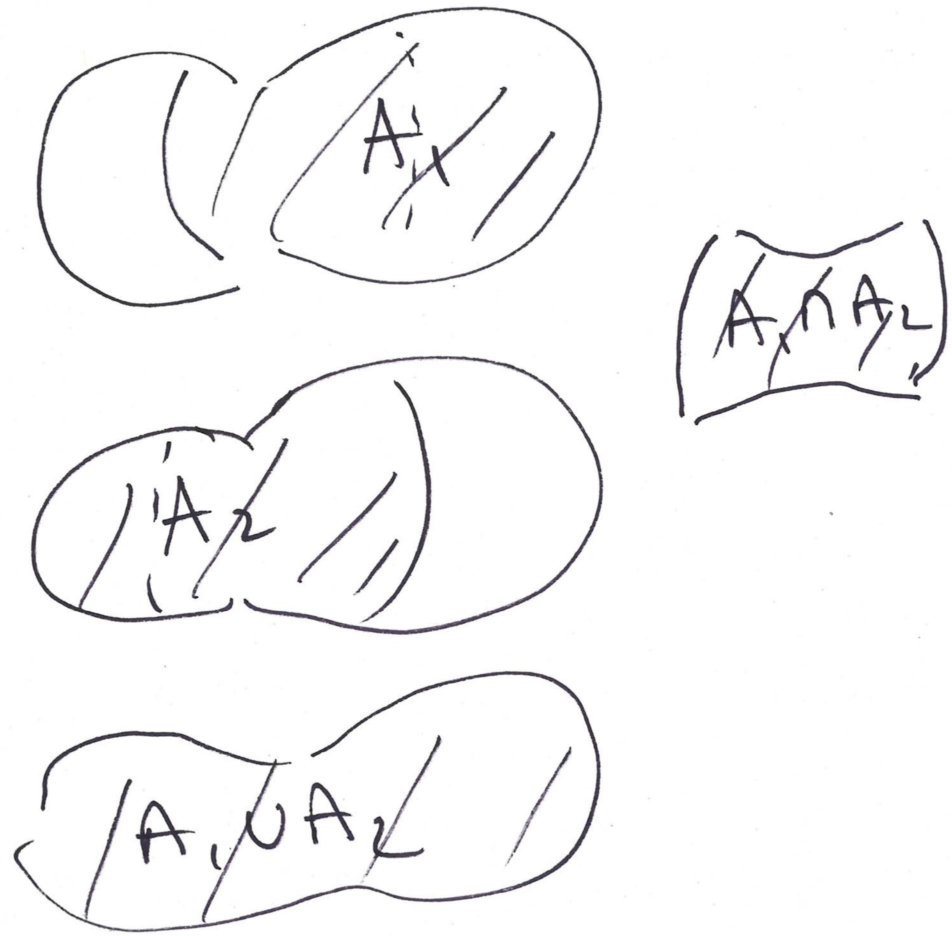
Property of the integral.

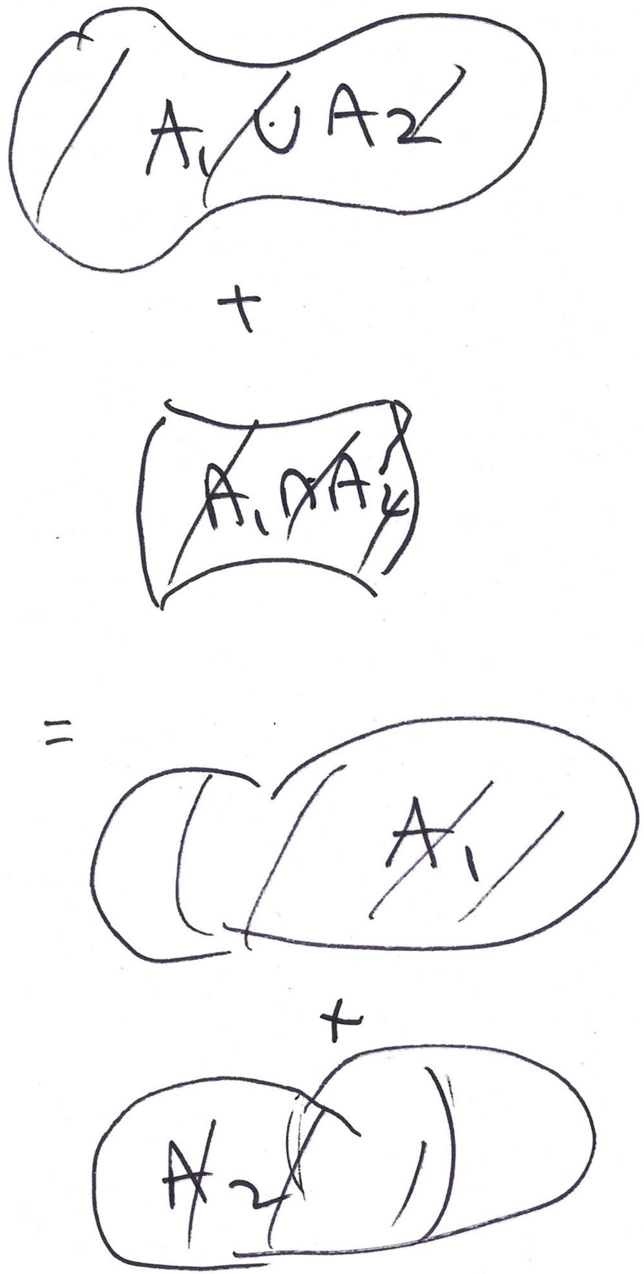
Domain additivity.

If $X = A_1 \cup A_2$ where A_1, A_2 bdd closed then for $f : X \to \mathbb{R}$

$$\int_{X = A_1 \cup A_2} f \, dx + \int_{A_1 \cap A_2} f \, dx = \int_{A_1} f \, dx + \int_{A_2} f \, dx.$$

e.g.





In particular if ~~$A_1 \cap A_2 =$~~

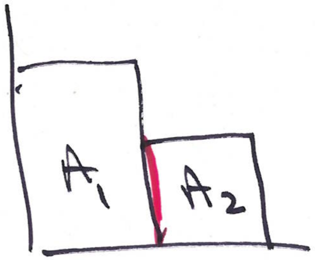
$$\int_{A_1 \cap A_2} f \, dx = 0 \quad \text{Then}$$

$$\int_{A_1 \cup A_2} f \, dx = \int_{A_1} f \, dx + \int_{A_2} f \, dx.$$

In fact if $\mathrm{Vol}_n (A_1 \cap A_2) = 0$

then $\displaystyle\int_{A_1 \cap A_2} f \, dx_1 \ldots dx_n = 0$

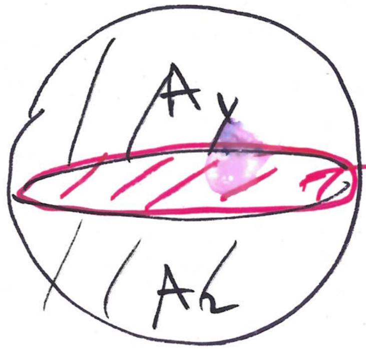
for any f.



$A_1 \cap A_2 = 1$

In 2-dimensions this intersection has 0 area.

$\underline{n=3}$



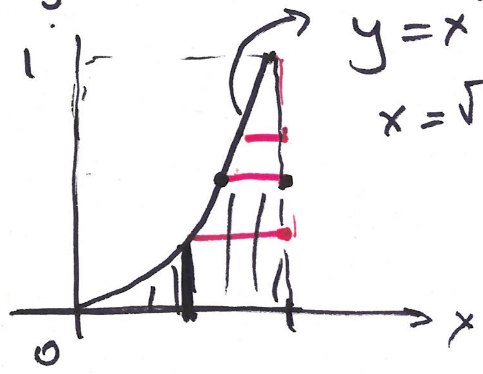
$\rightarrow A_1 \cap A_2$

In 3 dim,

$A_1 \cap A_2$ has 0 volume (3 dim'l volume)

Clicker question

$$\int_0^1 \left(\int_0^{x^2} f \, dy \right) dx$$



$y = x^2$

$x = \sqrt{y}$

$$\int_0^1 \left(\int_{\sqrt{y}}^1 f(x,y) \, dx \right) dy$$

Negligeble sets in $\mathbb{R}^n$

Defn 1) For $1 \leq m \leq n$ a parametrized m-set in $\mathbb{R}^n$ is a continuous function

$$\varphi : [a_1, b_1] \times \dots \times [a_m, b_m] \to \mathbb{R}^n$$

which is C^1 on $(a_1, b_1) \times \dots \times (a_m, b_m)$.

2) A set $Y \subset \mathbb{R}^n$ is negligeble if $\exists$ finitely many φ_i parametrized m_i-sets with $m_i \lneq n$ such that

$$Y \subset \bigcup \varphi_i(X_i)$$

where $\varphi_i : X_i \to \mathbb{R}^n$.

e.g. $n=1$, Y is a union of finitely many pts.

$n=2$ $Y \subset$ union of finitely many images of parametrized curves.

Y negligeble in $\mathbb{R}^2$.

Fact: If $Y \subset \mathbb{R}^n$ closed bdd and negligeble then $\int_Y f \, dx_1 \dots dx_n$ for any f is zero.

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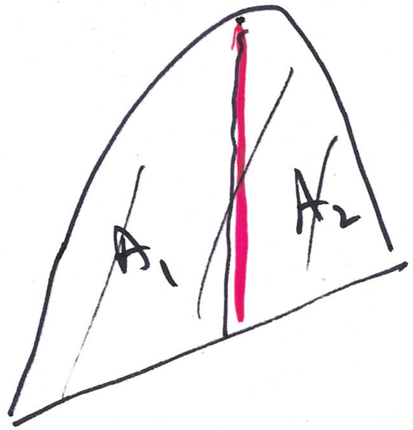
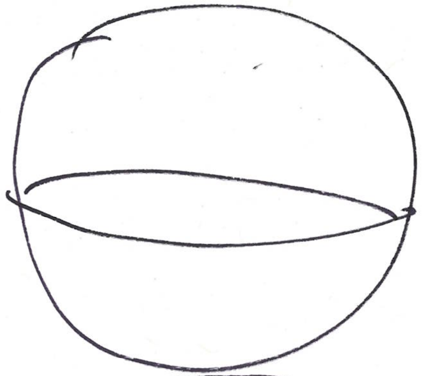
eg. $n = 2$

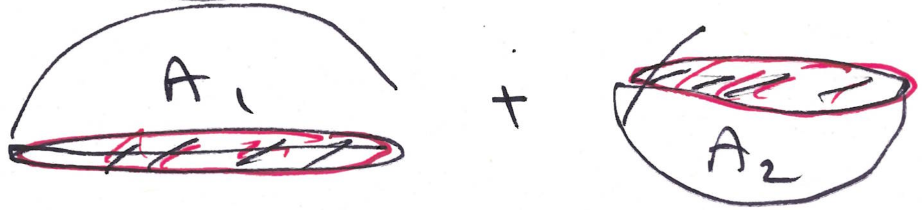


$A_1 \cap A_2 =$ 

negligable.

$3 = n$, $\mathbb{R}^3$





$A_1 \cap A_2 =$  neglegible

Applications of integral.

$D \subset \mathbb{R}^2$

$$\iint_D 1 \, dx\,dy = \text{Area } D.$$

$A \subset \mathbb{R}^n$, $f : A \to \mathbb{R}$

$f \geq 0$

$\mathbb{X} = \{(x, y) \in \mathbb{R}^{n+1} \mid x \in A, \ 0 \leq y \leq f(x)\}$

$n+1$ dim'l

$$\text{vol } \mathbb{X} = \int_A f(x) \, dx$$

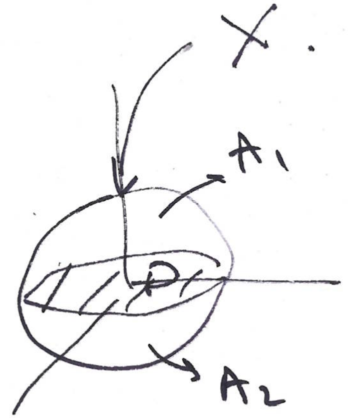
$n=2 \qquad f : \cancel{A} \longrightarrow \mathbb{R}.$

$$\operatorname{vol} X = \int_A f(x)\,dx$$

~~In particular we can use~~

Ex Find the volume of a sphere of radius 1 centered at 0 in $\mathbb{R}^3$.

$$X = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \le 1 \}.$$



Method 1.

$X = A_1 \cup A_2$

$A_1 = \{ (x,y,z) \mid z \ge 0,\ x^2+y^2+z^2 \le 1 \}.$

$A_2 = \{ \ldots \mid z \le 0 \}.$

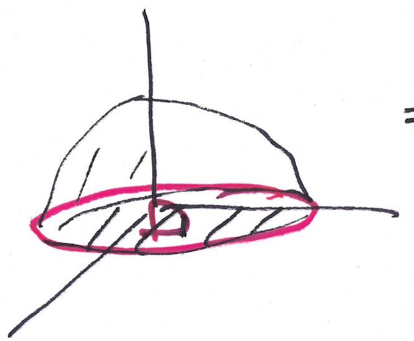
$D = A_1 \cap A_2 \subset$ xy-plane $\rightarrow$ disc of radius 1

It is negligeble.

Let

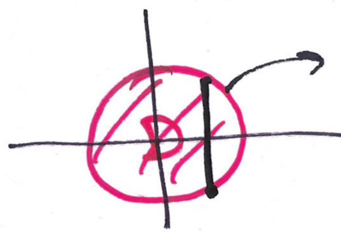
$$f : D \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto \sqrt{1-x^2-y^2}$$

$$\text{vol} A_1 = \int_D f \, dx\, dy$$



$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2} \, dy\, dx$$

$$= \cdots\cdots = \frac{4\pi}{3}.$$



$x^2+y^2=1$

method 2.

$$\text{vol(sphere)} = \text{vol}(X)$$

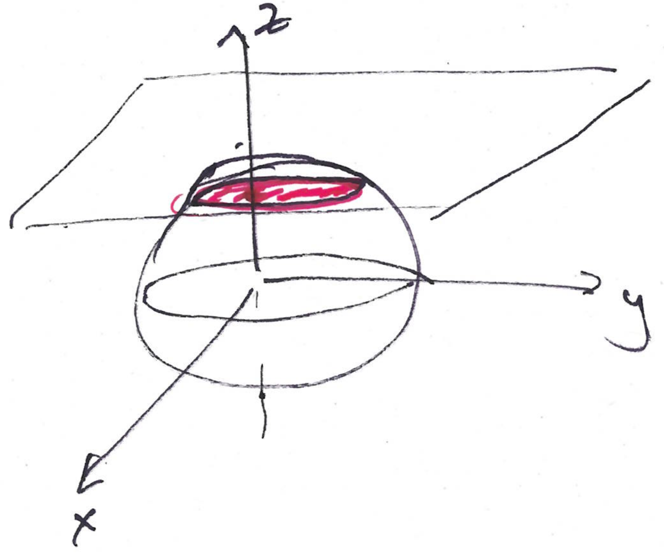
$$= \int_X 1 \, dx\, dy\, dz$$

We can use Fubini

$$3 = \cancel{}\, 2+1$$

$$X = \{((x,y),z) \mid x^2+y^2+z^2 \le 1\}.$$

$$\int_X 1 \, dx\,dy\,dz = \int_{-1}^{1} \left(\int_{X_z} 1 \, dx\, dy \right) dz$$



$$\int_{X_z} 1 \, dx\, dy =: g(z) = \mathrm{vol}(X_z)$$

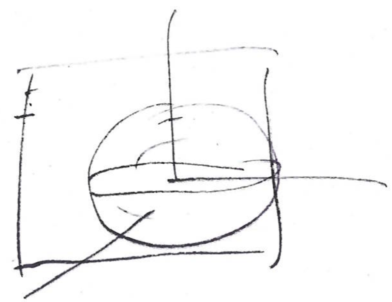
$X_z = \{(x,y,z) \in X\}$.

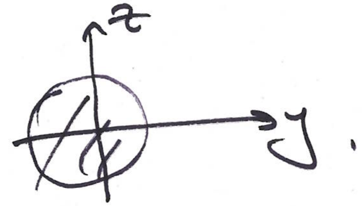
↳ points in X w/ last coordinate equal to z

which is a disc of radius $\sqrt{1-z^2}$.

$$\int_X 1 \, dx\, dy \, dz = \int_{-1}^{1} \underbrace{\left(\int_{X_z} 1 \, dx\, dy \right)}_{\text{area of a disc of radius } \sqrt{1-z^2}} dz$$

$$= \int_{-1}^{1} \pi (1-z^2) \, dz = \frac{4\pi}{3}.$$





§4.3 Improper Integrals.

Generalization of

$\int_a^\infty f(x)\,dx$ or $\int_{-\infty}^\infty f(x)\,dx$

for a 1 sonable func

$f : \mathbb{R} \to \mathbb{R}$.

let $X \subset \mathbb{R}^n$ non-compact sset.

$f : X \longrightarrow \mathbb{R}$ a func such that $\int_K f\,dx$ exists

for every compact set $K \subset X$.

and Suppose $\underline{f \geq 0.}$

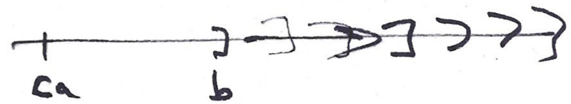
Suppose we have a sequence of regions X_k $k = 1, 2 \ldots$ such that

1) Each region X_k is closed bdd

② $X_{k+1} \supset X_k$

③ $\bigcup_{k=1}^\infty X_k = X$

$n = 1$ ~~Eg~~ $\lim_{b \to \infty} \int_a^b f(x)\,dx$ expanding intervals $[a, b]$



If $\lim\limits_{n\to\infty} \int_{X_n} f\,dx$ exists

then we say $\int_X f\,dx$ converges.

$$\int_X f\,dx := \lim_{n\to\infty} \int_{X_n} f\,dx$$

e.g. $X = \mathbb{R}^3$

$X_n = \{(x,y,z) \mid x^2+y^2+z^2 \leq n^2\}$

a seq. of expanding balls.

$\tilde{X}_n = \{(x,y,z) \mid |x| \leq n,\ |y| \leq n,\ |z| \leq n\}$

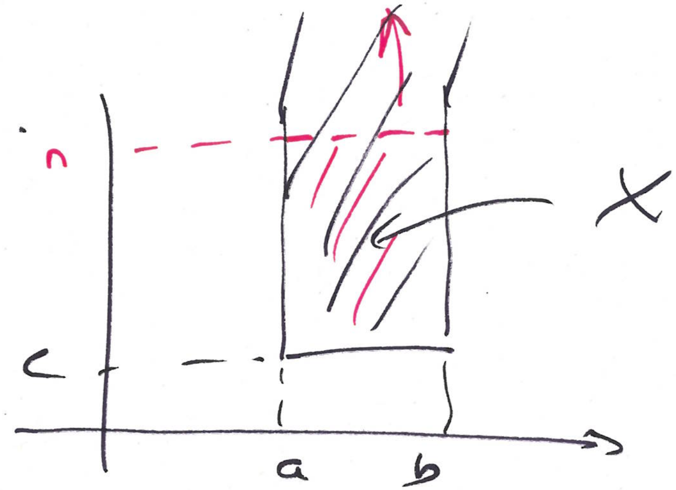
rectang. boxes.

If $\lim\limits_{n\to\infty} \int_{X_n} f\,dx$ exists

then we say $\int_{\mathbb{R}^3} f\,dx$

converges.

Ex.



$X = [a,b] \times [c, \infty)$

$$\int_X f\,dx\,dy := \lim_{n\to\infty} \int_{[a,b]\times[c,n]} f\,dx\,dy.$$

Ex ① $\int_X 1 \, dx\,dy$ diverges.

$$\int_{[a,b]\times[c,n]} 1 \, dx\,dy = \int_a^b \int_c^n 1 \, dy\,dx$$

$$= (b-a)(n-c) \longrightarrow \infty \text{ as } n \to \infty.$$

② $$\int_X \frac{1}{y^2} \, dx\,dy = \lim_{n\to\infty} \int_a^b \int_c^n \frac{1}{y^2} \, dy\,dx$$

$$= \lim_{n\to\infty} (b-a)\left(\frac{1}{c} - \frac{1}{n}\right)$$

$$= \frac{b-a}{c}$$

converges.

③ $X = \mathbb{R}^2$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} \, dx\,dy.$$

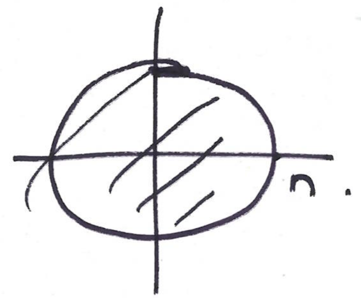
$$= \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2$$

We'll show:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} \, dx\,dy = \lim_{n\to\infty} \iint_{D_n} e^{-x^2-y^2} dx\,dy$$

$$= \pi$$

$D_n =$ 

For this we need change of variables formulas in n- dimensions.

§4.4 Change of variables.

$\underline{n=1}$ we had the change of variables or the substitution method which helped us to evaluate certain integrals.

$$\int f(y)\,dy = \int f(\varphi(x))\,\varphi'(x)\,dx$$

$y = \varphi(x)$

$dy = \varphi'(x)\,dx$

where $\varphi : X \longrightarrow Y$

$[a,b] \longrightarrow [c,d]$.

φ is bijective C^1, $\varphi' \neq 0$ for any $x \in [a,b]$

If φ is increasing then

$Y = [c,d] = [\varphi(a), \varphi(b)]$.

If φ is dec. then

$Y = [c,d] = [\varphi(b), \varphi(a)]$

$$\int_a^b f(\varphi(x))\varphi'(x)\,dx$$

$$= \begin{cases} \int_{\varphi(a)}^{\varphi(b)} f(y)\,dy & \text{if } \varphi \nearrow \\ \int_{\varphi(b)}^{\varphi(a)} f(y)\,dy & \text{if } \varphi \searrow \end{cases}$$

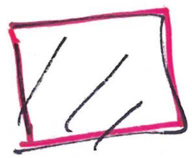
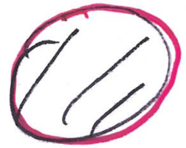
$$= \operatorname{sgn}\varphi' \int_{\varphi(a)}^{\varphi(b)} f(y)\,dy$$

$$\boxed{\int_{[a,b]=X} f(\varphi(x))\,|\varphi'(x)|\,dx = \int_{\varphi(X)=Y=[c,d]} f(y)\,dy}$$

In $\mathbb{R}^n$. Suppose we have

$\varphi : X \longrightarrow Y$
closed bdd → closed bdd

$X = X_0 \cup B$ $\qquad$ $Y = Y_0 \cup C$

$\varphi : X_0 \longrightarrow Y_0$ is C^1 bijec.

$\det J_\varphi(x) \neq 0 \quad \forall x \in X_0$.

Let $Y = \varphi(X)$ and

$f : Y \to \mathbb{R}$ cont. Then.

$$\boxed{\int_Y f(y)\,dy = \int_X f(\varphi(x))\,|\det J_\varphi|\,dx}$$