Improper Integrals

Let $X \subset \mathbb{R}^n$ non-compact set. $f : X \to \mathbb{R}$ a continuous, positive function. $(f \geq 0)$

Suppose X_k, $k = 1, 2, \dots$ a sequence of compact regions such that $X_k \subset X_{k+1}$ and $\bigcup_{k=1}^{\infty} X_k = X$

If $\lim_{k\to\infty} \int_{X_k} f\,dx$ exists then we say $\int_X f\,dx$ converges

$$\boxed{\int_X f\,dx := \lim_{k\to\infty} \int_{X_k} f\,dx.}$$

e.g $f : \mathbb{R}^2 \to \mathbb{R}$, $f \geq 0$

If $\lim_{k\to\infty} \int_{[-k,k]^2} f(x,y)\,dx\,dy$ exists then $\int_{\mathbb{R}^2} f\,dx\,dy$ converges.

Change of variables $\boxed{X_0}_B \xrightarrow{\varphi} (Y_0)_C$

Let $\varphi : \bar{X} \to \bar{Y}$ be a continuous map, where $\bar{X} = X_0 \cup B$, $\bar{Y} = Y_0 \cup C$ are closed and bounded sets with X_0, Y_0 open, B, C negligeble ssets of $\mathbb{R}^n$.

Suppose $\varphi : X_0 \to Y_0$ is C^1 and bijective with $\boxed{\det J_\varphi(x) \neq 0}$ $\forall\, x \in X_0$. Let $\bar{Y} = \varphi(\bar{X})$

Suppose $f : \bar{Y} \to \mathbb{R}$ continous

Then

$$\boxed{\int_{Y} f(y)\,dy = \int_{X} f(\varphi(x))\,|\det J_\varphi(x)|\,dx}$$

$n=1$ $Y = [c,d] = \varphi([a,b])$ $\varphi : [a,b] \to [c,d]$

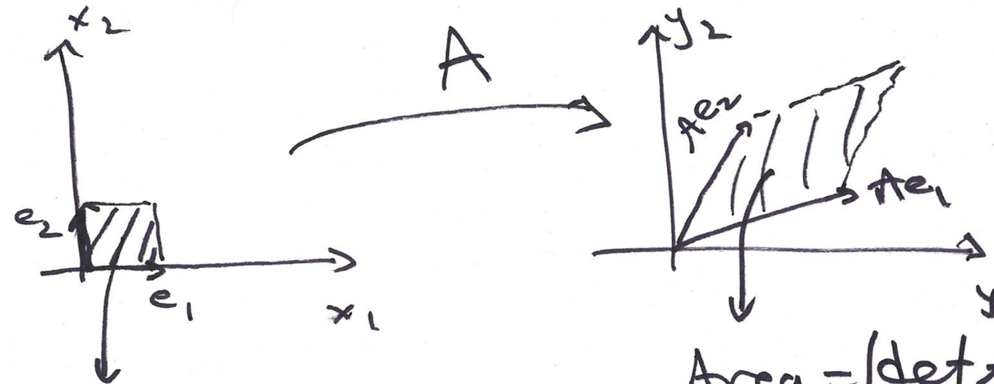
$\varphi'(x) \neq 0$ $\forall\, x \in [a,b]$. Then

$$\int_{[c,d] = Y = \varphi(X)} f(y)\,dy = \int_{[a,b] = X} f(\varphi(x))\,|\varphi'(x)|\,dx$$

Ex: $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$x \longrightarrow Ax$

A is $n \times n$ matrix, $\det A \neq 0$.



area $= 1$ $\qquad$ Area $= |\det A|$.

$\Delta V(x) = dx_1 dx_2$.

$\Delta V(y_1, y_2) = |\det A| \, \Delta V(x_1, x_2)$

$dy_1 \, dy_2 = \underbrace{|(\det A)|}_{|\det Jac_\varphi(x)|} dx_1 dx_2$.

$Jac_\varphi(x) = A$

So for $Y \subset \mathbb{R}^n$,

$\varphi : X \longrightarrow Y$

$x \longrightarrow Ax = y$

$X = A^{-1} Y$, $\quad AX = Y$.

if $f \equiv 1$

$\mathrm{vol}\, Y = (\det A) \, \mathrm{vol}(X)$.

$(\det A) \, \mathrm{vol}(A^{-1} Y)$.

or putting $X = A^{-1} Y$,

$$\boxed{\mathrm{vol}(A X) = (\det A) \, \mathrm{vol}(X)}$$

For instance if $X = [0,1]^n$

then $\mathrm{vol}\, X = 1$.

$\mathrm{vol}(A [0,1]^n) = |\det A|$

For example if

$$A = \begin{pmatrix} r & & & \\ & r & & \\ & & \ddots & \\ & & & r \end{pmatrix} \quad r > 1$$

dilation matrix.

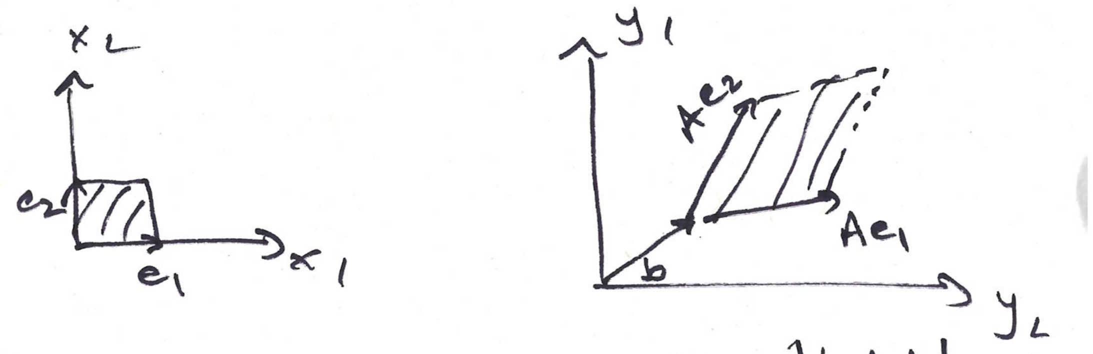
then $\mathrm{vol}(rX) = \underbrace{|\det A|}_{r^n} \mathrm{vol}\, X$ $= r^n \cdot \mathrm{vol}\, X$.

$$\mathrm{vol}(\text{sphere of radius } r) = r^3 \,\mathrm{vol}(\text{sphere of radius } 1)$$

$$= r^3 \frac{4\pi}{3}$$

All this still holds if we have an affine map.

$$\varphi : X \to Y$$
$$x \to Ax + b$$

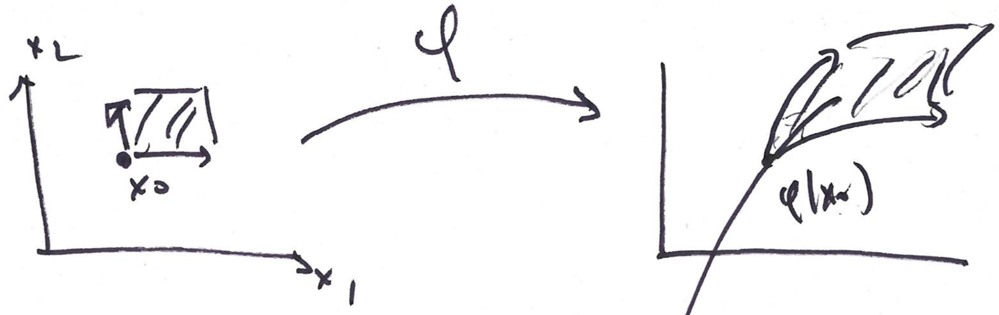


$$dx_1 dx_2 \longrightarrow dy_1 dy_2 = \underbrace{|\det A|}_{\mathrm{Jac}_\varphi(x) = A} dx_1 dx_2$$

In general $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ non-linear differentiable function. At each pt x_0, the differential $d\varphi(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map, its matrix representation is given by $J_\varphi(x_0)$.

if x is near x_0 then

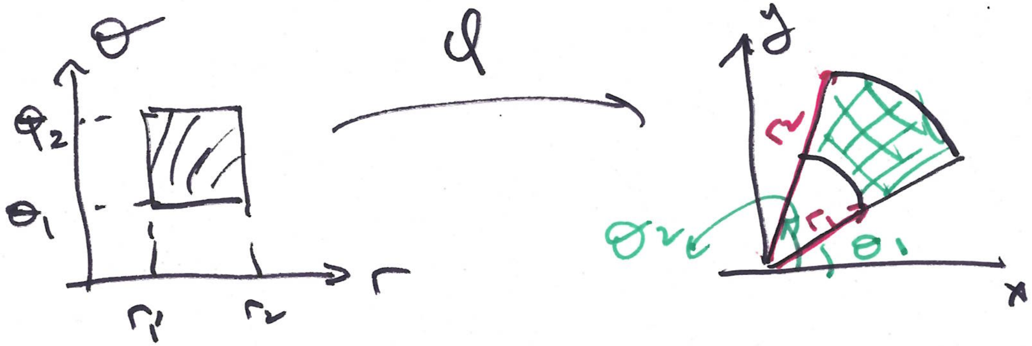
$$\varphi(x) \approx \underbrace{\varphi(x_0) + J_\varphi(x_0)(x - x_0)}_{\text{affine linear map}}$$



$$\Delta V(y_1,y_2) = |J_\varphi(x_0)|\,\Delta V(x_1,x_2)$$

Important Example.

Polar coordinates.

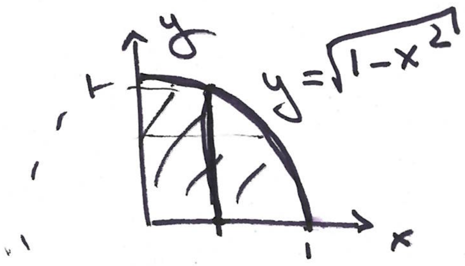


$$\begin{pmatrix} x \\ y \end{pmatrix} = \varphi\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \qquad J_\varphi = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\left|J_\varphi\begin{pmatrix} r \\ \theta \end{pmatrix}\right| = r = r\cos^2\theta + r\sin^2\theta$$

$$\boxed{dx\,dy = r\,dr\,d\theta}$$

Ex.: $\displaystyle\iint_D x\,dx\,dy$ where D is



Directly in rectangular coordinates $\left(\begin{array}{c} x^2+y^2=1 \\ \text{unit circle} \end{array}\right)$

$$\int_0^1\int_0^{\sqrt{1-x^2}} x\,dy\,dx$$

$$\int_0^1 x\left(\int_0^{\sqrt{1-x^2}} dy\right)dx = \int_0^1 x\Big[y\Big]_0^{\sqrt{1-x^2}}dx$$

$$= \int_0^1 x\sqrt{1-x^2}\,dx = \int_0^1 \frac{u^{1/2}}{2}\,du = \frac{1}{3}$$

$1-x^2 = u$
$-2x\,dx = du$

or in polar coordinates

$$\iint_D x\,dx\,dy = \int_0^{\pi/2}\int_0^1 r\cos\theta\, r\,dr\,d\theta$$

$$= \underbrace{\int_0^{\pi/2} \cos\theta\, d\theta}_{1} \underbrace{\int_0^1 r^2\,dr}_{\underbrace{\frac{r^3}{3}\Big|_0^1}_{1/3}}$$

$$= \frac{1}{3}.$$

Example. Find the volume between 2 surfaces given by $z = x^2+y^2$, $z = 50-x^2-y^2$

(See the picture)

we first find the intersection

$z = x^2+y^2$ and $z = 50-x^2-y^2$

$$x^2+y^2 = 50-x^2-y^2$$
$$\Rightarrow 2x^2+2y^2 = 50$$
$$\Rightarrow x^2+y^2 = 25.$$

They intersect in a circle of radius 5.

$$D = \{(x,y) \mid x^2+y^2 \le 25\}.$$

$$\text{Volume} = \iint_D (50-x^2-y^2)-(x^2+y^2)\,dx\,dy$$

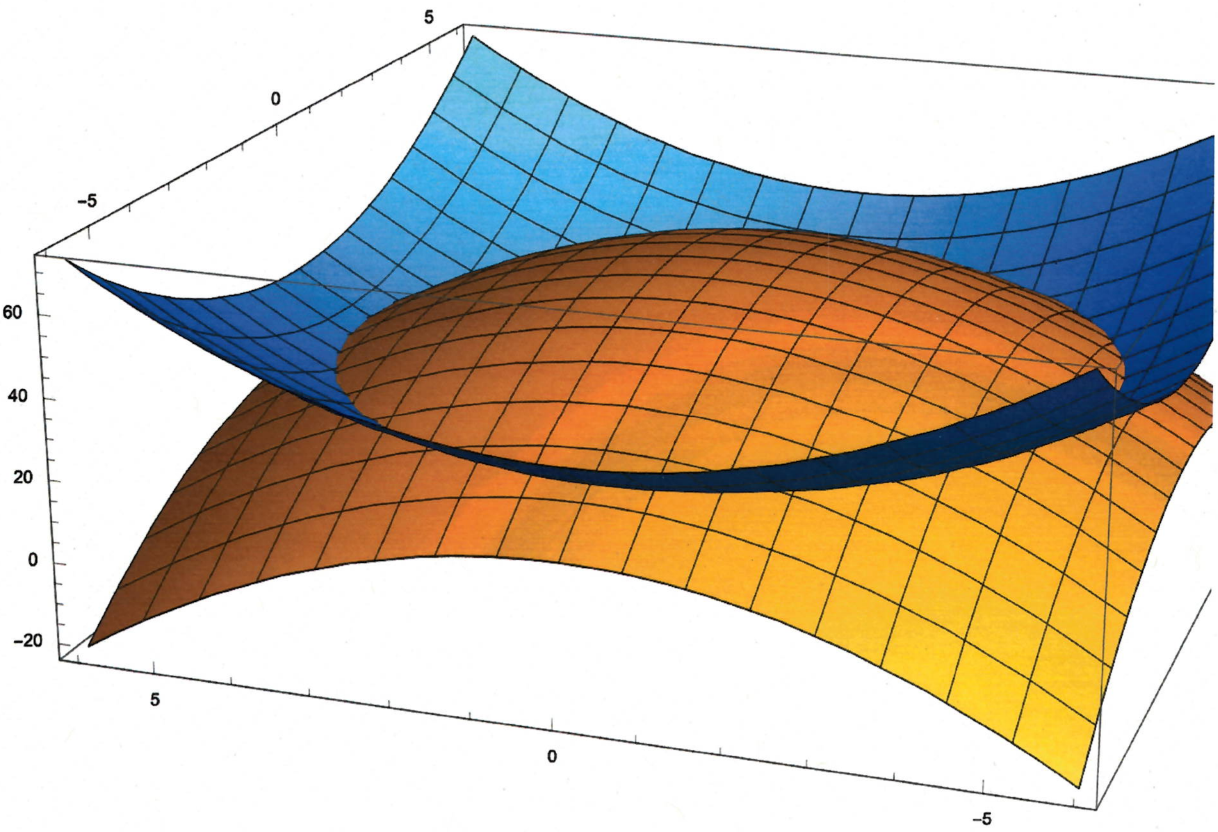
$$= \iint_D \underbrace{(50-2x^2-2y^2)\,\underbrace{dx\,dy}}.$$

$$= \int_0^{2\pi}\int_0^5 (50-2r^2)\, r\,dr\,d\theta.$$

$$= \dots = 625\pi.$$

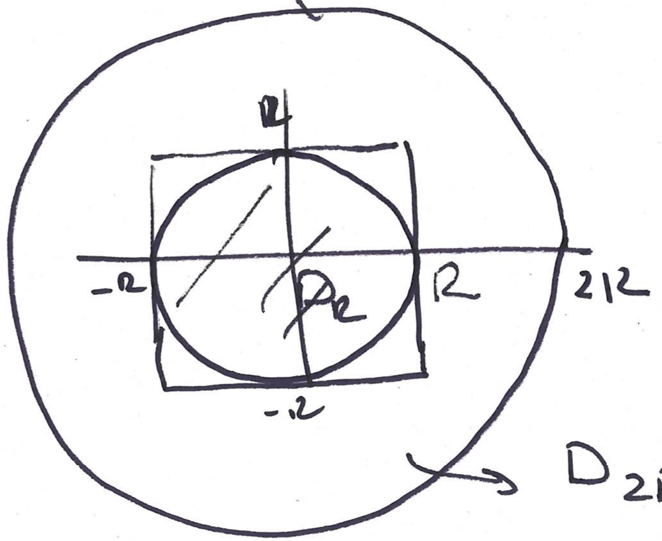
In[3]:= `Plot3D[{50 - x^2 - y^2, x^2 + y^2}, {x, -6, 6}, {y, -6, 6}]`

Out[3]=



Ex $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2$$



$$\iint_{D_R} e^{-x^2-y^2}\,dx\,dy \leq \int_{-R}^{R}\int_{-R}^{R} e^{-x^2-y^2}\,dx\,dy \leq \iint_{D_{2R}} \cdots dx\,dy$$

$$\iint_{D_R} e^{-x^2-y^2}\,\underbrace{dx\,dy}$$

$$= \int_0^R \int_0^{2\pi} e^{-r^2}\,r\,d\theta\,d\theta \qquad r^2 = u. \quad 2r\,dr = du$$

$$= \int_0^{2\pi} d\theta \int_0^R e^{-r^2}\,r\,dr = \frac{1}{2}\int_0^{2\pi} d\theta \int_0^{R^2} e^{-u}\,du$$

$$= \pi\left[-e^{-u}\Big|_0^{R^2}\right] = \pi(1-e^{-R^2}).$$

$$\iint_{D_{2R}} e^{-x^2-y^2}\,dx\,dy = \pi(1-e^{-4R^2}).$$

$$\pi(1-e^{-R^2}) \le \iint_{\square_R} \quad \le \pi(1-e^{-4R^2}).$$

let $R \to \infty$

$$\pi \le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy \le \pi$$

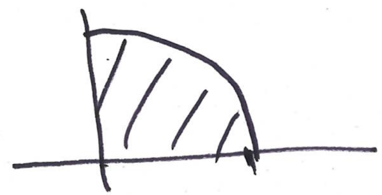
$$\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \pi$$

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}}$$

Clicker question.

$$\iint_X \frac{dx\,dy}{1+x^2+y^2}$$

$X = \{(x,y) \mid x^2+y^2 \le 1,\ 0 \le x,y \le 1\}$



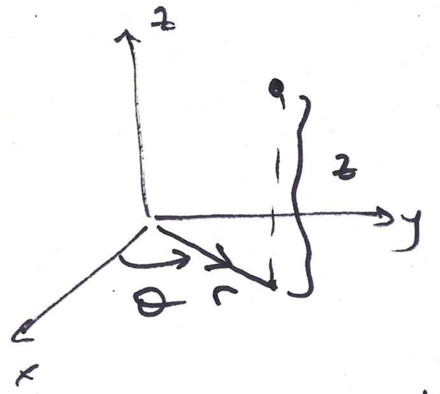
$$\int_0^{\pi/2}\underbrace{\int_0^1 \underbrace{\frac{1}{1+r^2}\,r}\,dr}\,d\theta$$

$$\left.\frac{\log(1+r^2)}{2}\right|_0^1 = \frac{\log 2}{2}$$

$$\frac{\pi}{2}\cdot\frac{\log 2}{2} = \frac{\pi \log 2}{4}.$$

$n=3$ Important examples.

① cylindrical coordinates.

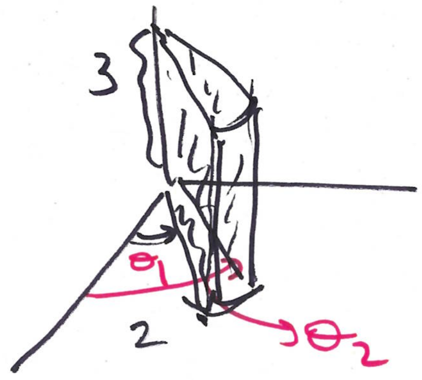


$$(r,\theta,z) \xrightarrow{\varphi} (x,y,z)$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

$r>0$, $\theta \in (0,2\pi)$, $z \in \mathbb{R}$.

$$|J_\varphi(r,\theta,z)| = r$$



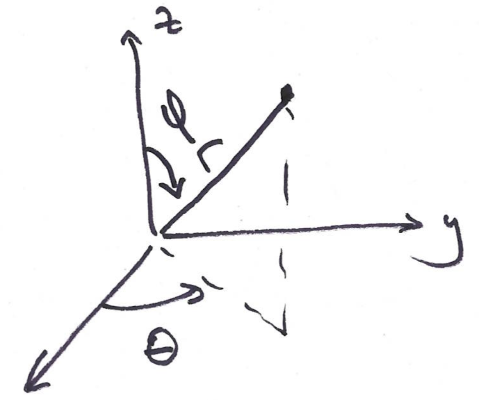
$0<r<2$

$\theta_1 \le \theta \le \theta_2$

$0 \le z \le 3$

$$\iint_X f\,dx\,dy\,dz = \int_0^2 \int_{\theta_1}^{\theta_2} \int_0^3 f(\,)\,r\,dz\,d\theta\,dr$$

② Spherical coordinates.



$$\varphi : \overset{r}{[0,\infty)} \times \overset{\theta}{[0,2\pi]} \times \overset{\varphi}{[0,\pi]} \to \mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}.$$

$x = r\cos\theta \sin\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\varphi$.

$|\det J_\varphi| = r^2 \sin\varphi$.

Ex. volume of unit sphere.

$$\iiint_{\text{sphere}} 1 \, dx\,dy\,dz$$

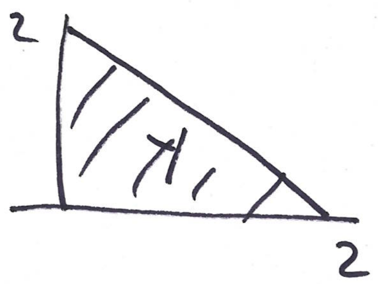
$$= \int_0^1 \int_0^{2\pi} \int_0^{\pi} 1 \; r^2 \sin\varphi \, d\varphi \, d\theta \, dr$$

$$= \underbrace{\int_0^1 r^2 dr}_{\frac{1}{3}} \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \underbrace{\int_0^{\pi} \sin\varphi \, d\varphi}_{2}.$$

$$= \frac{4\pi}{3}.$$

Example.

$$\iint_Y e^{(y-x)/(y+x)}\,dx\,dy.$$



let $y-x=u$
$y+x=v$

$$\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}u\\ v\end{pmatrix}$$

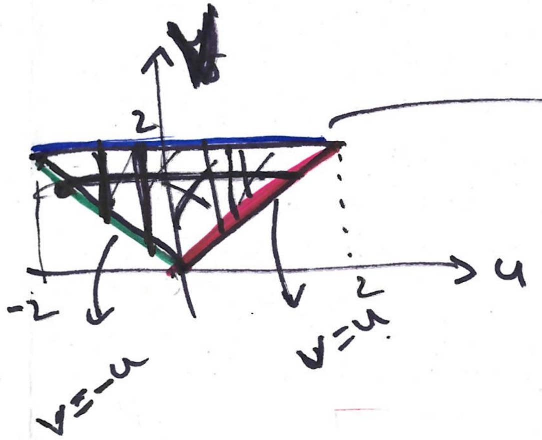
$$\begin{pmatrix}x\\ y\end{pmatrix}=-\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & -1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$$

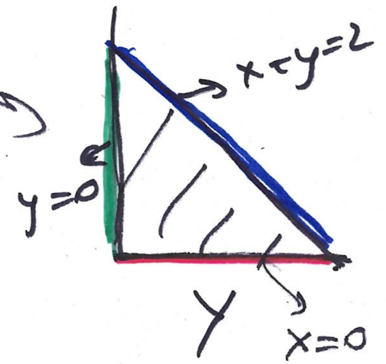
$$x=\frac{v-u}{2} \qquad y=\frac{u+v}{2}.$$

$$\varphi=(u,v)\longrightarrow(x,y)$$
$$(u,v)\longmapsto\left(\frac{1}{2}\frac{v-u}{\cancel{\ }},\ \frac{u+v}{2}\right)$$

$$J_\varphi=\begin{pmatrix}\frac{\partial x}{\partial u}, & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u}, & \frac{\partial y}{\partial v}\end{pmatrix}=\begin{pmatrix}-1/2 & 1/2\\ 1/2 & 1/2\end{pmatrix}$$

$$|\det J_\varphi|=\frac{1}{2}$$





$x=0$ line $\Rightarrow \left.\begin{matrix} u=y \\ v=y \end{matrix}\right\} \Rightarrow u=v.$

$y=0$ line $\Rightarrow \left.\begin{matrix} u=-x \\ v=x \end{matrix}\right\} \Rightarrow u=-v.$

$x+y=2 \Rightarrow v=x+y=2$

$$\iint\limits_{Y} e^{x-y/x+y}\,dx\,dy$$

$$\iint\limits_{X} e^{u/v}\,\frac{1}{2}\,du\,dv.$$

$$\frac{1}{2}\int_0^2\int_{-v}^{v} e^{u/v}\,du\,dv$$

$$\frac{1}{2}\int_0^2 \left(v e^{u/v} \Big|_{u=-v}^{u=v} \right) dv$$

$$\frac{1}{2}\int_0^2 v\,[e^{1} - e^{-1}]\,dv$$

$$= \frac{1}{2}\left(e - \frac{1}{e}\right) \underbrace{\int_0^2 v\,dv}_{\frac{v^2}{2}\Big|_0^2} = e - \frac{1}{e}.$$

or

$$\underbrace{\frac{1}{2}\int_{-2}^{0}\int_{v=-u}^{2} e^{u/v}\,dv\,du + \frac{1}{2}\int_{0}^{2}\int_{v=u}^{2} e^{u/v}\,dv\,du}_{\text{But this is much harder!}}$$

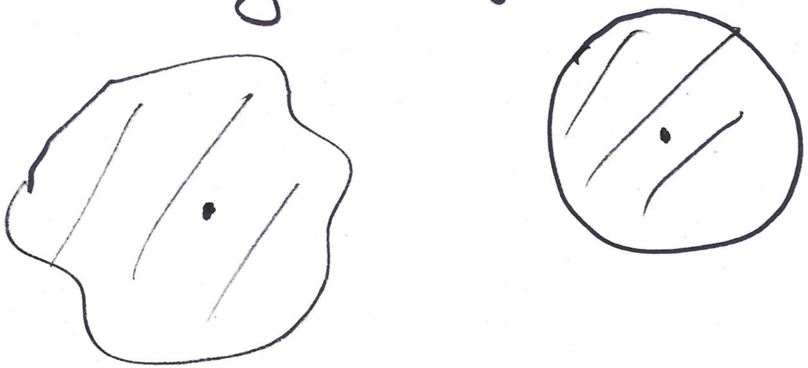
/11

§4.5. Other geometric applications of integral.

① Center of mass.

$X \subset \mathbb{R}^n$ bdd closed

Center of mass of X is a point where X is "perfectly balanced" assuming uniform density.



Coordinates of the center of mass $\tilde{X} = (\tilde{x}_1, \tilde{x}_2 \ldots \tilde{x}_n) \in \mathbb{R}^n$

center of mass of X

$$\tilde{x}_i(X) = \underbrace{\frac{1}{\operatorname{vol} X} \int_X x_i \, dx_1 \, dx_2 \ldots dx_n}_{\text{average of } x_i \text{ coordinates}}.$$

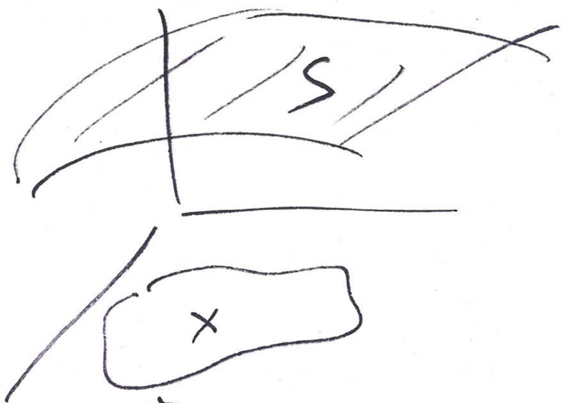
If the density changes from one point to another according to $f(x_1, \ldots x_n)$ then

$$\tilde{x}_i(X) = \frac{\int_X x_i f(x_1 \ldots x_n) \, dx_1 \ldots dx_n}{\int_X f(x_1 \ldots x_n) \, dx_1 \ldots dx_n}.$$

(2)

Surface Area $n=3$.

Suppose a surface in $\mathbb{R}^3$ is given by the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$



$$S = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in X, \; z = f(x,y)\}.$$

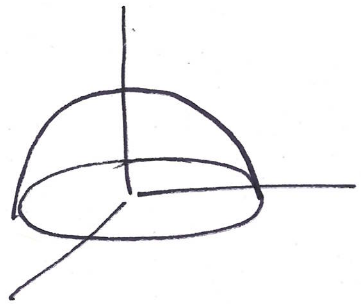
Then

$$\text{Area}(S) = \iint_X \sqrt{1 + \left(\frac{\partial f}{\partial x}(x,y)\right)^2 + \left(\frac{\partial f}{\partial y}(x,y)\right)^2}\, dx\, dy.$$

Example. Surface area of a sphere can be found using

$$f : X \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto \sqrt{1 - x^2 - y^2}$$

$$X = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$



$$\text{surface of sphere} = 2 \iint_X \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}\, dx\, dy.$$

$$= \cdots = 4\pi.$$

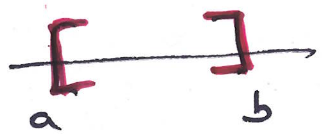
§4.6. Green's formula.

Fundamental thm of 1-variable Analysis

$$\boxed{\int_a^b f(t)\,dt = F(b) - F(a)}$$

where F is a primitive of f
$F' = f$

$$\int_a^b dF = F(b) - F(a).$$



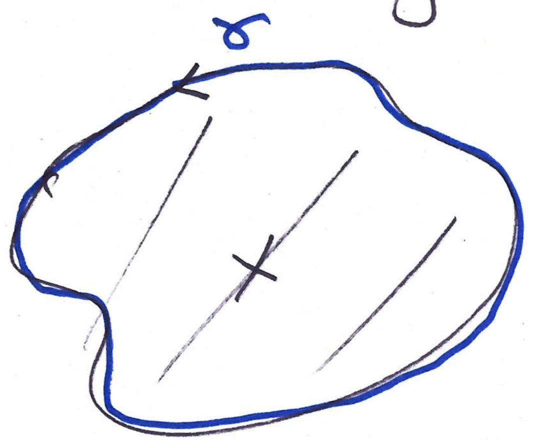
This relates the integral of f over an interval $[a,b]$ to the values of its primitive at the boundary pts of the interval, namely the points $\{a\}$, $\{b\}$.

Do similar formulas exist in higher dimensions?

Yes

Green's Formula is the simplest of many such formulas that relate an n-dim'l integral over a region D to a $n-1$ dim'l integral.

Green's thm (which is dim $n=2$) expresses a double integral over a (closed bdd) region X as a line integral taken along the closed curve γ, forming the boundary of the region X.



The most common form of Green's theorem is

$$\boxed{\iint_X \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy = \int_\gamma f \cdot ds}$$

$f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ vector field

To write the thm properly, we have 2 types of assumptions

1) The vector field $f = (f_1, f_2)$ has components such that $\frac{\partial f_2}{\partial x}, \frac{\partial f_1}{\partial y}$ exist in the region X.

The usual assumption is $f \in C^1$, then $\frac{\partial f_i}{\partial x}, \frac{\partial f_i}{\partial y}$ $i = 1, 2$ exist and continuous.

and $\text{curl}(f) = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}$ is continuous hence integrable