## Change of variables

$X = X_0 \cup B$, $\quad Y = Y_0 \cup C$

closed, bdd sets with

$X_0, Y_0$ open, $B, C$

negligeble ssets of $\mathbb{R}^n$

Suppose $\varphi : X_0 \to Y_0$ is $C^1$

and bijective with $\det J_\varphi(x) \neq 0$

$\forall x \in X_0$. Let $Y = \varphi(X)$

Suppose $f : Y \to \mathbb{R}$ continuous

Then

$$\boxed{\int_Y f(y)\,dy = \int_X f(\varphi(x))\,|\det J_\varphi(x)|\,dx}$$

## Important examples

① Polar coordinates

$\varphi : [0,\infty) \times [0, 2\pi) \to \mathbb{R}^2$

$(r, \theta) \to (r\cos\theta, r\sin\theta)$

$|\det|J_\varphi(r,\theta)|| = r$

$$\boxed{dx\,dy = r\,dr\,d\theta}$$

② Cylindrical coordinates.

$\varphi : [0,\infty) \times [0, 2\pi] \times \mathbb{R} \to \mathbb{R}^3$

$(r, \theta, z) \to (r\cos\theta, r\sin\theta, z)$

$|\det(J_\varphi)| = r$

$$\boxed{dx\,dy\,dz = r\,dr\,d\theta\,dz}$$

③ Spherical coordinates

$\varphi : [0,\infty) \times [0, 2\pi) \times [0,\pi) \to \mathbb{R}^3$

$(r, \theta, \varphi) \to \begin{pmatrix} r\cos\theta\sin\varphi \\ r\sin\theta\sin\varphi \\ r\cos\varphi \end{pmatrix}$

$|\det J_\varphi| = r^2\sin\varphi$

$$\boxed{dx\,dy\,dz = r^2 \sin\varphi\,dr\,d\theta\,d\varphi.}$$

# Green's Formula

$X$ closed bounded region in $\mathbb{R}^2$.

$\gamma$ a curve forming the boundary of $X$

$$\boxed{\iint_X \underbrace{\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)}_{\operatorname{curl}(f)} dx\,dy = \int_\gamma f \cdot ds} \qquad (G)$$

where $f : X \longrightarrow \mathbb{R}^2$

$$(x,y) \longrightarrow \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$$

a vector field.

There are implicit assumptions:

(1) We assume that vector field $f = (f_1, f_2)$ has component $f_1, f_2$ such that $\dfrac{\partial f_2}{\partial x}, \dfrac{\partial f_1}{\partial y}$ exist in the region $X$. The usual assumption is that $f \in C^1$.

Then $\dfrac{\partial f_i}{\partial x}, \dfrac{\partial f_i}{\partial y}$ $i = 1, 2$

exist and continuous so that

$\operatorname{curl}(f) = \dfrac{\partial f_2}{\partial x} - \dfrac{\partial f_1}{\partial y}$ is continuous.

Hence on the left hand side of $(G)$ the integral exists.

# Green's Formula

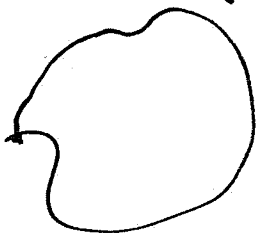
$\int$

② We also need the region
ⓐ $X$ to be closed and bounded and that its boundary $\partial X$ is a <u>simple</u> <u>closed</u> parametrized curve $\gamma$

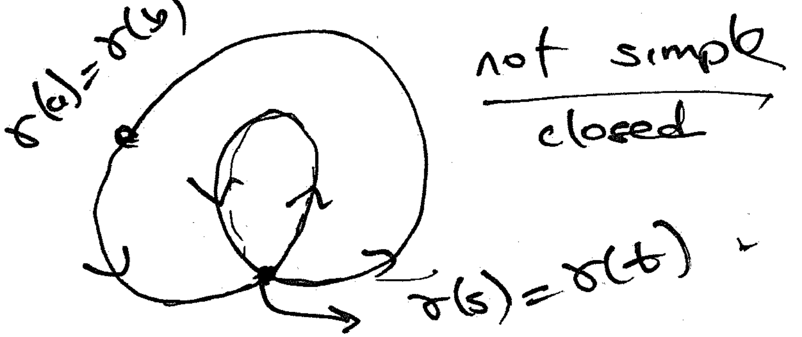
$\gamma : [0, b] \longrightarrow \mathbb{R}^2$

simple
closed

<u>closed</u> $\gamma(a) = \gamma(b)$

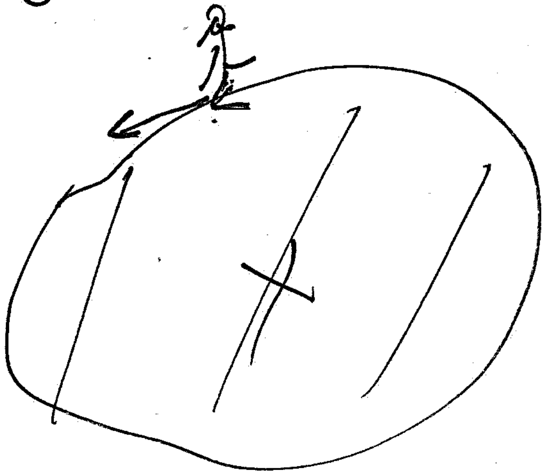


<u>simple</u> There are not any $a < s \neq t < b$ such that $\gamma(s) = \gamma(t)$

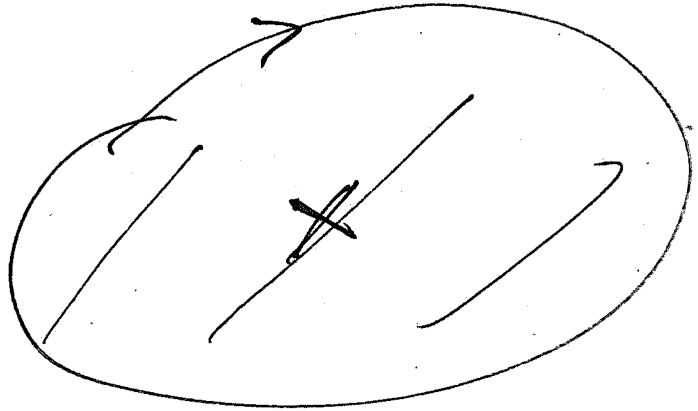



<u>not simple</u>
closed

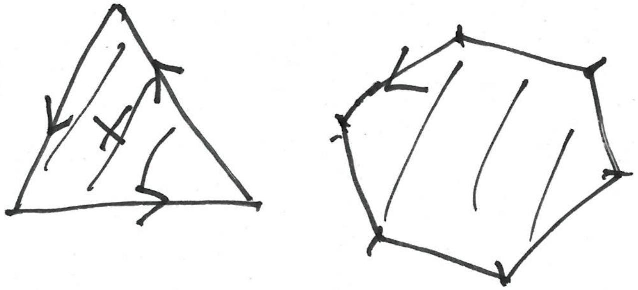
ⓑ $X$ is always to the left hand side of a tangent vector to the boundary.



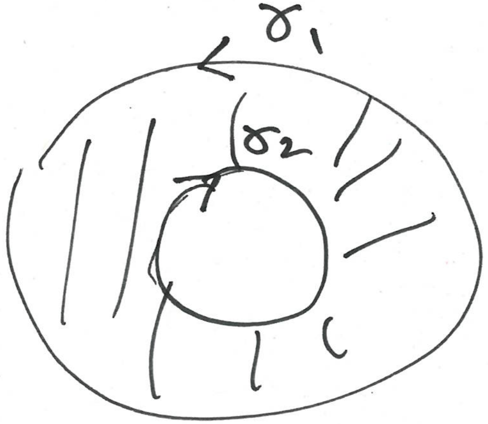

OK




not ok




We can also have a region bounded by a union of simple closed curves.




$\gamma = \gamma_1 \cup \gamma_2$

With these assumptions about $f = (f_1, f_2) \in C^1$

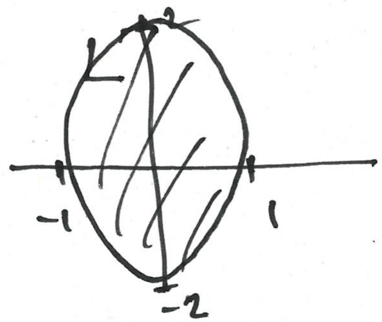
$X$ closed bdd with a simple closed boundary, $\gamma = \bigcup_{i=1}^{k} \gamma_i$.

Then we have

$$\boxed{\iint_X \operatorname{curl}(f)\, dx\, dy = \sum_{i=1}^{k} \int_{\gamma_i} f \cdot ds \,.} \qquad (G)$$

Ex: $f = (y + 3x,\ y - 2x)$

$X$ is the region bdd by the ellipse $x^2 + \frac{y^2}{4} = 1$



Then $\displaystyle\int_\gamma f \cdot ds$

$$= \iint_X \operatorname{curl} f \, dx\,dy.$$

$$\operatorname{curl} f = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}$$

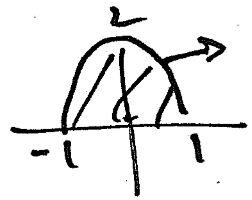
$$= -2 - 1 = -3$$

$$\int_\gamma f\,ds = \iint_X (-3)\,dx\,dy$$

$$= -3 \underbrace{\iint_X dx\,dy}_{\text{area of the ellipse.}} = -6\pi$$

$$= 2\pi$$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ area $= \pi ab$.



$x^2 + \frac{y^2}{4} = 1$

$y^2 = (1 - x^2)4$

$$\iint_X dx\,dy = 2\int_{-1}^{1}\int_{0}^{2\sqrt{1-x^2}} dy\,dx$$

or directly do the line integral.

$$\int_0^{2\pi} f(\gamma(t)) \cdot \gamma'(t)\,dt = \int_\gamma f\,ds.$$

$\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$

$t \longrightarrow (\cos t, 2\sin t)$

$$\int_0^{2\pi} (2\sin t + 3\cos t,\ 2\sin t - 2\cos t) \cdot (-\sin t, 2\cos t)\,dt.$$

$$= \int_0^{2\pi} -2\sin^2 t + 3\cos t \sin t + 4\sin t\cos t - 4\cos^2 t\,dt$$

$$= \cdots = -6\pi$$

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**Rk** In general Green's thm can be used to calculate the area of a region as a line integral

or we can use the Green's to calculate the line integral of the double integral of curl f looks simpler.

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$$\iint_X 1\,dx\,dy = \text{Area}(X) = \int_{\partial X} f \cdot ds.$$

if we can find $f : \mathbb{R}^2 \to \mathbb{R}^2$ s.t $\text{curl} f = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = 1$

We can take $f = (0, x)$ or $f = (-y, 0)$

**Ex** Find the area enclosed by the curve $\gamma(t) = \begin{pmatrix} t^2 \\ \frac{t^3}{3} - t \end{pmatrix}$

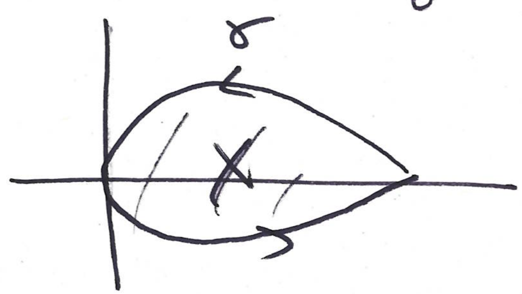
$-\sqrt{3} \leq t \leq \sqrt{3}$.

We can take $f = (0, x)$

then $\text{curl} f = 1$.

Area the region $X$ enclosed by $\gamma$ is $= \int_X 1\,dx\,dy$

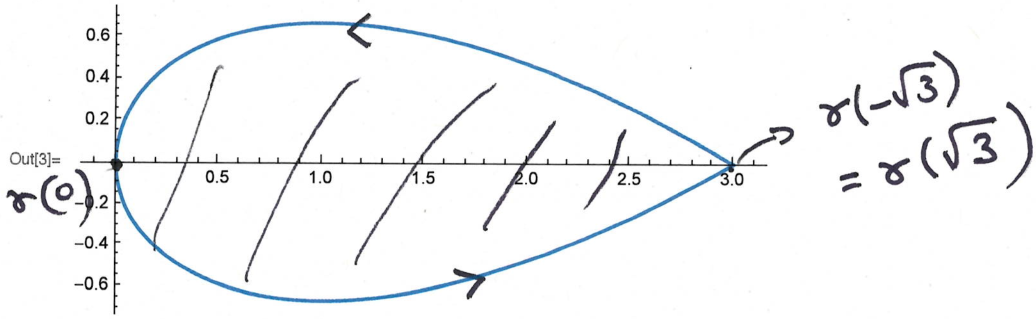
$$= \int_\gamma (0, x) \cdot ds.$$




(see the mathematica plot)

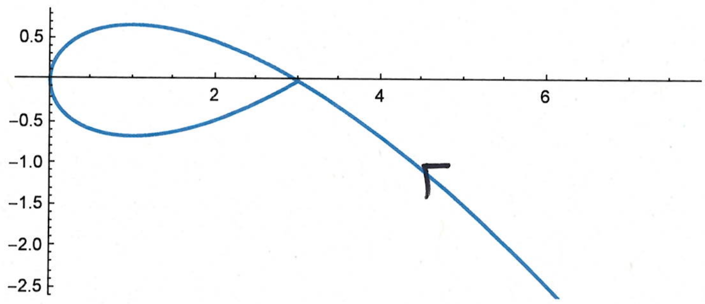
In[3]:= `ParametricPlot[{t^2, t^3 / 3 - t}, {t, -Sqrt[3], Sqrt[3]}]`

Out[3]=



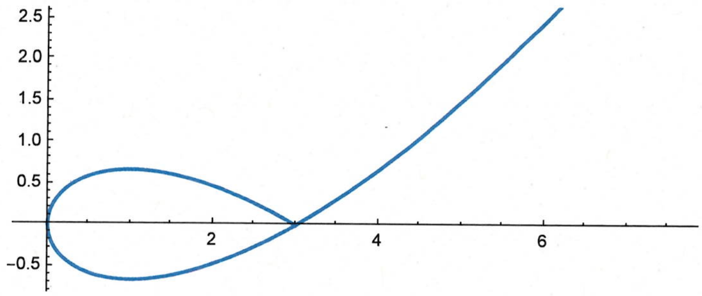

In[4]:= `ParametricPlot[{t^2, t^3 / 3 - t}, {t, -Sqrt[3] - 1, Sqrt[3]}]`

Out[4]=



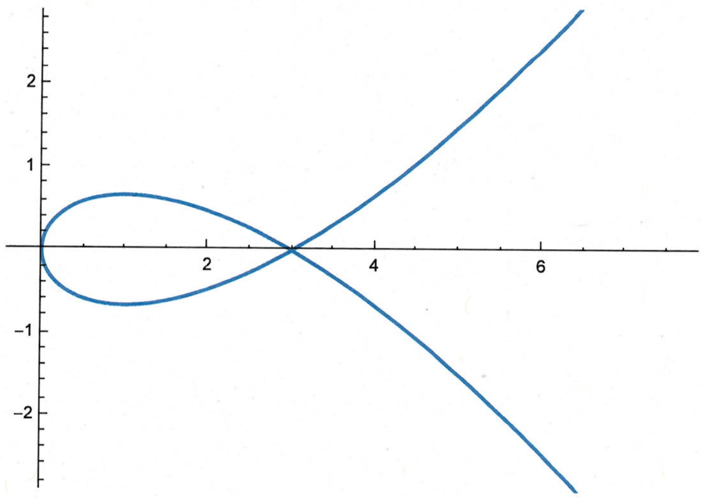

In[5]:= `ParametricPlot[{t^2, t^3 / 3 - t}, {t, -Sqrt[3], Sqrt[3] + 1}]`

Out[5]=



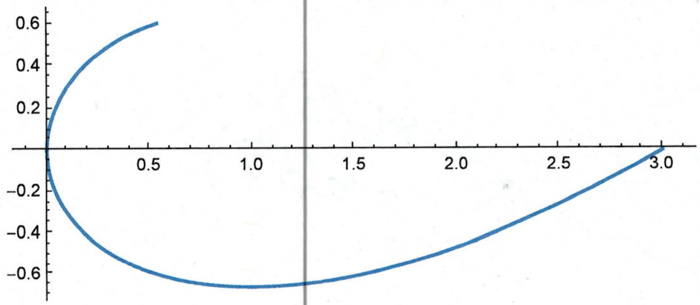

In[6]:= `ParametricPlot[{t^2, t^3 / 3 - t}, {t, -Sqrt[3] - 1, Sqrt[3] + 1}]`

Out[6]=



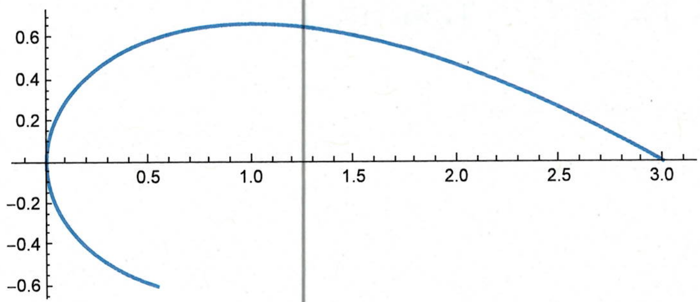
In[7]:= `ParametricPlot[{t^2, t^3/3 - t}, {t, -Sqrt[3] + 1, Sqrt[3]}]`

Out[7]=




In[8]:= `ParametricPlot[{t^2, t^3/3 - t}, {t, -Sqrt[3], Sqrt[3] - 1}]`

Out[8]=



Using Green's thm.

$$\text{Area} = \int_\gamma f \cdot ds \qquad f = (0, x)$$

$$\int_{-\sqrt{3}}^{\sqrt{3}} f(\gamma(t)) \cdot \gamma'(t)\, dt$$

$$\gamma'(t) = \begin{pmatrix} 2t \\ t^2 - 1 \end{pmatrix} \qquad f(\gamma(t)) = (0, t^2)$$

$$\int_{-\sqrt{3}}^{\sqrt{3}} \cancel{f} (0, t^2) \cdot (2t, t^2 - 1)\, dt$$

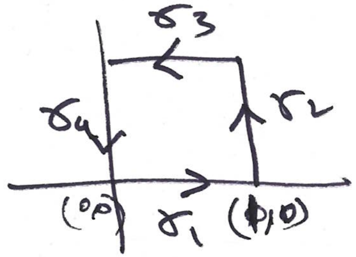
$$= \int_{-\sqrt{3}}^{\sqrt{3}} t^4 - t^2\, dt = \frac{t^5}{5} - \frac{t^3}{3} \Big|_{-\sqrt{3}}^{\sqrt{3}}$$

$$\ldots = \frac{8}{5}\sqrt{3}$$

or

**Ex** let $\gamma$ be the curve tracing the square with corners of $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$

$$f = (5 - xy - y^2,\ x^2 + 2xy)$$




$$\int_\gamma f\, ds = ?$$

$$\gamma_1 : [0,1] \to \mathbb{R}$$
$$t \to (t, 0)$$

we can do directly the line integral

we can use Green's thm

$$\int_\gamma f\, ds = \int_0^1\int_0^1 (\text{curl} f)\, dx\, dy.$$

$$= \int_0^1\int_0^1 \left(\frac{\partial}{\partial x}(x^2-2xy) - \frac{\partial}{\partial y}(5-xy-y^2)\right) dx\, dy$$

$$= \int_0^1\int_0^1 2x-2y-(-x-2y)\, dx\, dy.$$

$$\int_0^1\int_0^1 3x\, dx\, dy = \frac{3}{2}.$$

Rk

In general we might use Green's thm to calculate a 2 dim'l integral as a line integral

If we want to evaluate this

$$\iint_X g(x,y)\, dx\, dy$$

as a line integral

then we need to find a vector field $f=(f_1, f_2)$ s.t.

such that $\text{curl} f = g$.

ie $\dfrac{\partial f_2}{\partial x} - \dfrac{\partial f_1}{\partial y} = g(x,y)$

For example we can take $f_1 = 0$ then find $f_2$ such that $\dfrac{\partial f_2}{\partial x} = g(x,y)$

eg. $g(x,y) = 2xe^y$

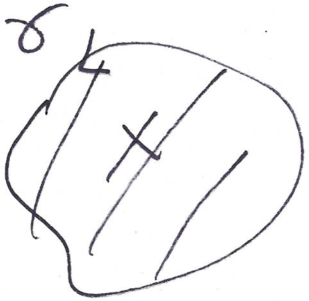
$\dfrac{\partial f_2}{\partial x} = 2xe^y \Rightarrow f_2 = x^2e^y$

Then $f = (0, x^2 e^y)$

then $\operatorname{curl} f = g$,

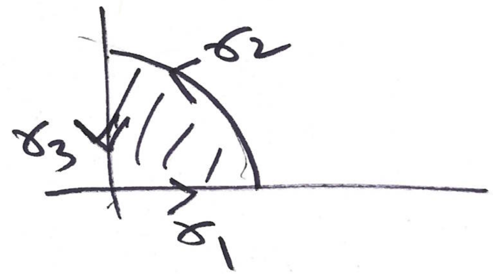
$$\iint_X g(x,y)\,dx\,dy$$

$$= \int_\gamma f\,ds \quad \text{where } \operatorname{curl} f = g.$$




**Example:**

Q Let $A = \mathbb{R}^2 \cap \{(x,y) : x \geq 0,\ y \geq 0,\ x^2+y^2 \leq 1\}$




and $\gamma$ be its oriented boundary.

Compute $\int_\gamma f\,ds$ where

$$f = (xy^2, 2xy)$$

a We can parametrize the boundary $\gamma$

$\gamma_1(t) = (t, 0) \qquad 0 \leq t \leq 1$

$\gamma_2(t) = (\cos t, \sin t) \qquad 0 \leq t \leq \pi/2.$

$\gamma_3(t) = (0, 1-t) \qquad 0 \leq t \leq 1$

$$\int_{\gamma} f\,ds = \int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f\,ds$$

$$\gamma_1 : \int_0^1 \underbrace{f(\gamma_1(t))}_{(0,0)} \cdot \gamma_1'(t)\,dt = 0$$

$$\gamma_3 = \int_0^1 \underbrace{f(\gamma_3(t))}_{0} \cdot \gamma_3'(t) = 0$$

$$\gamma_2 : \int_0^{\pi/2} \underbrace{(\cos t \sin^2 t,\ 2\cos t \sin t)}_{f(\gamma_2(t))} \cdot \gamma_2'(t)\,dt$$

$$= \int_0^{\pi/2} f(\gamma_2(t)) \cdot (-\sin t, \cos t)\,dt$$

$$= \int_0^{\pi/2} -\cos t \sin^3 t + 2\cos^2 t \sin t\,dt$$

$$= -\frac{1}{4}\sin^4 t - \frac{2}{3}\cos^3 t \Big|_0^{\pi/2}.$$

$$= -\frac{1}{4} + \frac{2}{3} = \frac{5}{12}.$$

(b) or using Green's thm.

$$\iint_A (\operatorname{curl} f)\,dx\,dy.$$

$$= \iint_A (2y - 2xy)\,dx\,dy.$$

using polar coordinates

$$\int_0^1 \int_0^{\pi/2} (2r\sin\theta - 2r^2\cos\theta\sin\theta)\,d\theta\,dr$$

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$$\int_0^1\int_0^{\pi/2} 2r^2 \sin\theta \, d\theta \, dr$$

$$- \int_0^1\int_0^{\pi/2} 2r^3 \cos\theta \sin\theta \, d\theta \, dr$$

$$= \cdots = 5/12.$$

## Another form of Green's formula.

**Defn.** For a v.field $f : \mathbb{R}^n \to \mathbb{R}^n$, $f \in C^1$, $f = (f_1, \dots, f_n)$ the divergence of $f$ is defined by

$$\boxed{\operatorname{div} f = \frac{\partial f_1}{\partial x_1} + \dots + \frac{\partial f_n}{\partial x_n}.}$$

$\operatorname{div} f : \mathbb{R}^n \to \mathbb{R}$ scalar valued.

<u>$n=2$</u> $\operatorname{div} f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}$.

we want to compute

$$\iint_X \operatorname{div} f \, dx \, dy.$$

We can use Green's thm with $\tilde{f}(x,y) = (-f_2, f_1)$

then $\tilde{f} \in C^1$

$$\operatorname{curl} \tilde{f} = \frac{\partial \tilde{f}_2}{\partial x} - \frac{\partial \tilde{f}_1}{\partial y}$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = \operatorname{div}(f)$$

So $\iint_X \operatorname{div} f \, dx dy = \iint_X \operatorname{curl} \tilde{f} \, dx dy$

$$= \int_{\partial X} \tilde{f} \cdot ds \quad ///$$

$$\gamma : [a,b] \longrightarrow \mathbb{R}^2$$
$$t \mapsto \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \end{pmatrix}$$


$$\int_\gamma \tilde{f}\, ds = \int_a^b \tilde{f}(\gamma(t)) \cdot \gamma'(t)\, dt$$

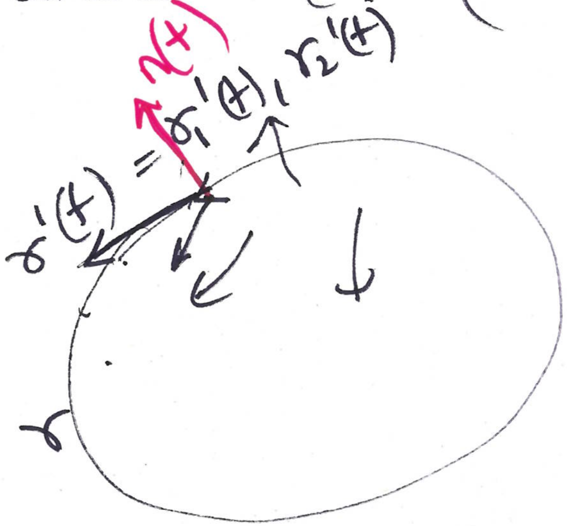
$$= \int_a^b \left(-f_2(\gamma(t)), f_1(\gamma(t))\right) \cdot \gamma'(t)\, dt$$

$$= \int_a^b -f_2(\gamma(t)) \cdot \gamma_1'(t) + f_1(\gamma(t)) \cdot \gamma_2'(t)\, dt$$

$$= \int_a^b \left(f_1(\gamma(t)), f_2(\gamma(t))\right) \cdot \overbrace{\left(\gamma_2'(t), -\gamma_1'(t)\right)}^{n(t)}\, dt$$

$$= \int_a^b f(\gamma(t)) \cdot n(t)\, dt$$

where $n(t) = \left(\gamma_2'(t), -\gamma_1'(t)\right)$




Note $\gamma'(t) \cdot n(t) = 0$.

$\|n(t)\| = \|\gamma'(t)\|$.

$n(t)$ is called the exterior normal to the curve.

**Thm.** Divergence-flux form or the normal form of Green's thm.

$$f = (f_1, f_2) : X \to \mathbb{R}^2$$

$$\iint_X \operatorname{div} f \, dx\,dy = \int_{\partial X} f \cdot d\vec{n}.$$

**Thm** curl-circulation form, tangential form of Greens thm.

$$\iint_X \operatorname{curl} f \, dx\,dy = \int_{\partial X} \vec{f} \cdot \vec{ds}.$$

If $n = 1$

Fund. thm. of calculus

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

$$F' = f.$$

$$\int_a^b F'(t)\,dt = F(b) - F(a).$$

$$\int_{[a,b]} F'(t)\,dt = F(b) - F(a)$$