Contents

1 – **7**

1 Symmetry 8

2 Isometries 11

3 Metadata 19

4 Set theory 20

5 Symmetries of polygons 27

2 – **31**

6 Platonic solids 32

7 Inscribed tetrahedra and cubes 38

8 Classification of Platonic solids - proof 42

9 Regular figures in higher dimensions 48

3 – **51**

10 The bare minimum for orientation 52

11 Fixed-point sets 56

12 Naming rigid motions 59

13 Hyperbolic space 71

4 – **73**

14 Motions fixing a point in $\mathbb{R}^2$ - proof 74

15 Motions fixing a point in $\mathbb{R}^3$ - proof 76

16 Composition of rotations 83

5 – **85**

17 When must isometries be equal? . . . 86
18 Abstract properties of symmetries . . . 92
19 Symmetries of the equilateral triangle . . . 93

6 – 99

20 Groups . . . 100
21 Isomorphisms . . . 103
22 Order. Cyclic groups . . . 105
23 Permutations . . . 108
24 Notation for permutations . . . 110

7 – 115

25 Subgroups . . . 116
26 Generators . . . 121
27 Cosets . . . 123
28 Divisibility of order . . . 125
29 Right and left multiplication . . . 127

8 – 129

30 Symmetries of the cube . . . 130
31 Subgroups of the cube group . . . 134
32 Product relationships among Platonic groups . . . 138
33 Conjugate elements . . . 140
34 Conjugate subgroups . . . 146

9 – 153

35 Homomorphisms . . . 154
36 Normal subgroups . . . 158
37 Quotient groups . . . 160
38 A quotient of the cube group . . . 163

10 – 167

39 Group actions . . . 168
40 Orbits and stabilizers . . . 170

41 Group action vocabulary 176

11 – 177

42 Transpositions and parity 178

43 The alternating group 182

12 – 185

44 Point groups 186

45 Point groups - exceptional 189

46 Point groups - axial type 198

47 Frieze groups 208

48 Point group theorems 212

13 – 213

49 Pyrite 214

50 Pyrite - preferred planes 218

14 – 225

51 Factoring isometries into reflections 226

52 Factoring isometries into rotations and translations 230

53 Completing the classification of proper motions 232

54 Proper motions of $\mathbb{R}^2$ - proof 233

55 Proper motions of $\mathbb{R}^3$ - proof 236

15 – 239

56 Isometries as linear maps 240

57 A finite group of isometries of $\mathbb{R}^n$ fixes a point 243

58 Semidirect product structure of Isom($\mathbb{R}^n$) 246

16 Appendices 253

59 Involutions and reflections 254

60 Quaternions 255

61 Metric spaces 256

62 Ten types of motion of $\mathbb{R}^3$ 260

63 Opposite pyritohedra 263

64 Exercises in search of a home 264

17 Bibliography **271**

65 Books 272

66 Articles, blogs, and references 273

67 Software and visualization 275

List of figures **277**

Figure credits **285**