Geometrie

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Notes taken by Alexander Venuleth

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You boil it in sawdust: you salt it in glue:
You condense it with locusts and tape:
Still keeping one principal object in view—
To preserve its symmetrical shape.

–Lewis Carroll

Contents

Chapter 1

—

1 Symmetry

Symmetries. Irregular figure. Triangles. Rotations, reflections.

Take a look at the salt crystal.[1] It's very symmetrical.

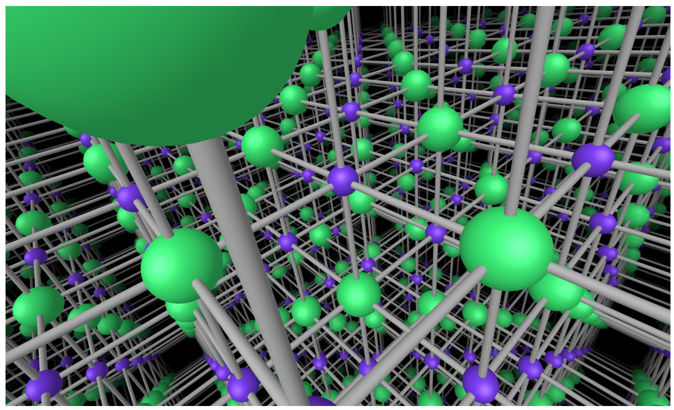


Figure 1.1: Salt. (Made with J. Weeks' Crystal Flight app)

In this course we will answer the following questions:

- How symmetric is a figure?
- How do we express this? With what mathematical structures can one measure and describe this? $\longrightarrow$ Group (of isometries)

Figure 1.2: Figures in $\mathbb{R}^2$.

The figure on the left is totally asymmetrical. The one on the right has a reflective symmetry in a vertical line.

[1] It's from Jeff Weeks' program Crystal Flight (iOS, macOS) at `http://www.geometrygames.org/CrystalFlight/index.html`.

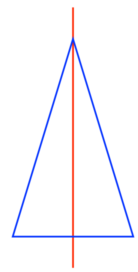


Figure 1.3: Reflection in a vertical line.

But it also has the identity function as a “symmetry” – the function that takes each point x to itself.

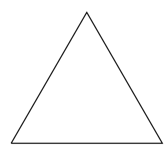


Figure 1.4: An equilateral triangle.

We observe that an equilateral triangle has 6 symmetries as follows:

The identity map, two rotations (by 120 and 240 degrees), three reflections

Figure 1.5: Two nontrivial rotations.

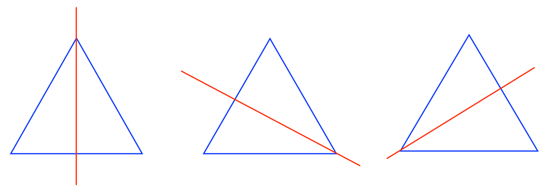


Figure 1.6: Three reflections.

We will develop tools to describe symmetries and find relations between them. We will classify isometries of Euclidean space and symmetries of figures. We will apply this to crystals (a bit).

Exercise 1.1 *Find two figures F_1, F_2 in $\mathbb{R}^2$ so that $F_1 \subseteq F_2$, but F_1 has more symmetries than F_2.*

Exercise 1.2 *Find a figure with exactly 3 symmetries.*

Exercise 1.3 *Find a figure with exactly 4 symmetries.*

2 Isometries

$\mathbb{R}^n$, distance, symmetries, figures, isometries, salt, fluorite, diamond, Penrose tiling, iron crystal.

References

- Knörrer, 1-4.
- Senechal, 20-23.

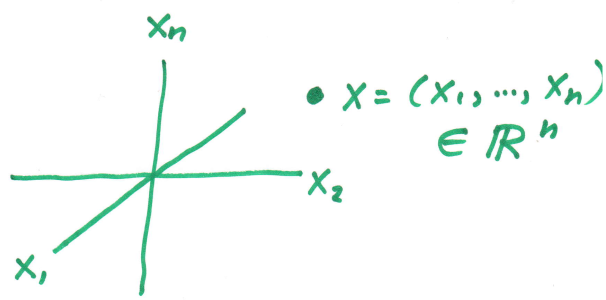


Figure 2.1: Point x in $\mathbb{R}^n$.

By $\mathbb{R}^n$ we mean the set of ordered n-tuples

$$x = (x_1, \dots, x_n),$$

of real numbers $x_1, \dots, x_n$. The numbers x_i vary freely in $\mathbb{R}$. The formula

$$x \in \mathbb{R}^n$$

reads[2]

$$x \text{ is an element of } \mathbb{R}^n.$$

[2] For more information on sets, functions, and all that, read ahead to the section on set theory.

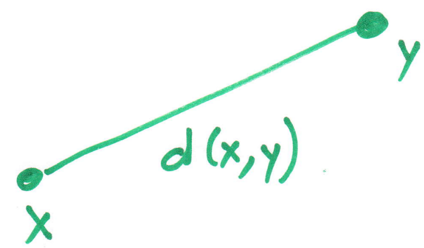


Figure 2.2: Distance between x and y.

Definition Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be two points in $\mathbb{R}^n$. The *distance* $d(x, y)$ between x and y is given by

$$d(x, y) \coloneqq \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

The function d is called the *Euclidean metric* on $\mathbb{R}^n$.

Definition A *figure* is any subset of $\mathbb{R}^n$.

Most of our figures will be polygons and polyhedra (flat sides), or lattices (crystals), but figures are allowed to be anything, including fractals and unbounded sets. The *Sierpinski space* is the limit of the following construction as the number of "levels" goes to infinity.

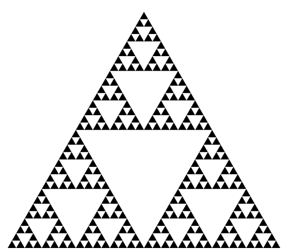


Figure 2.3: The Sierpinski triangle.

Definition An *isometry* or *rigid motion* of $\mathbb{R}^n$ is a surjective function[3]

$$\phi\colon \mathbb{R}^n \to \mathbb{R}^n, \qquad x \mapsto \phi(x),$$

[3] *Surjective* means that every point of $\mathbb{R}^n$ is hit by at least one point x under ϕ. *Injective* means that every point of $\mathbb{R}^n$ is hit by at most one point x under ϕ. *Bijective* means injective and surjective. A bijective function is a one-to-one correspondence.

that preserves the distance between points:

$$d(x, y) = d(\phi(x), \phi(y)), \qquad x, y \in \mathbb{R}^n. \tag{2.1}$$

Here, x is a point of $\mathbb{R}^n$, and $\phi(x)$ is its image under the mapping ϕ.

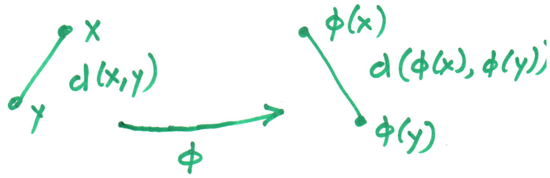


Figure 2.4: Distance is preserved.

Remarks:

1) Some authors reserve the term "motion" for isometries that can be achieved by continuous movement, such as translation or rotation, but we use "motion" for all isometries.

2) Because distance is preserved, an isometry is automatically injective.

3) Distance-preserving maps are automatically surjective when they are from $\mathbb{R}^n$ to $\mathbb{R}^n$.[4]

4) So isometries are bijective.

5) Isometries of $\mathbb{R}^n$ also preserve angles, areas, and volumes. We will accept these assertions of high-school geometry without proof.[5]

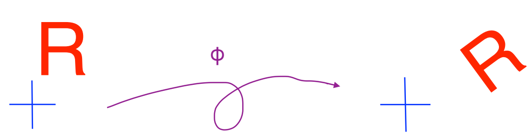


Figure 2.5: An isometry.

Example Here are some isometries of $\mathbb{R}^n$:

- The identity map $\mathrm{id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto x$
- Translations, rotations

[4]This is not obvious, but requires a proof – best done with linear algebra. So we could drop the surjectivity condition here. But we need it in more general cases, where the space is not $\mathbb{R}^n$.

[5]For general metric spaces (Appendix 47), such notions are more subtle or don't exist.

- Reflections in planes, lines, or points.

Then there are some exotic ones:

- Roto-reflections, glide reflections, screw motions.

These are illustrated in Section 12.

Example The following kinds of maps are generally not isometries:

> expansions, contractions, shears, projections, distortions, rips, tears, constant maps.

Definition The set of all isometries ϕ that map $\mathbb{R}^n$ to itself is written

$$\mathrm{Isom}(\mathbb{R}^n) := \{\text{all isometries } \phi : \mathbb{R}^n \to \mathbb{R}^n\}. \tag{2.2}$$

$\mathrm{Isom}(\mathbb{R}^n)$ is called the *Euclidean group*.

Definition A *symmetry* of a figure $F \subseteq \mathbb{R}^n$ is an isometry $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ that maps F to itself, that is, $\phi(F) = F$. The set of all symmetries of a figure F is written

$$\mathrm{Sym}(F) := \{\phi \in \mathrm{Isom}(\mathbb{R}^n) \,\big|\, \phi(F) = F\}. \tag{2.3}$$

We write $\mathrm{Sym}_{\mathbb{R}^n}(F)$ when we want to make the ambient space explicit.[6]

Example (Platonic solids) These figures have a lot of symmetries.

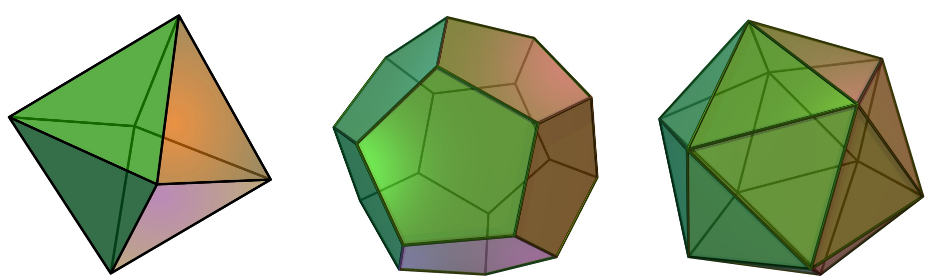


Figure 2.6: Octahedron, Dodecahedron, Icosahedron. (Cyp, Wikipedia)

Exercise 2.1 *How many?*

[6]The ambient space matters, but usually it will be clear from the context, so we will usually leave it off. See Exercises 5.4, 53.2.

Example (Crystals) Look at the salt crystal again. The greens are chlorine and the purples are sodium. It is an unbounded figure, with infinitely many symmetries – translations, rotations, and combinations of them.

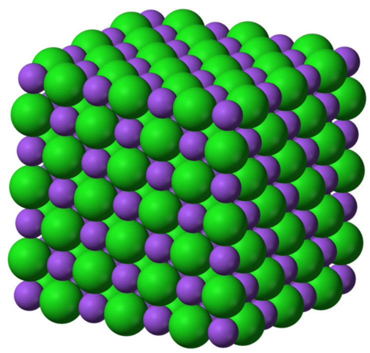


Figure 2.7: NaCl crystal. (Benjah-bmm27, Wikipedia)

Here is fluorite (CaF_2), with calcium atoms in red:

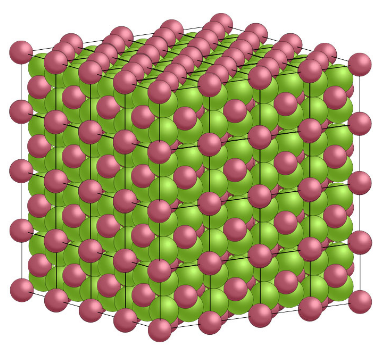


Figure 2.8: Fluorite crystal. (gotbooks.miracosta.edu, geologycafe.com)

Detail of the unit cell of fluorite, with calciums now in blue:

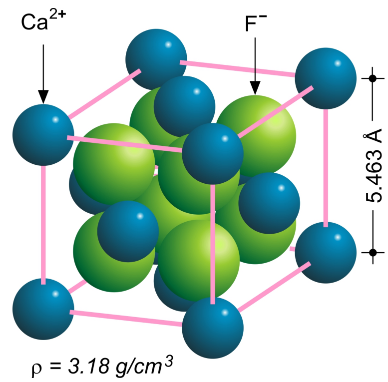


Figure 2.9: Fluorite unit cell. (ShutterWaves, Wikipedia)

Exercise 2.2 *Does fluorite have the same symmetries as salt or different?*

It sure looks different.[7]

The next one is harder to grasp. It is diamond. Only one unit cell is shown. You have to imagine it repeated throuch space. The symmetries are "similar to, but different" from the symmetries of NaCl or CaF_2. Each carbon atom connects to four others, in line with the fact that carbon has valence 4.

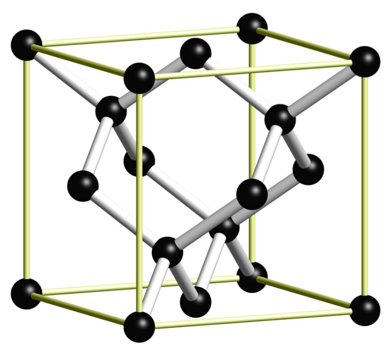


Figure 2.10: Diamond. (strpeter, tex.stackexchange.com)

You can see the whole crystal structure if you fly around in it. But it doesn't get much easier to grasp. The following crystal flight app is by J. Weeks. You have to choose the "diamond" setting.

[7]Trick question. It has the same symmetries, expressed by the space group $Fm\bar{3}m$, #225, but the atoms are positioned differently within the unit cell, as expressed by the Wyckoff positions of the atoms, which are 4a, 4b and 4a, 8c respectively. See Burns-Glazer, pp 192, 193, 345; *Fluorite structure*, Wikipedia; also International Tables of Crystallography, vol A (online 2016), p 688, vol A1 (online 2011), p 752, entries for group #225, accessible online through the ETH library.

- `http://www.geometrygames.org/CrystalFlight/index.html`

A screenshot:

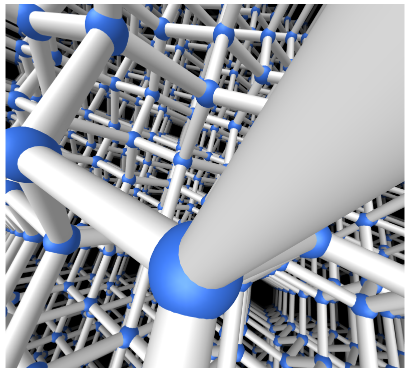


Figure 2.11: Diamond. (Made with J. Weeks' Crystal Flight app)

Example (Penrose tiling) Below is a Penrose tiling. Strictly speaking, it has no symmetries – meaning no global isometries. But it has arbitrarily large pieces that are isometric to arbitrarily large other pieces. It also has a "local" 5-fold symmetry that is forbidden to traditional crystals. That's why there was so much excitement in the 80s when real materials were discovered with such a "quasicrystal" symmetry. Unfortunately we won't be getting to this in the class. But almost. This one is associated to the golden ratio $(\sqrt{5}+1)/2 = 1.618....$

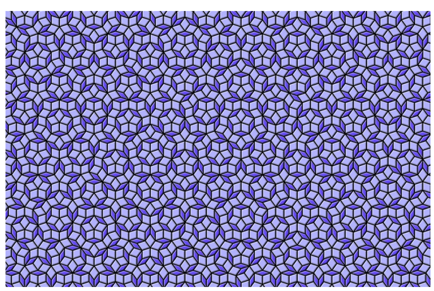


Figure 2.12: Penrose tiling. (Preshing on programming, 2011.)

If you print out a large piece of the Penrose tiling onto a transparency, and make two transparencies like this, you can overlap them so that it lines up over a large region, but not everywhere. This shows that a large piece gets repeated in a different position.

Exercise 2.3 *There is a nice explanation of the Penrose tiling at* *http://www.math.ubc.ca/~cass/courses/m308-02b/projects/schweber/penrose.html.*

It contains a .ps file to download and print. The .ps file is actually a computer program with a parameter /N that determines the iteration depth. Try printing the image for various values of /N.

Example (Silicon crystal) Here is a TEM (transmission electron microscope) image of a real silicon crystal.

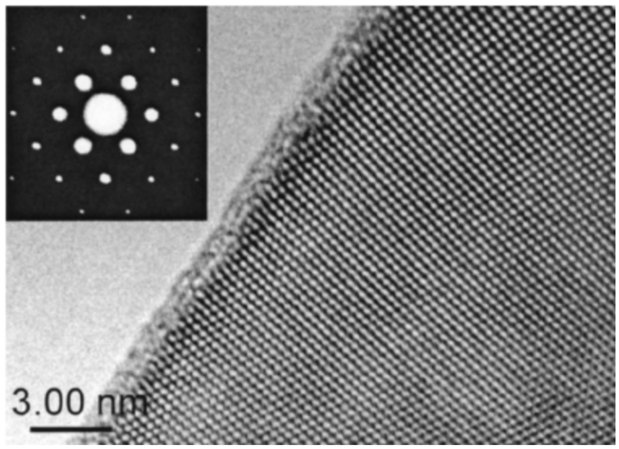


Figure 2.13: Silicon crystal in a nanowire. (D. Li et al)

3 Metadata

Tom Ilmanen, lecturer

Raphael Appenzeller, organizer

Lectures Tuesday 16-18 weekly:

15.09.; 22.09.; 29.09.; 06.10.; 13.10.; 20.10.; 27.10.; 03.11.; 10.11.; 17.11.; 24.11.; 01.12.; 08.12.; 15.12 (exam).

Exercise sections Monday 16-18 biweekly:

21.09.; 05.10.; 19.10.; 02.11.; 16.11.; 30.11.; 14.12.

Exercises are issued Wednesday week n, discussed in section Monday week $n+1$, due Monday week $n+2$, returned in section Monday week $n+3$, where n is odd.

Website: `https://metaphor.ethz.ch/x/2020/hs/401-1511-00L`

Exercises: `https://metaphor.ethz.ch/x/2020/hs/401-1511-00L`

Script: `https://metaphor.ethz.ch/x/2020/hs/401-1511-00L/literatur/script.pdf`

Forum: `https://forum.math.ethz.ch/t/geometrie-herbst-2020/277`

Exam: ~~15.12.20 in class~~. Tuesday 5. January 2021, 15.00-16.30, 90 min, Gebäude HIL, Hönggerberg.

References:

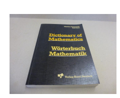
1) H. Knörrer, *Geometrie*, `https://www.springer.com/de/book/9783322939807`. The course corresponds roughly to Chapter 1, pp 1-64.

2) For group theory: D. Saracino, *Abstract Algebra: A First Course*, pp 1-132, `https://www.waveland.com/browse.php?t=483`. It looks friendly.

3) A lovely introduction to crystallography (goes beyond the class, but pleasant reading) is M. Senechal, *Crystalline Symmetries, An informal mathematical introduction*, 1990.

4) Additional reference books include Rotman, Brieskorn I, III, Burns-Glazer, Fischer, Jänich, and others. See Chapter 14 for more books, software, etc.

5) Mathematical dictionary: G. Eisenreich, R. Sube, *Dictionary of Mathematics; Wörterbuch Mathematik*, Verlag Harry Deutsch, 1987.



5) Images: Details of the picture credits for these notes are in Section 57.

4 Set theory

Sets, elements, subsets. Examples. Products of sets. Functions, graphs. Injective, surjective, bijective. Images and preimages. Composition. Equivalence relations. Divisibility.

References

- Saracino 1-3 (sets), 59-65 (functions), 80-82 (equivalence relations).
- Rotman, Appendix II (equivalence relations), Appendix III (functions).

Sets

A *set* is a collection of elements. The elements can be anything, including other sets. The order in which the elements are listed is not important. Nor do repetitions count. We use curly brackets for sets.

The empty set (the set with no elements) is written $\varnothing$ or $\{\}$.

Examples

- $\{1, 2, 3\} = \{3, 1, 2\} = \{2, 2, 1, 3\}$
- $\{a, b\} = \{c, d\}$ iff ($a = c$ and $b = d$) or ($a = d$ and $b = c$)
- $\{a, b\} = \{c\}$ iff $a = b = c$
- $\mathbb{N}_+ = \{1, 2, 3, 4, 5, \dots\}$
- $\mathbb{N}_0 = \{0, 1, 2, 3, 4, 5, \dots\}$
- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$
- The quaternions $\mathbb{H}$

□

We can also define sets by giving a domain A and a condition $P(x)$ in the form

$$\{x \in A : P(x)\},$$

where $P(x)$ is a proposition about x, that is, a function of x that takes the values “true” or “false”.

Examples

- $\{x \in \mathbb{R} : x^{10} > 100\}$
- $\{n \in \mathbb{N}_+ : n^2 < -2\} = \varnothing$
- $\{n \geq 2 : (q > 0 \ \& \ q|n) \Rightarrow (q = 1 \text{ or } q = n)\}$ (the prime numbers)

□

We write

$$x \in M$$

to mean "x is an element of A".

Examples

- $1 \in \{1, 2\}$
- $\pi \notin \mathbb{Q}$.

□

We write

$$A \subseteq B$$

to mean "A is a subset of B", that is,

$$A \subseteq B \quad \iff \quad (\forall a : a \in A \implies a \in B) \tag{4.1}$$

Note that $\forall$ means "for all" and $\exists$ means "there exists".

Examples

- $\{1, 2\} \subseteq \{1, 2, 3\}$
- $\{(a, b, c) : a, b, c \in \mathbb{N}_+, a^{17} + b^{17} = c^{17}\} \subseteq \varnothing$.

□

Let $n \geq 0$. An n-tuple is an ordered list of mathematical objects with n entries. Differently from a set, the order matters. Repetitions are allowed and they matter. We use parentheses to indicate an n-tuple. A 2-tuple is also called an *ordered pair*.

Examples

- $(2, 3, 5, 3)$ 4-tuple
- $(5, \sin(x), \{3, 5\})$ 3-tuple
- $()$ 0-tuple
- $(2, 3) \neq (3, 2)$

□

Definition Let A and B be sets. The *Cartesian product* of A and B is the set of ordered pairs (a, b) with $a \in A$ and $b \in B$, i.e.

$$A \times B \coloneqq \{(a, b) \mid a \in A \text{ and } b \in B\} \tag{4.2}$$

Similarly, we define

$$A_1 \times \ldots \times A_n$$

as the set of all n-tuples $(x_1, \ldots, x_n\}$, where $x_i \in A_i$, $i = 1, \ldots, n$. We write A^n for

$$\underbrace{A \times \ldots \times A}_{n\ times}.$$

We identify

$$A \times B \times C$$

with

$$(A \times B) \times C,$$

etc. This involves dropping some internal parentheses.

In particular, the set of all n-tuples of real numbers is written $\mathbb{R}^n$. In this lecture, we'll mostly be interested in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Functions

Definition A function f from a set X to a set Y is a rule that assigns to each x in X exactly one element y in Y. We write it as follows:

$$\begin{aligned} f\colon X &\to Y \\ x &\mapsto y = f(x) \end{aligned}$$

We also write

$$X \xrightarrow{f} Y.$$

If X is finite, a function can be defined by a table. The *graph* of f is the set of all ordered pairs $(x, f(x))$ such that $x \in X$:

$$(x, y) \in \operatorname{graph}(f) \qquad \Longleftrightarrow \qquad y = f(x).$$

The graph of f is a subset of $X \times Y$.

The set X is called the *domain* of f, written $\operatorname{dom}(f)$. The set Y is called the *target space*, written $\operatorname{target}(f)$. The *image* of f is the set

$$\operatorname{im}(f) := \{f(x) \big| x \in X\}.$$

More generally, the image of a subset $A \subseteq X$ is defined by

$$f(A) := \{f(x) \big| x \in A\}.$$

f is called *surjective* if

$$\operatorname{im}(f) = Y,$$

that is, every $y \in Y$ gets hit by some $x \in X$. In symbols:

$$f \text{ is surjective} \iff \forall y \in Y\ \exists x \in X\ f(x) = y. \tag{4.3}$$

Figure 4.1: Surjective.

Note that the definition of surjective depends on our choice of the target space Y. Therefore, strictly speaking, the definition of a function must include a specification of its target space, and two functions are not equal unless they have the same target space. Usually, but not always, we can overlook this.

Exercise 4.1 *Is the function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ surjective? What about $g : \mathbb{R} \to [0, \infty)$, $x \mapsto x^2$? Is f "equal" to g?*

The *preimage* of an element $y \in Y$ is the subset

$$\{x \in X : f(x) = y\}$$

of X. We write

$$f^{-1}(a)$$

for it. If $f^{-1}(y)$ consists of a single point x, we sometimes use $f^{-1}(y)$ to mean the element x rather than the set $\{x\}$.

Similarly, the preimage of a subset $B \subseteq Y$ is the subset

$$f^{-1}(B) := \{x \in X : f(x) \in B\}$$

of X.

The function f is called *injective* if for each $y \in Y$, $f^{-1}(y)$ consists of at most one element. That is, y gets hit by at most one element of X. In symbols:

$$f \text{ is injective} \iff (\forall x, x' \in X : f(x) = f(x') \implies x = x') \tag{4.4}$$

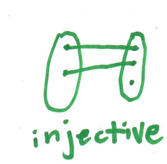


Figure 4.2: Injective.

A function f is called *bijective* if f is both injective and surjective. Such a function is also said to be a *one-to-one correspondence*.

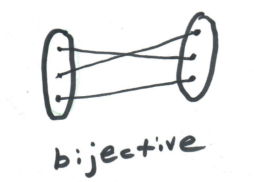


Figure 4.3: Bijective.

If f is bijective, there exists a function (the inverse of f)

$$f^{-1} : Y \to X$$

which takes each element $y \in Y$ to its (unique) preimage x in X.

Let $f : X \to Y$ be a function. If $A \subseteq X$, we define the *restriction*

$$f|A : A \to Y,$$

of f to A by

$$(f|A)(x) := f(x) \qquad \text{for all } x \in A.$$

If $A \neq X$, then $f|A$ has a different domain from f, so it's a different function.

Occasionally (!) we might want to explicitly redefine the target space as well. If $B \supseteq \mathrm{im}(f)$, we define the *co-restriction*

$$f \restriction B : X \to B,$$

of f to A by

$$(f \restriction B)(x) := f(x) \qquad \text{for all } x \in X.$$

It's the same "rule", but it has a different target space, so it's a different function. Officially, $f \restriction B \neq f$ unless $B = Y$. Usually this will not matter.

Let X, Y and Z be sets. Let

$$f\colon X \to Y, \qquad g\colon Y \to Z$$

be functions. The function

$$\begin{aligned} g \circ f\colon X &\to Z, \\ x &\mapsto g(f(x)), \end{aligned}$$

that assigns $g(f(x))$ to x is called the *composition* of f and g. To make the order precise, we say "f followed by g". In symbols

$$(g \circ f)(x) := g(f(x)).$$

We can also write the *commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g\circ f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

Figure 4.4: Commutative diagram.

Equivalence relations

A *relation* P is a function $P(x, y)$ with two arguments and values in $\{\text{true, false}\}$. Usually it is written with the relation sign in the middle. So

$$xPy$$

means x has the relation P to y. An example of a relation is

$$x \text{ is a sister of } y.$$

An *equivalence relation* is a relation $\cong$ such that:

$$\begin{array}{llll} x \cong x & & & \text{(reflexive)} \\ x \cong y & \iff & y \cong x & \text{(symmetric)} \\ x \cong y, y \cong z & \implies & x \cong z & \text{(transitive).} \end{array}$$

An example of an equivalence relation is

$$x \text{ and } y \text{ have the same parents.}$$

A *partition* or *decomposition* of a set X is a subdivision of X into disjoint subsets $(A_i)_{i\in I}$ whose union is X:

$$X = \bigcup\{A_i \mid i \in I\}, \qquad A_i \cap A_j = \varnothing \quad \text{for } i \neq j \in I.$$

The main fact about equivalence relations is that they induce a partition of the set on which they are defined, characterized by the condition

$$x \cong y \iff x \text{ and } y \text{ lie in the same element } A_i \text{ of the partition.}$$

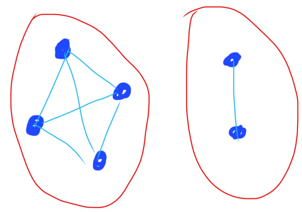


Figure 4.5: Equivalence relation.

See Saracino, pp 80-82, Rotman, Appendix III for details.

Divisibility

Definition Let $a, b \in \mathbb{Z}$. We say that a divides b if b is an integer multiple of a:

$$a \big| b \qquad \Longleftrightarrow \qquad \exists\, k \in \mathbb{Z} : b = ka \tag{4.5}$$

5 Symmetries of polygons

Polygons, dihedral group, two types of symmetries, Klein four-group.

References

- Knörrer, 8-9 (hexagon), 44.
- Saracino, 40 (Klein four-group).

Denote the regular n-sided polygon by P_n. It has equal sides and equal angles.

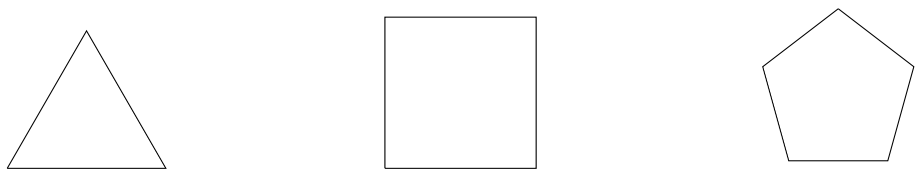


Figure 5.1: Regular 3-, 4-, und 5-gon.

The set of symmetries of the regular n-gon is called the *dihedral group* and is written

$$D_n = \mathrm{Sym}(P_n).$$

For example, the square has 8 symmetries:

- identity map
- 3 nontrivial rotations
- 4 reflections

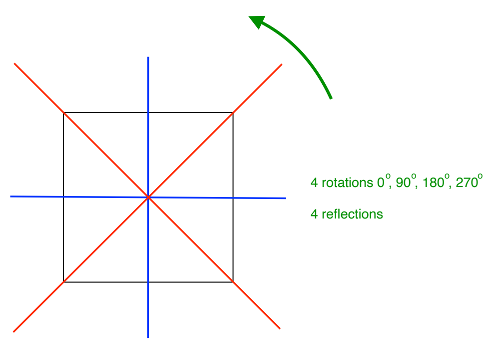


Figure 5.2: Symmetries of the square.

Exercise 5.1 *Check that* $\mathrm{Sym}(P_n)$ *contains* $2n$ *symmetries – indeed*

- n *reflections*
- $n-1$ *nontrivial rotations*
- *1 identity map.*

Types of reflections of polygons

We notice that the square has two different kinds of reflections:

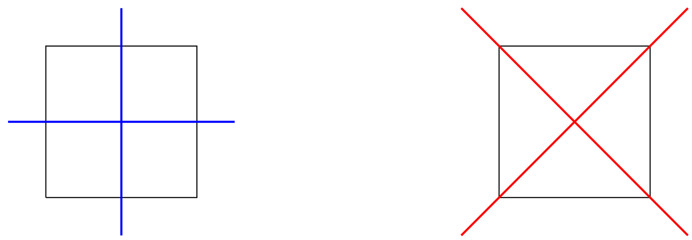


Figure 5.3: Reflections of a square.

On the other hand, the regular pentagon has only one kind:

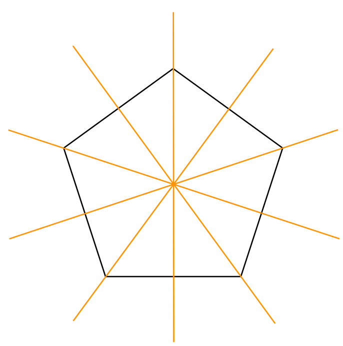


Figure 5.4: Reflections of the pentagon.

Exercise 5.2 *Observe that P_n has one kind of reflective symmetry when n is odd, but two kinds when n is even.*

The Klein four-group

The set $D_2 = \mathrm{Sym}(P_2)$ also exists, even though the 2-gon P_2 is degenerate. A "2-gon" is a line segment! D_2 consists of 4 elements

- identity
- reflection in a vertical line (green)
- reflection in a horizontal line (red)
- rotation by 180 about the midpoint M (brown).

It is also called the *Klein four-group*.[8]

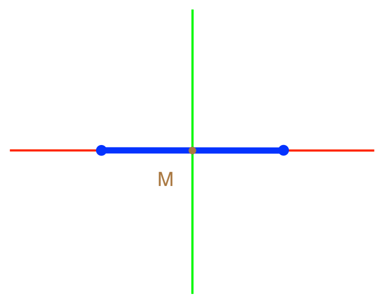


Figure 5.5: Klein four-group.

Exercise 5.3 *Note that D_2 is commutative, that is, $a \circ b = b \circ a$ for all $a, b \in D_2$.*

Exercise 5.4 *What is the difference between $\mathrm{Sym}_{\mathbb{R}}(P_2)$ and $\mathrm{Sym}_{\mathbb{R}^2}(P_2)$? Which one is the Klein four-group?*

[8] Saracino, p 40.

Chapter 2

—

6 Platonic solids

Platonic solids. Computing the number of symmetries. Duality.

References

- ◦ Knörrer, 60-61.
- ◦ Brieskorn I, 1-36 (background reading).
- ◦ Coxeter, pp 5-6, 15-16.

In addition to the above, there are many beautifully illustrated blogs and websites about Platonic solids – it's a favorite topic on the net. Some of the best of these are by J. Baez (going a little beyond the class):

- ◦ *Symmetry and the Fourth Dimension (Part 1)* , `https://johncarlosbaez.wordpress.com/2012/05/21/symmetry-and-the-fourth-dimension-part-1`.
- ◦ *Symmetry and the Fourth Dimension (Part 2)* , `https://johncarlosbaez.wordpress.com/2012/05/27/symmetry-and-the-fourth-dimension-part-2`.
- ◦ *Symmetry and the Fourth Dimension (Part 3)* , `https://johncarlosbaez.wordpress.com/2012/07/22/symmetry-and-the-fourth-dimension-part-3`.
- ◦ *Symmetry and the Fourth Dimension (Part 4)* , `https://johncarlosbaez.wordpress.com/2012/07/26/symmetry-and-the-fourth-dimension-part-4`.

We denote the cube by W, the tetrahedron by T, the octahedron by O, the dodecahedron by D and the icosahedron by I.

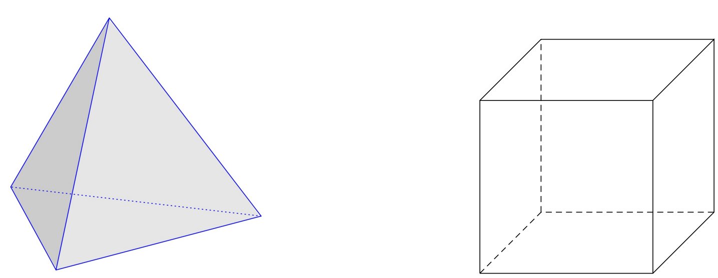


Figure 6.1: Tetrahedron. (Aldoaldoz, Wikipedia), cube

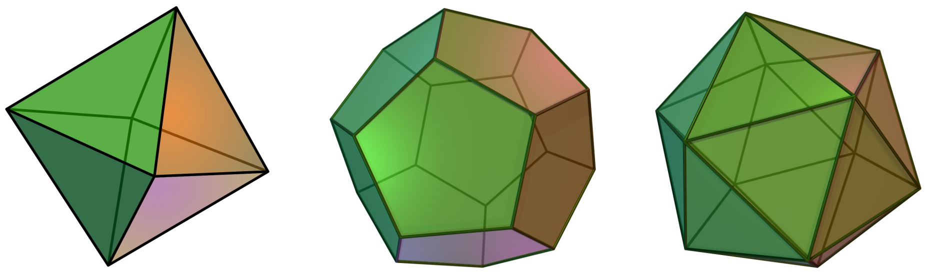


Figure 6.2: Octahedron, Dodecahedron, Icosahedron. (Cyp, Wikipedia)

Question. How many symmetries does the cube have?

Solution: Fix a reference face – say the top face.

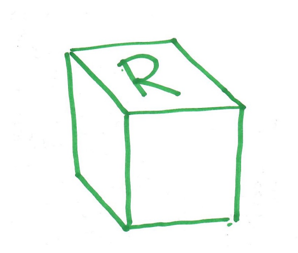


Figure 6.3: Reference face.

Then

a) Move the top face it to any one of 6 possible target faces.

b) Rotate or reflect the target face to itself in 8 possible ways – because $\mathrm{Sym}(\Box) = D_4$ has $2 \cdot 4 = 8$ elements.

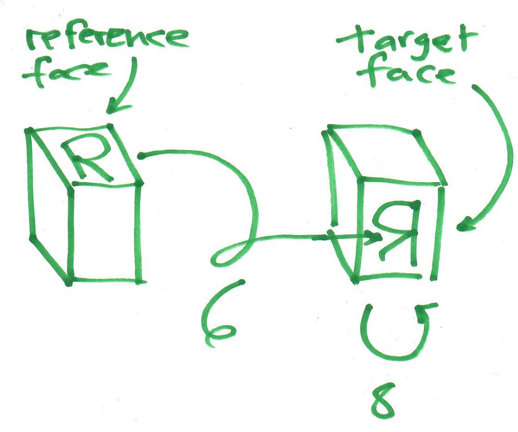


Figure 6.4: 48 motions.

The answer is

$$\begin{aligned}\#\mathrm{Sym}(W) &= \#(\text{faces of W}) \cdot \#\mathrm{Sym}(\square)\\ &= 6 \cdot 8\\ &= 48.\end{aligned}$$

See Knörrer pp 34-35 for a similar argument, justified more carefully.

Exercise 6.1 *Compute the number of symmetries of T, W, O, D, I.*

Answers: 24, 48, 48, 120, 120.

We discover something very curious – the cube and the octahedron have the same number of symmetries! What is going on here?

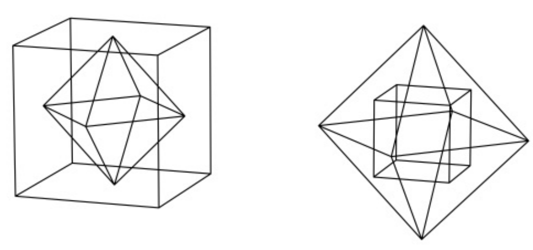


Figure 6.5: Octahedron inscribed in a cube and vice-versa. (Knörrer/Brieskorn)

We can inscribe an octahedron inside a cube in such a way that they have exactly the same set of symmetry motions. That is, we can arrange that

$$\mathrm{Sym}(W) = \mathrm{Sym}(O).$$

The Kaleidotile app explores a two-dimensional configuration space of figures with the same set of symmetries as the cube and the octahedron.

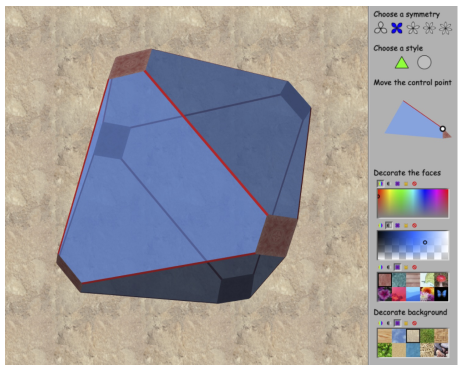


Figure 6.6: A two-dimensional configuration space of figures. (J. Weeks Kaleidotile app)

You can make screenshots to illustrate your homework. Here is the app.

- Kaleidotile (iOS, macOS, Windows): `http://www.geometrygames.org/KaleidoTile`

This is closely related to the fact that W and O are *dual polyhedra*. If we count the faces, edges, and vertices of W and O, we get

cube	faces	edges	vertices	
	6	12	8	
	vertices	edges	faces	octahedron

Duality is well-expressed by the figure, which shows how each face of the cube corresponds to a vertex of the octahedron, each edge of the cube to an edge of the octahedron, and each vertex of the cube to a face of the octahedron. Each figure can be computed from the other (by purely geometric operations), so they contain the same geometric information in some sense, and it is not surprising that they have the same set of symmetries.

The dodecahedron and the icosahedron are also dual, so that

$$\mathrm{Sym}(I) = \mathrm{Sym}(D),$$

with a similar picture. There's a nice video that illustrates this at

- https://www.youtube.com/watch?v=WgcJLVCoc8w

It was made by Debra Borkovitz with SketchUp. We can watch it right here and now.

The numbers read

dodecahedron	faces	edges	vertices	
	12	30	20	
	vertices	edges	faces	icosahedron

Exercise 6.2 *What is the dual of the tetrahedron?*

Certain viruses have perfect icosahedral or dodecahedral symmetry. This is discussed in Brieskorn pp 16-17.

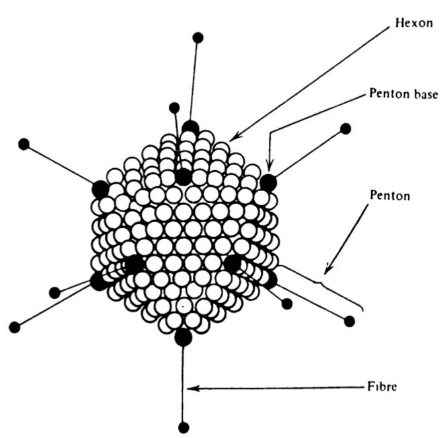


Figure 6.7: Icosahedral adenovirus. (R. C. Valentine & H. G. Pereira, *J. mol. biol.* **13** 13 1965, via Brieskorn, p 17)

An interactive video with much more information about virus symmetry is at

- http://viruspatterns.com.

The corona virus has far more spikes than an icosahedron has corners:

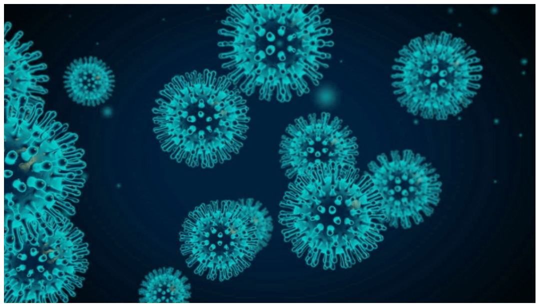


Figure 6.8: Virus with many spikes. (Provenance unknown)

Exercise 6.3 *Could there be a figure in $\mathbb{R}^3$ with thousands of symmetries (but not "prismlike")?*

A polygon in $\mathbb{R}^2$ can have arbitrarily many symmetries, and so can a prism or pyramid in $\mathbb{R}^3$. These are "prismlike" and it makes the problem too easy. Our intention is to exclude this case.

For $\mathbb{R}^3$, this question is answered in Theorems **??** and **??**. Mathematically, it is quite a deep question, and leads to further things such as the Bieberbach theorems.[1]

[1] For a deep answer, see Thurston, p 217, Cor. 4.1.11, where "prismlike" is replaced by "having a large abelian subgroup".

7 Inscribed tetrahedra and cubes

Inscribing tetrahedra in the cube. Inscribing tetrahedra and cubes in the dodecahedron.

References

- Knörrer, p 24, example 5.
- G. Egan, animated compounds, `https://blogs.ams.org/visualinsight/2015/05/15/dodecahedron-with-5-tetrahedra`

Consider the cube W with corners $E_W = \{(\pm 1, \pm 1, \pm 1)\}$. The tetrahedron can be found inside the cube with the following corners:

$$E_T = \{(1,1,1), (1,-1,-1), (-1,1,-1), (-1,-1,1)\}.$$

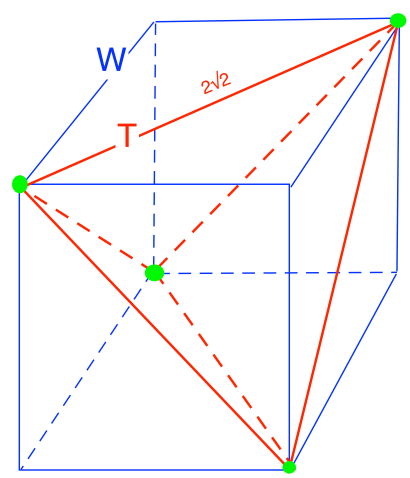


Figure 7.1: Inscribing the tetrahedron in the cube.

Evidently every symmetry of the tetrahedron carries the cube to itself.[2] It is therefore also a symmetry of the cube. So

$$\mathrm{Sym}(T) \subseteq \mathrm{Sym}(W) \cong \mathrm{Sym}(O). \tag{7.1}$$

Now $\#\mathrm{Sym}(T) = 24$, whereas $\#\mathrm{Sym}(W) = 48$. So exactly half of the symmetries of the cube preserve the tetrahedron. The other half of the symmetries

[2] One way to see it is to list each symmetry of the tetrahedron and check that it preserves the cube. It's a little easier to start listing the symmetries of the cube until you find all the symmetries of the tetrahedron.

of W take T to an "opposite" tetrahedron $T' = -T$ (where $-$ represents the inversion through the origin).

Figure 7.2: The tetrahedron and its "opposite". (Watchduck, Wikipedia)

For the dodecahedron we have a similar but even more interesting construction. We can place a tetrahedron inside a dodecahedron.

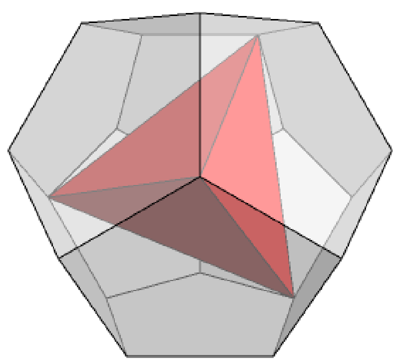


Figure 7.3: The tetrahedron in the dodecahedron. (G. Egan)

Question: Can we use this tetrahedron to show that the tetrahedral symmetries are a subset of the dodecahedral symmetries?

Apparently not, because the mirror reflections of the tetrahedron don't preserve the dodecahedron.[3] Yet it seems that at least some of the tetrahedral symmetries preserve the dodecahedron.

We can go further with inscribed tetrahedra. The dodecahedron has 20 vertices. They can be grouped into five groups of 4 vertices each in a very symmetrical

[3] This corrects an earlier version of the notes, where I said the opposite.

way. Each group of 4 is the vertex set of a regular tetrahedron. As a result, we can inscribe *five* tetrahedra in the dodecahedron.

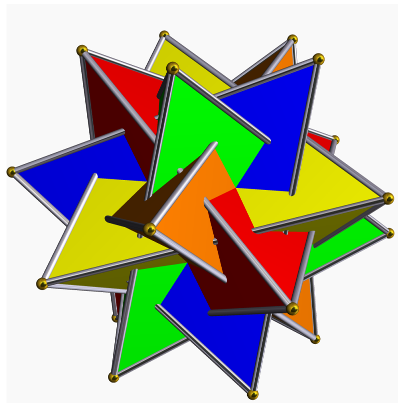


Figure 7.4: Five tetrahedra inscribed in the dodecahedron. (R. Webb, Wikipedia)

There are actually five more "opposite" tetrahedra in the icosahedron. Can you find them?

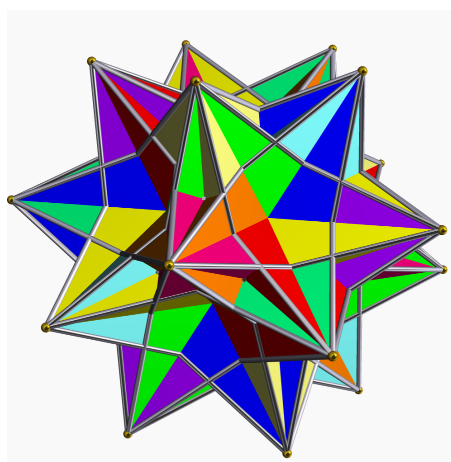


Figure 7.5: Compound of ten tetrahedra. (R. Webb, Wikipedia)

Indeed, the tetrahedra match each other in five pairs of "opposite" tetrahedra, related by the antipodal map (the inversion through the origin).

Each pair of tetrahedra defines a cube. So there are five cubes inscribed in the dodecahedron. Note that each vertex is used twice.

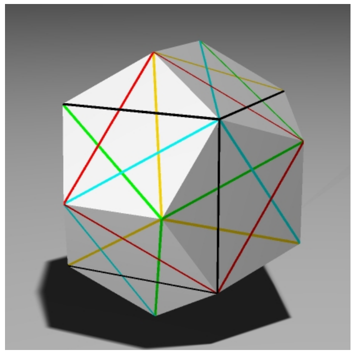


Figure 7.6: Five cubes in the dodecahedron. (S. Tatham)

Again, placing a cube in the dodecahedron is deceptive: the cube symmetries are not a subset of the dodecahedral symmetries.

Exercise 7.1 *What can we do instead?*

We will answer this in Section 37.

8 Classification of Platonic solids - proof

Polyhedra. Proof of the classification of regular polyhedra.

References

- H. S. M. Coxeter, *Regular polytopes,* 1973, pp 5-6, 15-16 (quick proof of the classification, proof of existence, vertex figures).
- I. Lakatos, *Proofs and refutations,* 1976. (deeper reading)
- B. Johnson, *Classifying regular polytopes in dimension 4 and beyond*, `https://digitalcommons.wou.edu/cgi/viewcontent.cgi?article=1116&context=aes` (contains a proof of the classification).
- *Regular polyhedron*, Wikipedia (glance at).
- *Vertex figure*, Wikipedia (glance at).

A *polyhedron* is a bounded, connected body in space that is bounded by a finite number of polygons, called *faces*. The faces meet along *edges*, and the edges meet at *vertices*. The *degree* of a vertex is the number of edges meeting at the vertex.

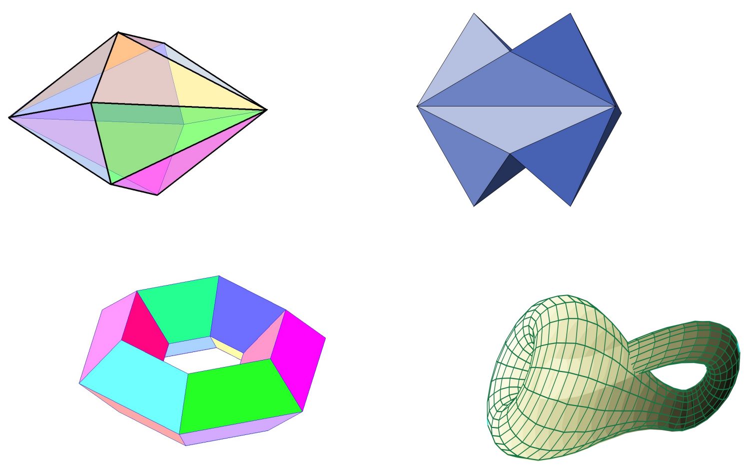


Figure 8.1: Convex, non-convex, non-simply-connected, self-intersecting. (Quatrostein, D. Eppstein, T. Ruen, Tttrung; Wikipedia)

We require that the faces meet only along whole edges, and that the surface of the polyhedron is not self-intersecting. This implies in particular that the surface of the polyhedron bounds a solid body, rather than merely being a surface hanging in space (like the Klein bottle).

We won't require convexity for polyhedra (but for *regular* polyhedra, it will follow from the definition of "regular"). The figures show that the precise definition of a polyhedron is rather subtle. For a philosophical discussion of the issues involved in refining our definitions, see Lakatos, *Proofs and Refutations*.

Definition A *regular polyhedron* is a polyhedron H in $\mathbb{R}^3$ such that the symmetries take[4]

- Every vertex to every other vertex,
- Every edge to every other edge,
- Every face to every other face,

and such that

- Every face is a regular polygon.

As a consequence of the definition,

- The vertices of H all have the same degree,
- The edges of H all have the same length and dihedral angle,
- The faces of H all have the same number of sides.

Indeed, the faces are identical regular polygons. In particular

- All corner angles are the same.

It follows:

- Every "vertex figure" is regular.

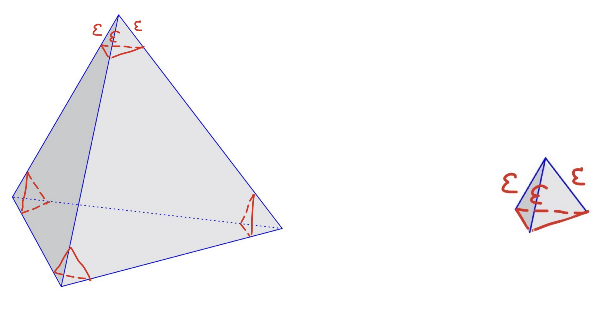


Figure 8.2: Four identical vertex figures of the tetrahedron; the top vertex figure.

Indeed, the vertex figures are identical and regular.

Explanation:

[4]We have taken the transitivity conditions from Brieskorn I, p 9 and Knörrer, p 60, but have added the requirement that every face be a regular polygon in order to simplify the proof.

By definition, a *vertex figure* is the shape of the polyhedron in the vicinity of a vertex. Specifically, it is the figure defined by the vertex, plus the points at distance ε along each edge that meets the vertex, where ε is smaller than the minimum edge length.[5]

A vertex figure is *regular* if the corner angles around the vertex are all the same, and the dihedral angles around the vertex are all the same. This is the same as the vertex figure being an upright pyramid over a regular polygon.

The vertex figures of a regular polyhedron are therefore, as claimed, regular and identical.

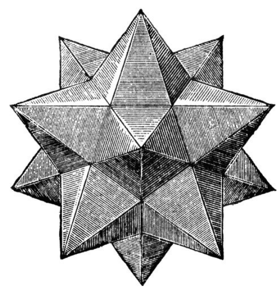


Figure 8.3: Why isn't the small stellated dodecahedron a regular polyhedron, according to our definition? (William & Robert Chambers Encyclopaedia, 1881)

Let S be a regular polyhedron. Let k be the number of sides of a typical face P of S. Let l be the degree of a typical vertex. The pair

$$(k, l)$$

is called the *Schläfli symbol* of S. The five Platonic solids above correspond to the Schläfli symbols

$$(3,3), \qquad (3,4), \qquad (3,5), \qquad (4,3), \qquad (5,3).$$

Theorem 8.1 *(Theaetetus) The regular polyhedra in $\mathbb{R}^3$ are precisely the Platonic solids:*

tetrahedron, octohedron, icosahedron, cube, dodecahedron.

The proof first appears in Euclid's Elements, Book XIII. See also H. S. M. Coxeter, pp 5-6, and B. Johnson, *Classifying regular polytopes in dimension 4 and beyond*.

Besides the geometric proof given below, there is a topological proof just using Euler's formula $\#faces - \#edges + \#vertices = 2$.

[5] There are multiple definitions in the literature. See Coxeter, pp 15-16, and *Vertex figure*, Wikipedia, for alternate definitions.

Proof There are three steps in the proof.

1) Existence of the Platonic solids. We omit this.[6]

2) Uniqueness of the Platonic solids: they are the only regular figures with the given Schläfli symbols.

3) No other Schläfli symbols are possible.

For 2):

Let S be a regular polyhedron with Schläfli symbol (k, l). Then the pair (k, l) determines the geometry of S uniquely, as follows:

It tells us that we must fit l regular k-gons around each vertex. Because the vertex figures are all the same, there is only one way to do this. So (k, l) determines the geometry around each vertex, and it determines the locations of the next neighboring vertices. By continuing the construction all the way around the polyhedron, the polyhedron S is determined uniquely. That is, there is at most one polyhedron with Schläfli symbol (k, l).

So for each (k, l) in the above list, the corresponding regular polyhedron S is one of the five classical bodies.

For 3):

It remains to show that no other (k, l) pairs are possible. This is the most interesting step.

Clearly $k \geq 3$, $l \geq 3$.

If $k = 6$, then P is a regular hexagon, with corner angle $120°$. The only way that several copies of P can fit around a vertex to form an embedded polyhedron with equal dihedral angles is to have three hexagons meet at a vertex ($l = 3$). But then S is flat near each vertex, and the pattern extends to form a hexagonal tesselation of the plane. This is not permitted. So $k = 6$ is excluded.

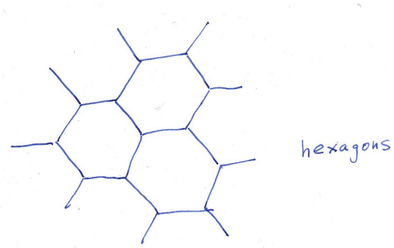


Figure 8.4: Hexagonal tiling.

Similar (but worse) objections apply to $k \geq 7$. Indeed, the corner angle of P

[6] Coxeter does it by constructing them geometrically. This is nontrivial only for the icosahedron and dodecahedron. See Exercise 53.5 for another method.

now exceeds 120, so the sum of the corner angles is more than 360, so the surface must wiggle back and forth like a crumpled thing near a vertex. In particular, some dihedral angles are more than 180 (they are indented!) and some are less. In particular, they are not all equal. This is not allowed. So $k \geq 7$ is excluded.

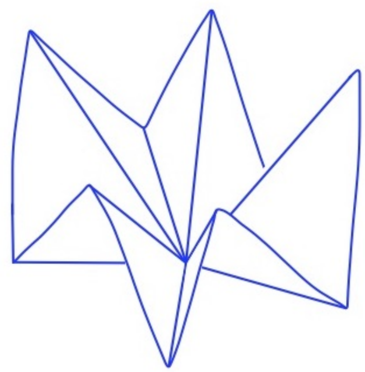


Figure 8.5: The angles about a vertex add up to more than 360 degrees, so the surface crinkles up.

Let's view the app by T. Hutton at

- `http://timhutton.github.io/hyperplay`.

Under the assumption that $k = 3$ (triangles), it starts with a plane tiled by triangles ($l = 6$). You can vary the dihedral angle with the mouse to flash through the icosahedron ($l = 5$) to an octahedron ($l = 4$) to a tetrahedron ($l = 3$). Between these, you see "non-closed figures".

On the other side, where the total angle around the vertex would be greater than 360 if you try to keep it in Euclidean space, it switches to the hyperbolic plane, and the tiled flat plane becomes a tiled hyperbolic plane.

Back to the proof. We get

$$k = 3, 4, \text{ or } 5.$$

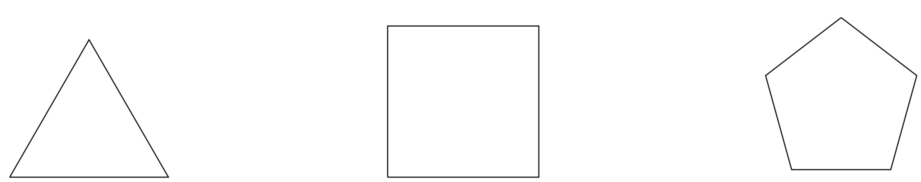


Figure 8.6: Only these faces are possible.

Let's continue to apply the "fitting around a vertex" principle. Let α be the corner angle of P. We have

$$\begin{aligned} k &= 3, \qquad \alpha = 60^\circ, \\ k &= 4, \qquad \alpha = 90^\circ, \\ k &= 5, \qquad \alpha = 108^\circ. \end{aligned}$$

In order to fit around a vertex, the sum of the corner angles of the polygons that meet at the vertex must be less than 360°. So α must satisfy

$$l\,\alpha < 360^\circ.$$

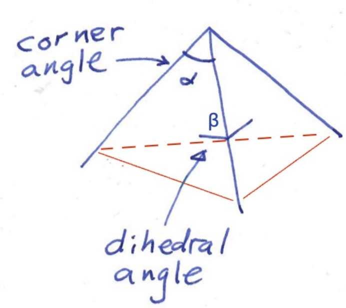


Figure 8.7: Corner angle and dihedral angle.

By inspection, this is only possible for the following combinations:

$$\begin{aligned} k &= 3, \qquad l = 3, 4, 5, \\ k &= 4, \qquad l = 3, \\ k &= 5, \qquad l = 5, \end{aligned}$$

which is exactly the given list. So the given list of (k, l) values is exhaustive, and we are done.

□

Exercise 8.1 *Besides the geometric proof given above, there is a topological proof just using Euler's formula*

$$F - E + V = 2$$

for simply-connected polyhedra. Find this proof.

9 Regular figures in higher dimensions

Polytopes, regular polytopes, hypercube, n-simplex, n-cube, n-orthoplex, 24-cell, 120-cell, 600-cell.

References

- J. Baez, *Platonic Solids in All Dimensions*, `http://math.ucr.edu/home/baez/platonic.html`.
- J. Baez, *The 600-Cell (Part 1)*, `https://johncarlosbaez.wordpress.com/2017/12/16/the-600-cell`.
- B. Johnson, ibid.
- Wikipedia articles: *Regular polytope*, *Tesseract* (hypercube), *16-cell*, *24-cell*, *120-cell*, *600-cell*.
- H. S. M. Coxeter, *Regular Polytopes,* 1973.

The situation in higher dimensions is quite remarkable.

The generalization of a polyhedron to n dimensions is called a *polytope*.

A k-dimensional polytope has a finite number of faces which are $(k-1)$-dimensional polytopes. With some care, this gives a recursive definition of a polytope in all dimensions.

A regular k-dimensional polytope S is one whose faces are regular $(k-1)$-dimensional polytopes, and whose symmetry group $\mathrm{Sym}(S)$ satisfies the following.

(1) Every face of S can be taken to every other face of S.

(2) Every symmetry of a face of S extends to a symmetry of S.

This definition is more elegant than the one we gave in three dimensions (which looks rather ad hoc by comparison), but it's equivalent.

For more information see H. S. M. Coxeter, *Regular Polytopes*, 1973. He defines a regular polytope a bit differently than we do (see Coxeter, pp 128-130), but the definitions are equivalent.

What are the regular polytopes in each dimension?

Here are some obvious ones.

(A) The regular n-simplex generalizes the triangle and the tetrahedron. ($n+1$ vertices)

(B) The n-cube (also known as the n-dimensional hypercube) generalizes the square and the cube. (2^n vertices)

(C) The n-orthoplex (otherwise known as the n-dimensional cross-polytope or hyperoctahedron) generalizes the square and the octahedron. ($2n$ vertices)

That gives at least three regular polytopes in each dimension.

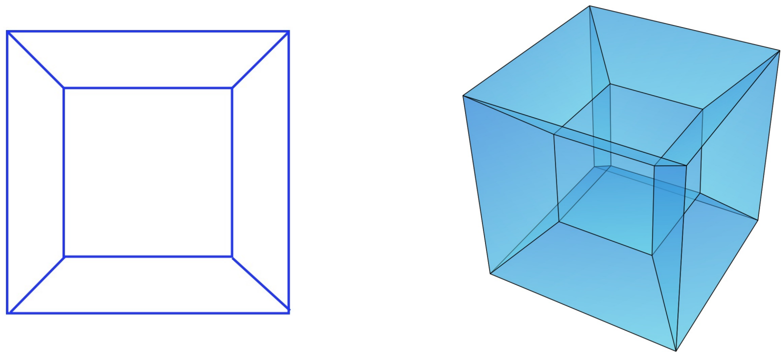


Figure 9.1: The cube and the hypercube, projected onto the plane. (Hypercube by Goffrie, Wikipedia).

In dimension four, we can build three additional regular polytopes with the following descriptions.

(i) 24-cell: 24 faces, 96 2-faces, 96 edges, 24 vertices. Each face is an octahedron.

(ii) 120-cell: 120 faces, 720 2-faces, 1200 edges, 600 vertices. Each face is a dodecahedron.

(iii) 600-cell: 600 faces, 1200 2-faces, 720 edges, 120 vertices. Each face is a tetrahedron.

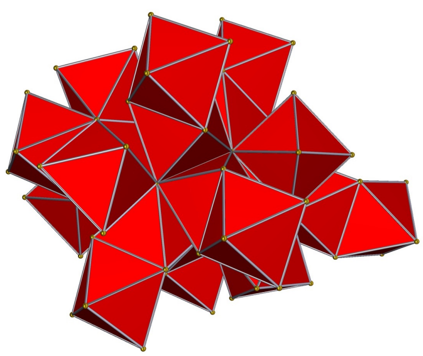


Figure 9.2: The 3-dimensional surface of the 24-cell, cut along the 2-dimensional faces (triangles) and flattened onto $\mathbb{R}^3$. (R. Webb, Wikipedia)

Here is the remarkable theorem.

Theorem 9.1

- *In $\mathbb{R}^4$, the regular polytopes are*

  4-simplex, 4-cube, 4-orthoplex, 24-cell, 120-cell, 600-cell.

- *In $\mathbb{R}^n$, $n \geq 5$, the regular polytopes are*

  n-simplex, n-cube, n-orthoplex.

Remarkably, exotic regular polytopes stop existing in dimensions higher than four!

Exercise 9.1 *How many faces does each of the following n-dimensional regular polytopes have?*

(a) n-simplex,

(b) n-cube,

(c) n-orthoplex.

Exercise 9.2

(a) Compute the number of symmetries of an n-cube.

(b) List the symmetries of the n-cube in matrix form.

Chapter 3

—

10 The bare minimum for orientation

Orientation-preserving and orientation-reversing isometries. Composition rule.

References

- K. Jänich, *Lineare Algebra*, Springer, 11th edition, 2010, pp 70-73, 157.
- G. Fischer, *Lineare Algebra: Eine Einführung für Studienanfänger*, 18th edition, Springer, 2014, pp 212-221.

An isometry ϕ of $\mathbb{R}^n$ is *orientation preserving* (OE) or *proper* if it takes right hands to right hands and left hands to left hands. It is *orientation reversing* (OU) or *improper* if it takes right hands to LEFT hands and left hands to RIGHT hands.

Effectively, an orientation-reversing isometry applies a mirror to everything.

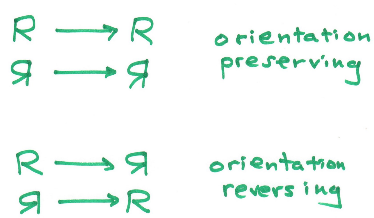


Figure 10.1: Orientation-preserving and orientation-reversing in $\mathbb{R}^2$.

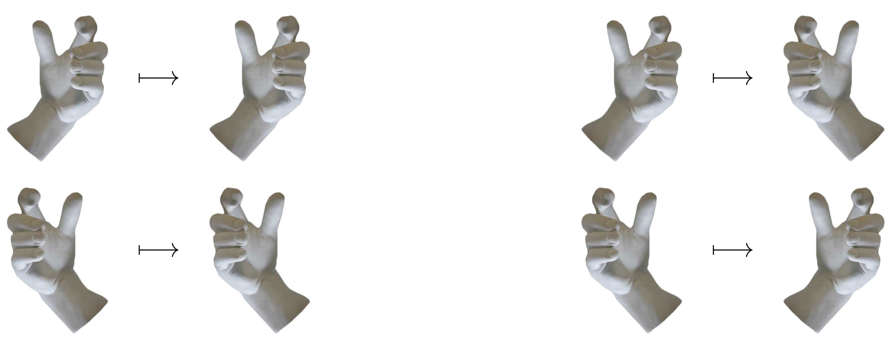


Figure 10.2: Orientation-preserving and orientation-reversing in $\mathbb{R}^3$. (www.houzz.com)

Example

- Rotations are orientation-preserving.
- A reflection of $\mathbb{R}^3$ in a plane is orientation-reversing.
- A reflection of $\mathbb{R}^3$ in a line is orientation-preserving. (It is the same as a 180-degree rotation about the line.)
- An inversion of $\mathbb{R}^3$ in a point is orientation-reversing.

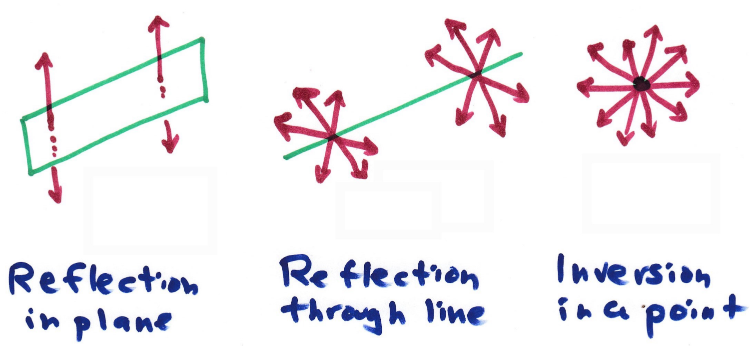


Figure 10.3: Reflection, reflection and inversion in $\mathbb{R}^3$.

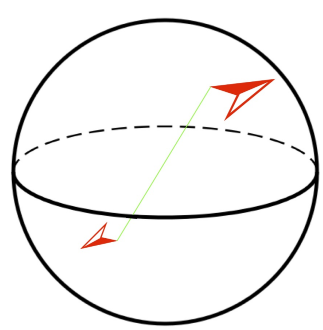


Figure 10.4: An inversion is orientation-reversing. (mathblog.com, modified)

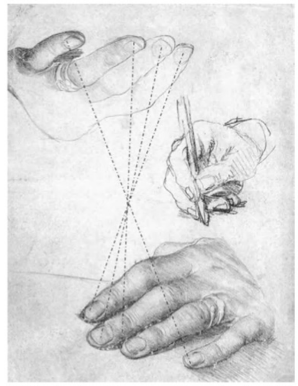


Figure 1.11 The hands of Erasmus of Rotterdam by Holbein, 1523.

Figure 10.5: What is wrong with this picture? (Burns-Glazer, p 15)

We have the following rules:

$$OE \circ OE = OE \tag{10.1}$$
$$OE \circ OU = OU \tag{10.2}$$
$$OU \circ OE = OU \tag{10.3}$$
$$OU \circ OU = OE \tag{10.4}$$

It's a matter of parity, or lex talionis – two wrongs DO make a right.

We can summarize this in a table:

$\circ$	E	U
E	E	U
U	U	E

We have written "E" for "OE" and "U" for "OU" in order to make the table easier to read.

Orientation-preserving isometries are also called "proper motions". Orientation-reversing isometries are often called "improper motions". Antique language.

We write
$$\mathrm{Isom}_+(\mathbb{R}^n)$$
for the set of orientation-preserving isometries of $\mathbb{R}^n$. More generally, if S is a set of isometries of $\mathbb{R}^n$, we write S_+ for the orientation preserving subset:

$$S_+ := \{\phi \in S \,\big|\, S \text{ is orientation-preserving}\} = S \cap \mathrm{Isom}_+(\mathbb{R}^n).$$

Not every map can be classified as either orientation-preserving or orientation-reversing. For example, the map that crushes everything down to a point is of ambiguous orientation. Similarly, linear maps of determinant zero, topological maps that have folds, and discontinuous functions, do not have a well-defined effect on orientation.

All in all, the existence of intermediate or ambiguous states adds to the diversity and interest of the landscape, but does not prevent us from affirming the essential distinction.

11 Fixed-point sets

Fixed points, fixed point sets, affine subspaces, fixed points of isometries.

References

- Brieskorn III, p 15, Prop. 1.8(i).

The *fixed-point set* of a map ϕ is the set

$$\mathrm{Fix}(\phi) := \{x \in \mathbb{R}^n : \phi(x) = x\}.$$

Its elements are called *fixed points*.

Figure 11.1: Typical fixed point.

Example Rotations and reflections have fixed points, translations do not.

Proposition 11.1 *The fixed point set of an isometry of $\mathbb{R}^2$ or $\mathbb{R}^3$ is empty, a point, a line, a plane, or all of space.*

Points, lines, and planes (not necessarily passing through the origin), and all of space are examples of *affine subspaces*.

The following lemma is useful.

Lemma 11.2 *If $p \neq q$ are fixed by ϕ, then every point on the line L determined by p and q is fixed by ϕ.*

Proof *Claim:* If $r \in L$, then r is uniquely characterized (in $\mathbb{R}^n$) by the two distances

$$d(r,p) \quad \text{and} \quad d(r,q).$$

Proof of Claim: Indeed, L is precisely the set of points r where the triangle inequality for

$$p, q, r$$

becomes an equality (in one of three ways). Then if $r \in L$, the position of r along L is determined by knowing the two numbers $d(r, p)$ and $d(r, q)$. This proves the Claim.

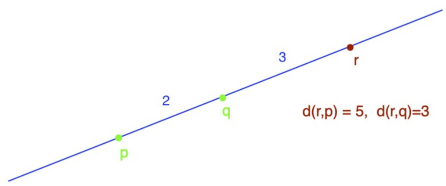


Figure 11.2: Two points p, q determine the position of each point r on the line through them.

But these two distances are preserved by ϕ, and p, q are fixed by ϕ. Therefore r is fixed by ϕ.

□

Proof of the Proposition Consider cases.

Figure 11.3: Empty or one point.

If ϕ has exactly 0 or 1 fixed points, we are done.

Figure 11.4: Two distinct points.

If ϕ has two distinct fixed points p_1, p_2, then by the Lemma, ϕ fixes every point on the line L through p_1, p_2. If that is all, we are done.

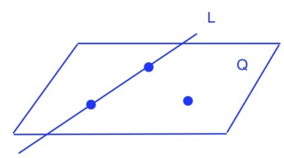


Figure 11.5: Three independent points.

Otherwise ϕ has a fixed point p_3 not on L. Then applying the Lemma repeatedly, ϕ fixes every point on the plane Q determined by p_1, p_2, p_3. If that is all, we are done.

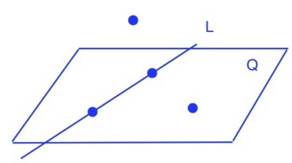


Figure 11.6: Four independent points.

Otherwise ϕ has a fixed point p_4 not on Q. Then applying the Lemma repeatedly, ϕ fixes every point in space, and we are done.

□

12 Naming rigid motions

Rotations, reflections, inversions, improper rotations, roto-reflections, roto-inversions, translations, glide reflections, screw motions.

References

- *Euclidean group*, Wikipedia
- Senechal, 23-27.
- Burns-Glazer, 8-20,93-97,97-100 (don't pay too much attention to the notation)

An isometry of $\mathbb{R}^n$ is often called a *rigid motion* or *motion*.

We will classify rigid motions in two broad ways:

- Orientation-preserving versus orientation-reversing.
- Fixed-point set.

Here are two charts:

$\mathbb{R}^2$	fixed points		fixed-point free
OE	point: $\mathbb{R}^2$:	rotation inversion identity	translation
OU	line:	line reflection	glide reflection

$\mathbb{R}^3$	fixed points		fixed-point free
OE	axis: $\mathbb{R}^3$:	rotation line reflection identity	screw motion glide line-reflection translation
OU	point: plane:	roto-reflection (= roto inversion) inversion plane reflection	glide reflection

The notation below is not standard, it's just for the course.

Rotations

Data in $\mathbb{R}^2$*:* Point $p \in \mathbb{R}^2$, angle $\theta \in \mathbb{R}$.

Then

$$R_\theta(p)$$

is the counterclockwise rotation about p by the angle θ. We have $R_\theta(p) = R_{\theta+360}(p)$. A special case is the identity $R_0(p) = \mathrm{id}_{\mathbb{R}^2}$ for any p.

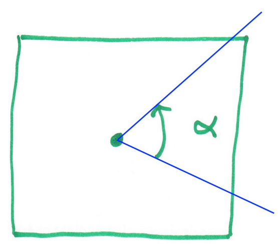


Figure 12.1: Clockwise rotation by the angle α in $\mathbb{R}^2$.

Data in $\mathbb{R}^3$*:* Axis A in $\mathbb{R}^3$, angle $\theta \in \mathbb{R}$.

Then

$$R_\theta(A)$$

is the rotation about the axis A by the angle θ.

An axis, by definition, is a *directed line*, meaning a line with an arrow on it. We use the *right-hand rule* to determine the direction of rotation. You will need a 200 franc note for this exercise. A right hand is depicted. The vertical thumb is the directed line. The direction of rotation about the thumb is given by the curling fingers.

Figure 12.2: Right-hand rule. (LeftoverCurrency.com)

Again, the identity shows up as the trivial rotation $R_0(A) = \mathrm{id}_{\mathbb{R}^3}$ for any A.

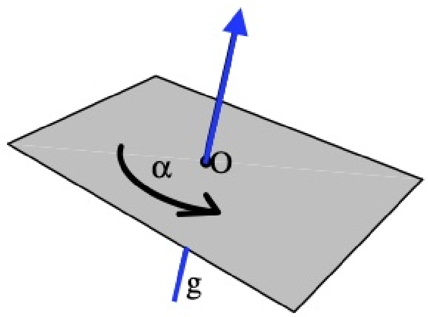


Figure 12.3: Rotation by the angle α about an axis g in $\mathbb{R}^3$. (Knörrer, modified)

Rotations are orientation-preserving. The fixed-point set is the rotation point or axis:

$$\operatorname*{Fix}_{\theta}(p) = \{p\}, \quad \operatorname*{Fix}_{\theta}(A) = A.$$

A special case of a rotation in $\mathbb{R}^3$ is rotation by 180 degrees, or equivalently, reflection through a line.

Data: A line L in $\mathbb{R}^3$.

It is

$$R_{180}(L) = \sigma_L.$$

Reflections and inversions

Data in $\mathbb{R}^2$*:* Point p or line L in $\mathbb{R}^2$.

When we reflect in a point p it is usually called an inversion. Inverting through a point p in the plane is the same as a 180-degree rotation about p.

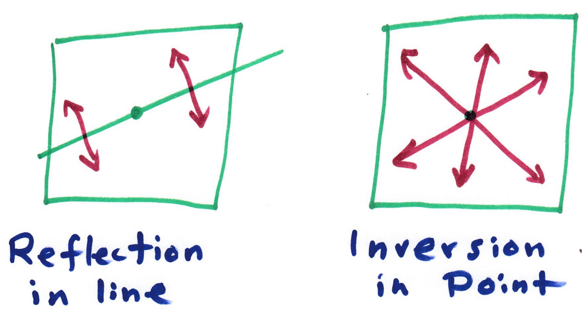


Figure 12.4: Reflection and inversion in $\mathbb{R}^2$.

Data in $\mathbb{R}^3$*:* Point p, line L, or plane E in $\mathbb{R}^3$.

When we reflect through a point it is usually called an inversion.

Reflecting through a line L in space is the same as rotating by 180 degrees about L.

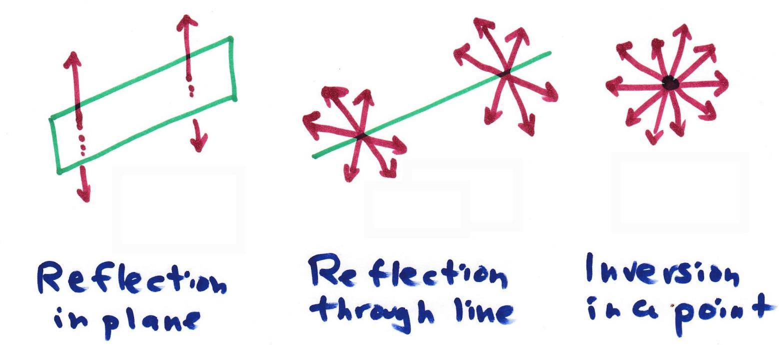


Figure 12.5: Reflection and inversion in $\mathbb{R}^3$.

We write

$$\sigma_p = Z_p, \quad \sigma_L, \quad \sigma_E$$

for reflections. In all cases, the fixed point set is the set we reflect through:

$$\mathrm{Fix}(\sigma_p) = \{p\}, \quad \mathrm{Fix}(\sigma_L) = L, \quad \mathrm{Fix}(\sigma_E) = E.$$

When is a reflection orientation-reversing?

You can verify for yourself:

- Reflecting in a line in $\mathbb{R}^2$ is orientation-reversing.
- Inverting in a point in $\mathbb{R}^2$ is orientation-*preserving*.

- Reflecting in a plane in $\mathbb{R}^3$ is orientation-reversing.
- Reflecting in a line in $\mathbb{R}^3$ is orientation-*preserving*.
- Inverting in a point in $\mathbb{R}^3$ is orientation-reversing.

Exercise 12.1 *(Linear algebra formula) Let E be a hyperplane through the origin in $\mathbb{R}^n$. Let e be a unit normal to E. Let σ_E be reflection across E. Verify the linear algebra formula*

$$\sigma_E(x) = x - 2\langle x, e\rangle e, \qquad x \in \mathbb{R}^n,$$

where $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ is the usual inner product on $\mathbb{R}^n$.

Roto-reflections and roto-inversions

So far, all of our motions have had a fixed point. Are there any more isometries with this property?

In $\mathbb{R}^3$, indeed there are.

Data: Axis A and plane E in $\mathbb{R}^3$, angle $\theta \in \mathbb{R}$, and we require

$$A \perp E.$$

Then

$$M_\theta(A, E) := \sigma_E \circ R_\theta(A) = R_\theta(A) \circ \sigma_E$$

is called a *roto-reflection*. It is rotation about A by angle θ, followed by reflection across E. Or the other way around. Note that the order of composition doesn't matter.

A roto-reflection is orientation-reversing. p is the only fixed point.

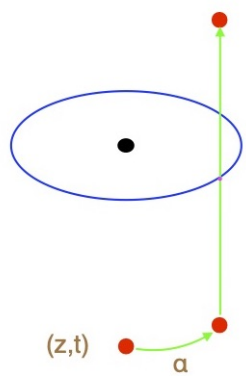


Figure 12.6: Roto-reflection.

One way to visualize it: decompose $\mathbb{R}^3$ as an orthogonal direct sum $\mathbb{R}^3 = A \oplus E$. Let $p = A \cap E$. Then the action splits into two parts, which operate independently from each other: it flips A about p and it rotates E about p.

Next consider

Data: Axis A and point p in $\mathbb{R}^3$, angle $\theta \in \mathbb{R}$, and we require

$$p \in A.$$

Then

$$N_\theta(A, p) := Z_p \circ R_\theta(A) = R_\theta(A) \circ Z_p$$

is called a *roto-inversion*. It is rotation about A by θ, followed by inversion in p. The order of composition doesn't matter, because Z_p commutes with every isometry that fixes p.

Roto-inversions are orientation-reversing. p is the only fixed point.

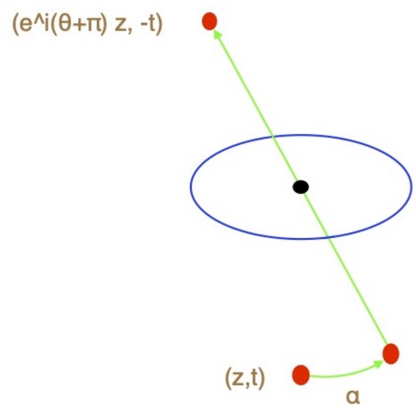


Figure 12.7: Roto-inversion.

Every roto-reflection is a roto-inversion, and vice-versa, as follows. If we write $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$, $p = 0$, then a roto-reflection M_θ takes

$$(z, t) \mapsto (e^{i\theta} z, -t)$$

whereas a roto-inversion N_θ takes

$$(z, t) \mapsto -(e^{i\theta z}, t) = (e^{i(\theta+\pi)} z, -t).$$

So

$$M_\theta(A, E) = N_{\theta+\pi}(A, p),$$

where E and p determine each other by $p = A \cap E$.

So: although the individual maps are notated differently, the set of all roto-reflections is the same as the set of all roto-inversions. Roto-reflections and roto-inversions are also called *improper rotations*.

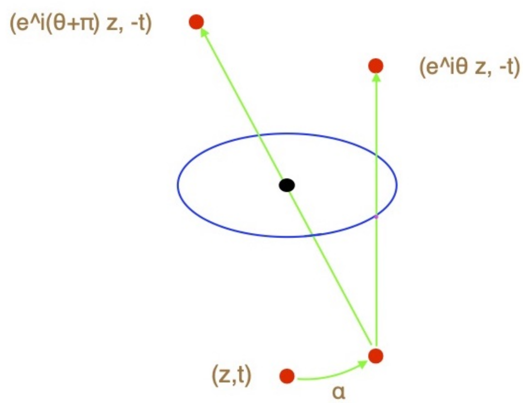


Figure 12.8: Roto-reflection versus roto-inversion.

A roto-reflection or roto-inversion preserves the following figure (an antipyramid marked with arrows indicating the direction of motion):

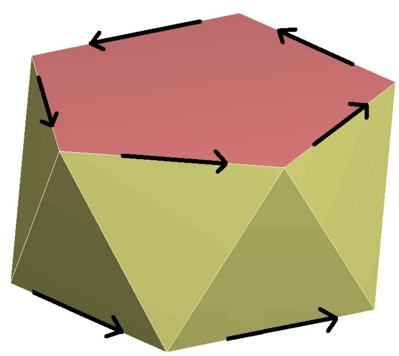


Figure 12.9: Marked antiprism. (Cyp, Wikipedia)

Exercise 12.2 *What is the order of a roto-reflection of angle $360/n$?*

Tricky.

Translations

So much for isometries with fixed points. We now turn to motions that are fixed-point free. The most obvious ones are translations.

Data: Vector a in $\mathbb{R}^2$ or $\mathbb{R}^3$

Let

$$T_a$$

denote the isometry that moves every point by the vector a. T_a is called *translation by a*.

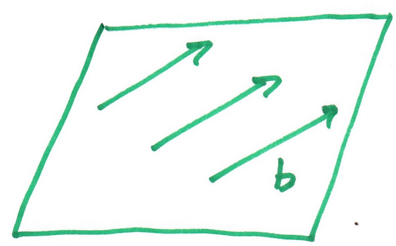


Figure 12.10: Translation by b.

Evidently $T_0 = \text{id}$. We have

$$T_a \circ T_b = T_{a+b}.$$

Indeed, all translations commute.

T_a has no fixed points unless $a = 0$. Translations are orientation-preserving.

Screw motions

Screw motions exist in $\mathbb{R}^3$.

Data: Axis A and vector t in $\mathbb{R}^3$, angle $\theta \in \mathbb{R}$, where we require that t is parallel to A:

$$t\,||A.$$

Then a *screw motion* is

$$S_\theta(A,t) = R_\theta \circ T_t = T_t \circ R_\theta,$$

namely translation along A by t, followed by rotation about A by θ. The motion leaves the axis invariant. It acts as translation along the axis.

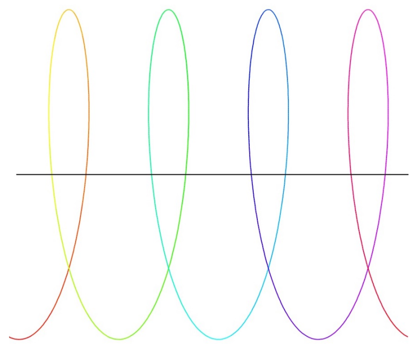


Figure 12.11: Effect of a screw displacement. (Zerodamage, Wikipedia)

A screw motion preserves a figure like the following:

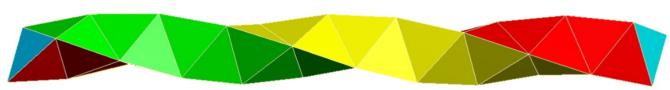


Figure 12.12: A Boerdijk-Coxeter helix. (T. Ruen, Wikipedia)

A screw motion is orientation-preserving. It has no fixed points (provided $t \neq 0$).

Note that translation is a special case of screw motion (with $t = 0$). So is rotation (with $\theta = 0$).

Another special case of a screw motion, worth noting separately, is what I call a "glide line-reflection". It has

Data: Line L and vector t in $\mathbb{R}^3$, where t is parallel to L.

It is

$$S_{180}(L, t) = \sigma_L \circ T_t = T_t \circ \sigma_L,$$

that is, translation along L by the vector t, followed by a reflection through L.

Glide-reflections

Less obvious are the glide-reflections. These occur in both $\mathbb{R}^2$ and $\mathbb{R}^3$.

Data in $\mathbb{R}^2$: A line L and a vector t in $\mathbb{R}^2$, where we require that t is parallel to L:

$$t \,||\, L.$$

Then define

$$G_t(L) = \sigma_L \circ T_t = T_t \circ \sigma_L.$$

A glide-reflection in $\mathbb{R}^2$ is translation by t, followed by reflection across L. The order of application does not matter. A glide reflection in $\mathbb{R}^2$ preserves the line L. It acts as a translation on L.

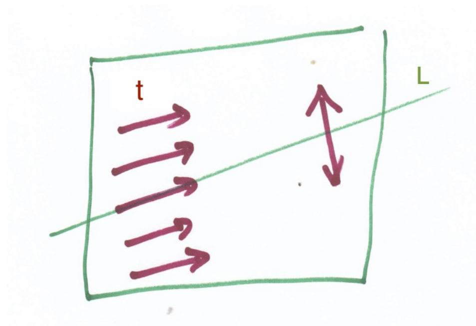


Figure 12.13: Glide reflection in $\mathbb{R}^2$.

It has the following effect on the letter R.

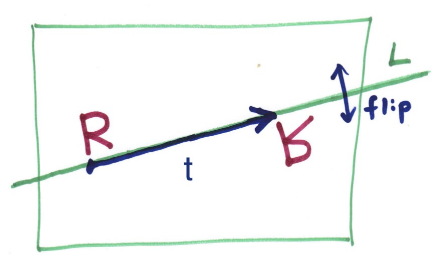


Figure 12.14: Effect of glide reflection in $\mathbb{R}^2$.

Data in $\mathbb{R}^3$*:* A plane E and a vector t in $\mathbb{R}^3$, where we require that t is parallel to E:

$$t \,||E.$$

Then define

$$G_t(E) = \sigma_E \circ T_t = T_t \circ \sigma_E.$$

A glide-reflection in $\mathbb{R}^3$ is translation by t, followed by reflection across E. The order of application does not matter. A glide reflection in $\mathbb{R}^3$ preserves the plane E. It acts as a translation on E.

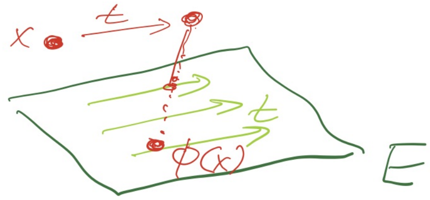


Figure 12.15: Glide-reflection in $\mathbb{R}^3$.

Glide-reflections are orientation-reversing. Glide-reflections have no fixed points ($t \neq 0$).

That's all.

Summary

Summarizing:

Theorem 12.1 *In $\mathbb{R}^2$, we have 4 kinds of rigid motion:*

- *Rotations (including the identity and inversion through a point)*
- *Reflections*
- *Translations*
- *Glide-reflections*

We will prove the fixed-point case of Theorem 12.1 as Theorem 14.1. See also Knörrer p 5 Satz 1.1. We will prove the fixed-point free, orientation-preserving case of this Theorem in Section **??**. We leave the non-orientable, fixed-point free case to the reader.

Theorem 12.2 *In $\mathbb{R}^3$, we have 4 kinds of rigid motion:*

With fixed point, orientation-preserving:

- *Rotations (including the identity and reflections through lines)*

With fixed point, orientation-reversing:

- *Roto-reflections aka roto-inversions (including inversions in points and reflections in planes)*

Fixed-point-free, orientation-preserving:

- *Screw motions (including translations and glide line-reflections)*

Fixed-point-free, orientation-reversing:

- *Glide reflection*

We will prove the fixed-point case of Theorem 12.2 as Theorem 14.2. See also Knörrer pp 13-15, Satz 1.2. We will prove the fixed-point free, orientation-preserving case of Theorem 12.2 in Section **??**. See also Knörrer, p 15, Remark 4. We leave the non-orientable, fixed-point free case to the reader.

Lattice of isometry types

Here are the subset relations between the 6 different types of isometry of $\mathbb{R}^2$. For example, you can consider the identity as being a special case of a rotation, a plane reflection as being a special case of a glide-reflection, etc.

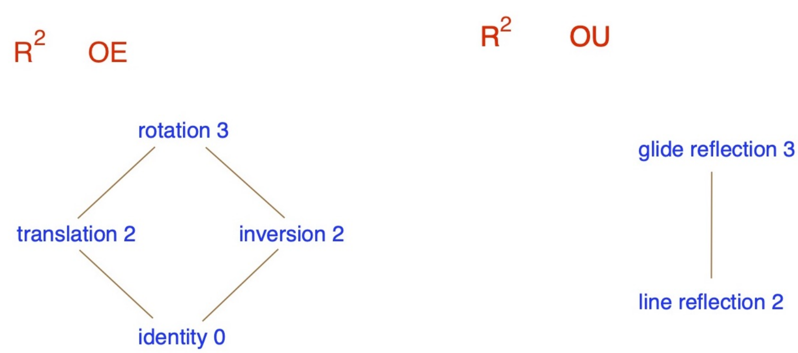


Figure 12.16: Subset relations in $\mathbb{R}^2$.

The numbers are the number of degrees of freedom (number of real variables) required to specify each type of isometry. For example, it takes two numbers to specify a line reflection in $\mathbb{R}^2$ – an angle for the direction of a unit normal e, and an offset to describe the displacement of the line parallel to itself from a given position (in the direction of e).

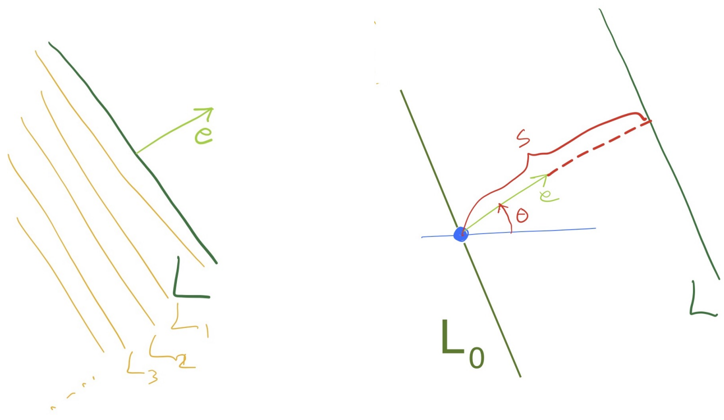


Figure 12.17: Specifying a line in $\mathbb{R}^2$ with two parameters.

Here are the subset relations between the 10 different types of isometry of $\mathbb{R}^2$.

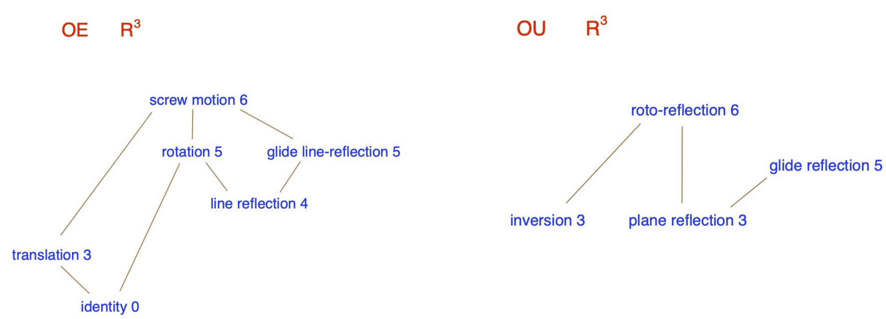


Figure 12.18: Subset relations in $\mathbb{R}^3$.

See Section 48 for more information.

13 Hyperbolic space

It's bigger than Texas.

For comic relief, let's fly around in hyperbolic space.

- Flying in curved space (iOS, macOS, Windows):
  `http://www.geometrygames.org/CurvedSpaces`

Hyperbolic space is an infinite space. It forms one of a triumvirate:

- Sphere S^n curvature = 1 (compact)
- Euclidean space $\mathbb{R}^n$ curvature = 0 (noncompact, flat)
- Hyperbolic space H^n curvature = -1 (noncompact, huge)

Like the sphere and Euclidean space, H^n has a lot of isometries. The classification of its isometries is similar to that of Euclidean space – there are rotations and "translations" – but hyperbolic space is much larger in some sense and the "translations" are more splayed out – they spread apart faster and don't commute. The volume of H^n grows exponentially with radius.

Also like Euclidean space (and the sphere), you can tesselate hyperbolic space with identical pieces like in a crystal. That is what we're seeing in the simulation above – a tiling by regular dodecahedra (depending on what I decided to show). This would be impossible in Euclidean space. In contrast to Euclidean space, hyperbolic space has infinitely many regular tilings (and infinitely many crystal structures).

Another way to get a sense of how very *big* hyperbolic space is is to play the game HyperRogue (Windows, Linux, Android, iOS, OSX):

- `https://roguetemple.com/z/hyper`.

For more information on hyperbolic geometry, see

- J. R. Weeks, *The Shape of Space,* CRC press, 2019.
- A. F. Beardon, *The Geometry of Discrete Groups*, 1983.
- W. P. Thurston, *Three-dimensional Geometry and Topology,* Princeton University Press, 1997.

`http://www.stevejtrettel.com/living-in-hyperbolic-space.html`

Chapter 4

—

14 Motions fixing a point in $\mathbb{R}^2$ - proof

Proof of the classification of rigid motions of $\mathbb{R}^2$ fixing a point.

References

- Knörrer, pp 5-9.

In the last chapter we listed a lot of isometries of $\mathbb{R}^3$, but how do we know we have all of them? Perhaps we could compose them to get new ones that we overlooked.

This can be solved by linear algebra. But we aim to give purely geometric proofs, which perhaps gives more insight.

Two theorems:

Theorem 14.1 *An isometry of R^2 that fixes a point is a rotation or a reflection.*

Theorem 14.2 *An isometry of R^3 that fixes a point is a rotation or a roto-reflection.*

We will prove Theorem 14.1 below and Theorem 14.2 in the next section.

Proof (Theorem 14.1)

We organize it according to the fixed-point set. In view of Proposition 11.1, this is either a point, a line, or the whole plane.

1) If $\mathrm{Fix}(\phi)$ is the whole plane, then ϕ is the identity map and we are done with this case.

2) Suppose $\mathrm{Fix}(\phi)$ is a line L. Consider a point $p \in \mathbb{R}^2$ not lying on L and consider where p can map to.

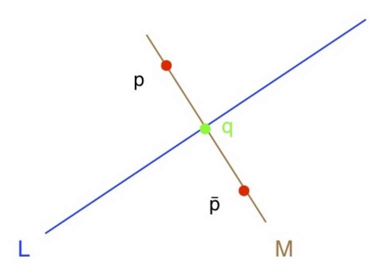


Figure 14.1: Two possible images of p under ϕ.

Let M be the line through p that meets L perpendicularly, say at the point q. Since $\phi(L) = L$ and $\phi(q) = q$, it follows that

$$\phi(M) = M.$$

Either ϕ fixes every point of M, or ϕ flips M about q. So either

$$\phi(p) = p$$

or

$$\phi(p) = \sigma_L(p),$$

because these are the only two points on M with distance $d(p, q)$ to q.

But $\phi(p) = p$ is impossible, because ϕ also fixes L, and then by Proposition 11.1, ϕ would fix the whole plane. But we are in the case $\mathrm{Fix}(\phi) = L$.

So $\phi(p) = \sigma_L(p)$. Indeed, this must occur for every p. Therefore $\phi = \sigma_L$ and we are done with this case.

3) If $\mathrm{Fix}(\phi)$ is a point, the proof of similarly difficulty and is left to the reader. See Knörrer pp 6-8 (Case 2).

□

15 Motions fixing a point in $\mathbb{R}^3$ - proof

Axis lemma. Proof of the classification of rigid motions of $\mathbb{R}^3$ fixing a point.

References

- Knörrer, 10-16, 63-64, 302.

We aim to prove Theorem 14.2: *A rigid motion of $\mathbb{R}^3$ that fixes a point is a rotation or a roto-reflection.*

This is an easy consequence of the following lemma:

Lemma 15.1 (Axis lemma) *(Euler) (Knörrer p 11 Lemma 3) Let ϕ be an isometry of R^3 that fixes a point a. Then ϕ preserves an axis L: $\phi(L) = L$.*

If Z is the inversion through a, the Lemma asserts that either ϕ or $\phi \circ Z$ fixes L.

This very important theorem has a number of proofs, ranging from algebra through geometry to topology.

1. Linear algebra. Wlog $a = 0$. Let A be the matrix of ϕ. The key is to solve the cubic polynomial $\det(A - \lambda I) = 0$ for $\lambda = \pm 1$.[1]

2. Topology. Any homeomorphism (i.e. bijective bicontinuous map) from S^2 to itself must fix a point or take it to its antipode.[2]

3. Geometry. We will now present a very beautiful geometric proof that we found in Knörrer pp 11-13.

Proof (Lemma 15.1) 1. Let ϕ be an isometry of $\mathbb{R}^3$ that fixes 0. Let S be the unit sphere

$$S = \{x \in \mathbb{R}^3 : d(x, 0) = 1\}.$$

Note that f takes S to itself since ϕ is an isometry.

Let us look at the point $p \in S$ that is moved the least distance by ϕ. Define

$$f : S \to \mathbb{R}, \qquad f(x) := d(x, \phi(x)), \qquad x \in S.$$

Let p be a point in S that minimizes f:

$$f(p) = \min_{x \in S} f(x). \tag{15.1}$$

To be sure that p exists, we need three theorems from analysis:

[1] Watch for it in linear algebra, or see Knörrer p 302.

[2] See Brouwer Fixed Point Theorem, Knörrer pp 63-64.

1) f is continuous.

2) A continuous function on a compact set attains a minimum.

3) S is compact.

We will assume these without proof. So assume p satisfies (15.1).

2. Consider the value of $f(p)$. Note that $0 \le f(p) \le 2$.

Case $f(p) = 2$*:* Then $\phi(p) = -p$, and indeed $\phi(q) = -q$ for every $q \in S$. So ϕ is the antipodal map Z. So ϕ preserves every axis. So we are done.

Case $f(p) = 0$*:* Then $\phi(p) = p$, so the axis through p and $-p$ is fixed by ϕ. So we are done.

Case $0 < f(p) < 2$*:* This is the most interesting case and will take up the rest of the proof.

Then $\phi(p) \neq p, -p$. So p, $\phi(p)$, and 0 are not collinear. Then there is a well-defined plane E passing through p, $\phi(p)$, and 0. E intersects S in a great circle β containing p and $\phi(p)$. There is a unique shortest geodesic segment γ from p to $\phi(p)$, of length $f(p)$, lying in β and defining the direction you have to move along S to get from p to $\phi(p)$.

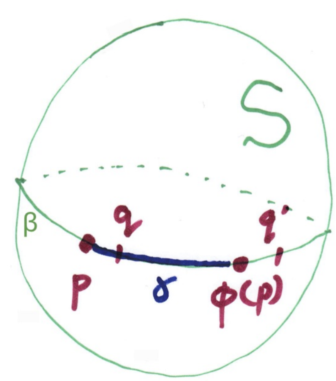


Figure 15.1: Moving p along γ.

Select a point q on γ very close to p. That is, q is slightly towards $\phi(p)$ along γ from p. Suppose q is at distance ε from p.

Where is $\phi(q)$?

Let q' be the point lying on $\beta \setminus \gamma$ at distance ε from $\phi(p)$. Note that q' is just beyond $\phi(p)$ from p.

Take ε so small that q' is on the short geodesic between $\phi(q)$ and $-p$. That is, q' is not beyond $-p$. This makes the proof easier to visualize.

Claim 1: $\phi(q) = q'$.

Proof of Claim 1: On the one hand, because ϕ is an isometry, $\phi(q)$ is at distance ε from $\phi(p)$. That is, $\phi(q)$ lies on the circle

$$C = C_\varepsilon(\phi(p)) = \{x \in S \,:\, d(x, \phi(p)) = \varepsilon\}$$

of radius ε about $\phi(p)$.

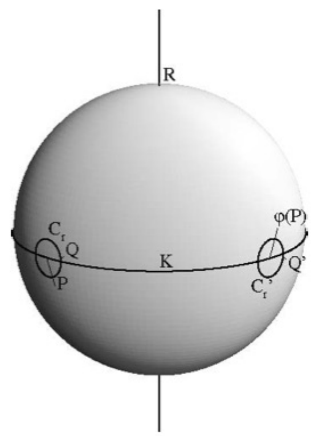


Figure 15.2: Where q lies. (Knörrer)

On the other hand,

$$d(\phi(q), q) = f(q) \geq f(p) = d(\phi(p), p)$$

because p minimizes the distance moved by ϕ. So $\phi(q)$ lies on or outside the circle

$$C' = C_{f(p)}(q) = \{x \in S \,:\, d(x, q) = f(p)\}$$

of radius $f(p)$ about q.

But here's the point – the circle C is tangent to C' from the inside, and the point of tangency is q' !

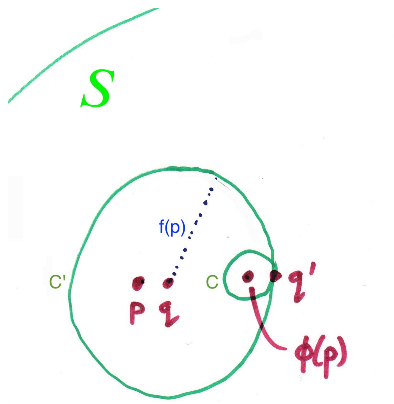


Figure 15.3: Tangent circles.

So the only point that can both lie on C and lie on or outside of C' is the point q'. Therefore,

$$\phi(q) = q',$$

as desired. This proves Claim 1.

3. Let L be the line perpendicular to the plane E at 0. L meets S at the north and south poles.

Claim 2: ϕ preserves L.

Proof of Claim 2: Consider the 5 points

$$p, q, \phi(p), \phi(q) = q', 0,$$

sitting in the plane E. We have

p, q, 0 determine the plane E.

Because ϕ is an isometry,

$\phi(p)$, $\phi(q)$, $\phi(0)$ determine the plane $\phi(E)$.

But $\phi(q) = q'$ and $\phi(0) = 0$, and

$\phi(p)$, q', 0 determine the plane E.

It follows that

$$\phi(E) = E,$$

so

$$\phi(L) = L.$$

Note that in this case ϕ reverses L, rather than fixing L, since $f(p) > 0$. This proves Claim 2. So we are done.

□

We will see that $f(p) = 0$ corresponds to rotations, whereas $f(p) > 0$ corresponds to roto-inversions. To understand the case $f(p) > 0$, consider a roto-inversion

$$Z \circ R_{180-\delta},$$

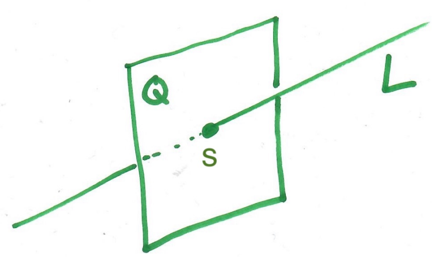
where δ is small. It functions as a rotation by δ along the equator. So the minimum of f will be very close to δ, attained along the equator, but f will be equal 2 at the poles.

In order to exploit the Axis Lemma, we need the following easy lemma.

Lemma 15.2 *An isometry of $\mathbb{R}^3$ that fixes an axis is either a rotation or a plane reflection.*

We prove this by reducing it to the two-dimensional classification theorem.

Proof Suppose χ is an isometry of $\mathbb{R}^3$ that fixes the axis L. What can χ be?



Let Q denote any plane perpendicular to L, meeting L at some point s. Then χ takes Q to Q, so $\lambda := \chi|Q$ is a well-defined isometry

$$\lambda : Q \to Q$$

that fixes s.

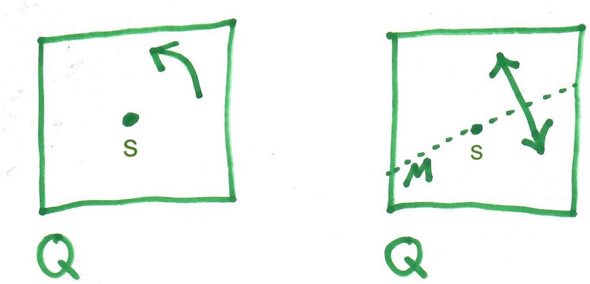


Figure 15.4: Case 1, Case 2.

So using the classification of two-dimensional motions fixing a point[3] λ is either a rotation of Q about s by some θ, or a reflection of Q in a line M lying in Q and passing through s.

Now λ really depends on s, so we should write λ_s, Q_s, θ_s, M_s. The reader should take care to convince him or herself that the character of λ_s does not vary from one s to another. That is, λ_s is either a rotation of Q_s by the same angle θ for all s, or λ_s is a reflection of Q_s across a line M_s where all M_s are parallel. Indeed, if the action of λ_s varied from s to s, then some distances would not be preserved and ϕ would not be an isometry.

So we have the following two cases.

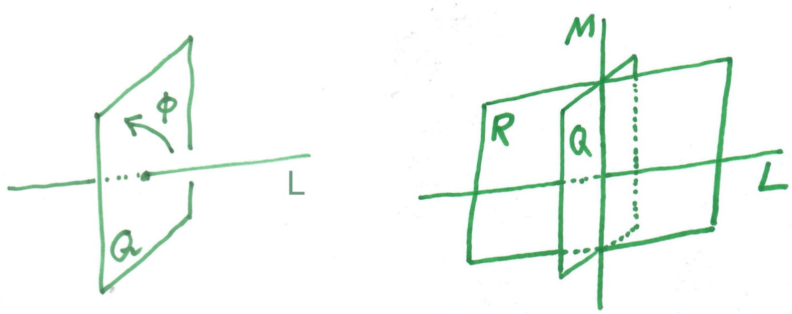


Figure 15.5: Case 1, Case 2.

If λ is a rotation of Q about s, then χ is a rotation of $\mathbb{R}^3$ about L.

If λ is the reflection of Q in M, then χ is the reflection of $\mathbb{R}^3$ in the plane R determined by L and M.

[3]Theorem 14.1, see also Knörrer p 5 Theorem 1.1.

□

Proof of Theorem 14.2 Let ϕ be an isometry of $\mathbb{R}^3$ that fixes a point a. Let Z be the inversion through a. By Lemma 15.1, either ϕ or $\phi \circ Z$ fixes an axis.

Suppose ϕ fixes an axis. Then by Lemma 15.2, ϕ is either a rotation or a plane reflection. But a plane reflection is a special case of a roto-inversion. So we are done.

Suppose $\phi \circ Z$ fixes an axis. Then by Lemma 15.2, $\phi \circ Z$ is either a rotation or a plane reflection. Then

$$\phi = (\phi \circ Z) \circ Z$$

is either a roto-inversion or a plane reflection composed with an inversion in a point of the plane. But a roto-inversion is a roto-reflection. And a plane reflection composed with an inversion in a point of the plane is a line-reflection, which is a rotation. In both cases, we are done.

□

16 Composition of rotations

Proof that the composition of rotations is a rotation.

References

- Knörrer, 43.

The composition of rotations is a rotation. This is not obvious; it's called Euler's theorem. It can be made to follow from the Classification Theorem 12.2, but because of its importance, we sketch several proofs, including an independent geometric proof.

Theorem 16.1 (Euler) *Let ϕ and ψ be rotations fixing p. Then $\phi \circ \psi$ is a rotation fixing p.*

Three proofs:

Proof 1: Linear algebra – soon you will be able to do this with matrices (or abstract properties of linear transformsations). Watch for it.

Proof 2: It follows from Theorem 14.2, because $\phi \circ \psi$ is clearly an *isometry* fixing 0. It is orientation-preserving. So by Theorem 14.2, it must be a rotation. □

Proof 3: A direct geometric proof, taken from Knörrer p 43, Theorem 1.6. We give it now.

We first need to know

$$\text{reflection} \circ \text{reflection} = \text{rotation}$$

unless the planes are parallel. Indeed,

Lemma 16.2 *Let P and Q be planes that meet in the axis $A = P \cap Q$. Then*

$$\sigma_Q \circ \sigma_P \ \ \textit{is a rotation about } A. \tag{16.1}$$

Proof We can see it from Figure 16.1.

□

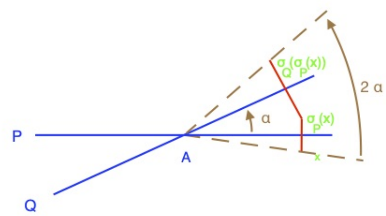


Figure 16.1: Composing reflections.

We can go further: if the planes meet at angle α, $0 < \alpha \leq \pi/2$, then the rotation is by an angle 2α.

If we compose the rotations in the other order, we get a rotation by -2α (i.e. same amount but in the opposite direction).

Proof (Geometric proof of Theorem 16.1) The case in which the axes of ϕ and ψ coincide is trivial. So we'll consider the case where the axes are distinct. Suppose ϕ has axis A and ψ has axis B. Let α be the rotation angle of ϕ and β the rotation angle of ψ.

By the hypothesis, A and B have an intersection point. So there exists a plane E determined by A and B. Let F be the plane containing A that makes an angle $\alpha/2$ with E. Let G be the plane containing B that makes an angle $-\beta/2$ with E. Note that $F \neq G$.

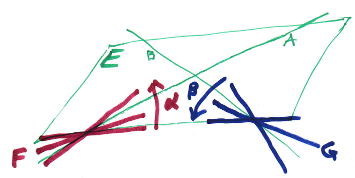


Figure 16.2: Expressing rotations via reflections.

Let σ_E, σ_F, σ_G be the reflections in E, F, G respectively. From Lemma 16.2 et seq., we see that

$$\phi = \sigma_E \circ \sigma_F$$

and

$$\psi = \sigma_G \circ \sigma_E$$

taking the sign into account. Then

$$\begin{aligned}\psi \circ \phi &= (\sigma_G \circ \sigma_E) \circ (\sigma_E \circ \sigma_F) \\ &= \sigma_G \circ (\sigma_E \circ \sigma_E) \circ \sigma_F \\ &= \sigma_G \circ \mathrm{id} \circ \sigma_F \\ &= \sigma_G \circ \sigma_F\end{aligned}$$

But the composition of two reflections is a rotation. Indeed, $\sigma_G \circ \sigma_F$ is a rotation about the intersection line $G \cap F$ by Lemma 16.2. That completes the proof.

□

Exercise 16.1 *If the composition of two plane reflections is a rotation, what is the composition of two inversions? (Brieskorn III, p 21, Satz 1.15(vi))*

Chapter 5

—

17 When must isometries be equal?

Noncollinear points. Noncoplanar points. Coincidence lemmas in $\mathbb{R}^2$, $\mathbb{R}^3$. Affine-independent points. Affine subspaces. Coincidence lemmas in $\mathbb{R}^n$.

We'll need the following to help us get *all* the symmetries of a figure.

$\boxed{\text{Question: When must two isometries coincide?}}$

Let's start in $\mathbb{R}^2$.

Figure 17.1: Collinear and non-collinear points.

Definition Points A, B, C in $\mathbb{R}^n$ are called *collinear* provided there is a line that contains all three of them. Otherwise they are *non-collinear.*

We now come to the crucial proposition.

Proposition 17.1 ($n = 2$) *Two isometries of $\mathbb{R}^2$ that take the same values on 3 non-collinear points must be equal everywhere.*

Another way to put it is: an isometry from $\mathbb{R}^2$ to $\mathbb{R}^2$ is determined by its values on 3 noncollinear points.

We have two proofs of this. One is very quick, using Proposition 11.1. It is given below.

The second is self-contained. It may give more geometric insight. It's in Section 49.

First proof of Proposition 17.1 Let x_1, x_2, x_3 be non-collinear points in R^2. Let ϕ and ψ be isometries of $\mathbb{R}^2$ with

$$\phi(x_i) = \psi(x_i), \qquad i = 1, 2, 3.$$

Then

$$(\psi^{-1} \circ \phi)(x_i) = x_i, \qquad i = 1, 2, 3.$$

So x_1, x_2, x_3 are fixed points of $\psi^{-1} \circ \phi$. But by Proposition 11.1, the fixed-point set is either empty, a point, a line, or all of $\mathbb{R}^2$. The first three possibilities are excluded since the points are not collinear. So $\mathrm{Fix}(\psi^{-1} \circ \phi) = \mathbb{R}^2$. So $\psi^{-1} \circ \phi = \mathrm{id}_{\mathbb{R}^2}$. So

$$\phi = \psi$$

and we are done.

□

The same method immediately yields

Proposition 17.2 **($n = 3$)** *Two isometries of $\mathbb{R}^3$ that take the same values on 4 non-coplanar points must be equal everywhere.*

Higher dimensions

We will state the results of this section without proof and we also draw on linear algebra.

The analog of a point, line, or plane in $\mathbb{R}^n$ is a *k-dimensional affine subspace*, or *k-plane* for short. It is essentially a k-dimensional flat object that goes on forever. An $(n-1)$-plane in $\mathbb{R}^n$ is called a *hyperplane*.

We will use the following working definition of an affine subspace from linear algebra (without proof):

Definition (1) An affine subspace of $\mathbb{R}^n$ is a set of the form

$$x + V$$

where V is a linear subspace of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$.

(2) Or equivalently: An affine subspace of $\mathbb{R}^n$ is a subset W of $\mathbb{R}^n$ that is closed under taking affine combinations.

What is an “affine combination”?

Definition A point x of $\mathbb{R}^n$ is called an *affine combination* of points $x_1, \ldots, x_k$ in $\mathbb{R}^n$ if

$$x = t_1 x_1 + \ldots + t_k x_k,$$

where

$$t_1 + \ldots + t_k = 1.$$

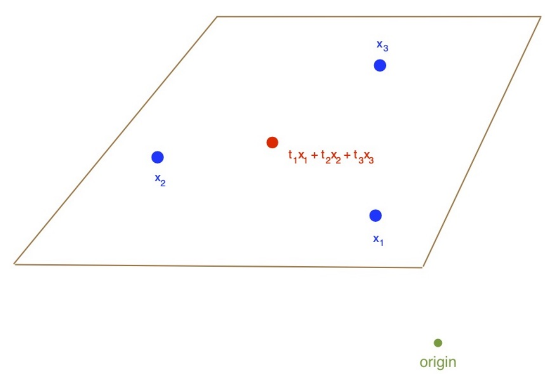


Figure 17.2: Affine combination.

That is, x is a weighted average of the points $x_1, \ldots, x_n$, where negative weights are allowed. For example, a line has the form $\{tx + (1-t)y \,|\, t \in \mathbb{R}\}$, where $x \neq y \in \mathbb{R}^n$. A plane has the form $\{sx + ty + (1-t)z \,|\, s, t \in \mathbb{R}\}$, where $x, y, z \in \mathbb{R}^n$ are not collinear.

The affine notions are more general than the linear ones. The main idea is: for the affine notions, we don't care where the zero point is.

From another perspective: Linear subspaces solve systems of homogeneous linear equations, whereas affine subspaces solve systems of inhomogeneous linear equations.

The generalization of "collinear" or "coplanar" is "affine-independent".

Definition (1) $x_1, \ldots, x_{k+1}$ are *affine-independent* if and only if they are not contained in a common $(k-1)$-dimensional affine subspace.

(2) Or equivalently, $x_1, \ldots, x_{k+1}$ are affine-independent if and only if no one of them is an affine combination of the remaining k points.

The affine combinations of $k+1$ affine-independent points in $\mathbb{R}^n$ sweep out a k-dimensional affine subspace. If they are not affine-independent, they sweep out a lower-dimensional affine subspace.

Then we can state:

Theorem 17.3 *The fixed-point set of an isometry of* $\mathbb{R}^n$ *is an affine subspace of* $\mathbb{R}^n$*.*

Theorem 17.4 *Two isometries of* $\mathbb{R}^n$ *that take the same values on* $n+1$ *affine-independent points must be equal everywhere.*

Although we have not proven these statements, the reader should have no difficulty applying them to geometry-style proofs.

18 Abstract properties of symmetries

Group-like properties.

References

- ◦ Knörrer, 20-21.
- ◦ Senechal, 27-28.

Let F be any figure. We observe

$$\phi, \psi \text{ symmetries of } F \qquad \Longrightarrow \qquad \phi \circ \psi \text{ is a symmetry of } F.$$

An isometry is bijective, and we see that

$$\phi \text{ symmetry of } F \qquad \Longrightarrow \qquad \phi^{-1} \text{ exists and is a symmetry of } F.$$

Additionally

$$\text{the identity is always a symmetry of } F.$$

Finally, all maps satisfy the associative law

$$(\phi \circ \psi) \circ \chi = \phi \circ (\psi \circ \chi). \tag{18.1}$$

These properties define a group; the formal definition will be given later.

19 Symmetries of the equilateral triangle

The group table for the equilateral triangle. Completeness of the list.

References

- Knörrer, 3-4, 8.

We will study the six symmetry operations of the equilateral triangle Δ (Section 1). We will show two things:

Claim 1: The list is closed with respect to composition and inversion.

Claim 2: The list is complete (it contains all the symmetries of Δ).

Towards Claim 1

To this end, we will organize the six known symmetries of Δ in a table. First, let's name them.

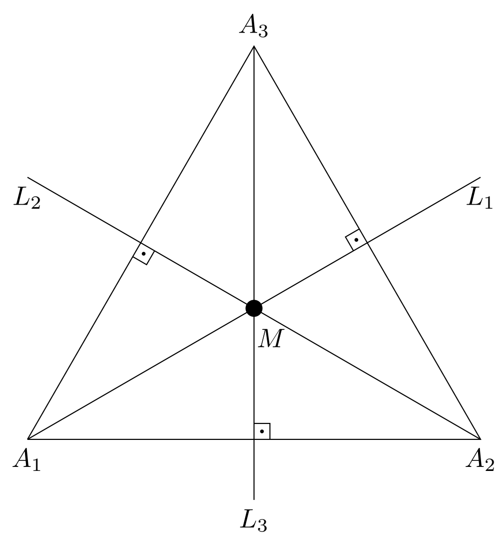


Figure 19.1: An equilateral triangle with mirror axes.

The line L_i passes through the midpoint M and through the vertex A_i. It meets the segment that connects the other two vertices perpendicularly at its midpoint. We now name the symmetries that we know, as follows:

$$\begin{aligned}
I &= \text{identity} \\
1 &= \text{reflection across } L_1 \\
2 &= \text{reflection across } L_2 \\
3 &= \text{reflection across } L_3 \\
A &= \text{counterclockwise rotation about } M \text{ by } 120^\circ \\
B &= \text{counterclockwise rotation about } M \text{ by } 240^\circ.
\end{aligned}$$

We collect these in a set

$$G \coloneqq \{I, 1, 2, 3, A, B\}.$$

Next we claim that G is closed with respect to composition and inversion. To this end we produce a multiplication table.

The first row and column are automatic, since the elements are composed with the identity:

$$\begin{aligned}
I \circ \phi &= \phi \qquad \forall \phi \in G \\
\phi \circ I &= \phi \qquad \forall \phi \in G
\end{aligned}$$

Next we calculate:

$$\begin{aligned}
&A \circ A = B && 120 + 120 = 240 \\
&A \circ B = B \circ A = I && 120 + 240 = 360 = 0 \\
&B \circ B = A && 240 + 240 = 480 = 120
\end{aligned}$$

and obviously

$$1 \circ 1 = 2 \circ 2 = 3 \circ 3 = I$$

Collecting these calculations in a table, we get so far:

$\circ$	I	A	B	1	2	3
I	I	A	B	1	2	3
A	A	B	I			
B	B	I	A			
1	1			I		
2	2				I	
3	3					I

Figure 19.2: Partial multiplication table #1.

Composition of reflections

The composition of (unequal) reflections is more complicated. We'll calculate $1 \circ 2$. Let's observe the effect on the vertices A_1, A_2, A_3. We have

$$A_1 \xrightarrow{2} A_3 \xrightarrow{1} A_2$$
$$A_2 \xrightarrow{2} A_2 \xrightarrow{1} A_3$$
$$A_3 \xrightarrow{2} A_1 \xrightarrow{1} A_1$$

so

$$1 \circ 2 \ : \ \begin{cases} A_1 \to A_2 \\ A_2 \to A_3 \\ A_3 \to A_1 \end{cases}$$

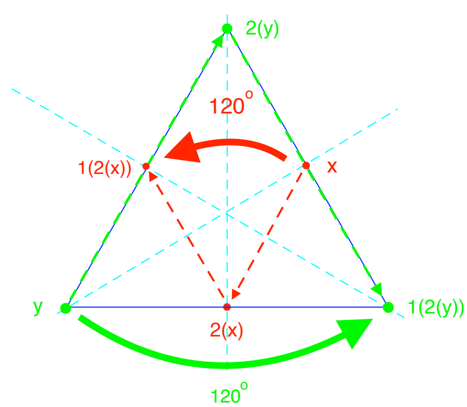


Figure 19.3: Effect of $1 \circ 2$.

Compare this with the rotations. The 120-degree rotation A has exactly the same effect on the vertices:

$$A \ : \ \begin{cases} A_1 \to A_2 \\ A_2 \to A_3 \\ A_3 \to A_1 \end{cases}$$

But recall Proposition 17.1:

> *Two isometries of the plane that agree on three non-collinear points must agree everywhere.*

It follows that

$$1 \circ 2 = A.$$

This is consistent with the principle from Lemma 16.2:

$$\text{reflection} \circ \text{reflection} = \text{rotation}.$$

Indeed, if L and M are lines at angle α, then $\sigma_L \circ \sigma_M$ is a rotation by 2α. Since L_1 and L_2 meet at 60 degrees (measured counterclockwise from 2 to 1), $1 \circ 2$ is a counterclockwise rotation by 120 degrees.

If we compose the rotations in the other order, we get a rotation by -2α. It follows that $2 \circ 1$ is a rotation by -120, which is 240, which we called B.

So

$$2 \circ 1 = B,$$

so

$$1 \circ 2 \neq 2 \circ 1$$

so multiplication is not commutative! We are surprised to discover this (not). We add this information to the table:

$\circ$	I	A	B	1	2	3
I	I	A	B	1	2	3
A	A	B	I			
B	B	I	A			
1	1			I	$\boxed{A}$	
2	2			$\boxed{B}$	I	
3	3					I

Figure 19.4: Partial multiplication table #2.

And now we fill out the rest by pattern-matching. By the above, it's clear that the composition of any two reflections is A, B, or I. Now it's generally true in such tables that

Each row and each column contains each element exactly once. (19.1)

We will prove this later (Proposition 20.4). We'll use it before proving it, but its predictions can be verified directly. This principle forces

$$1 \circ 3 = B, \qquad 1 \circ 3 = A, \qquad 3 \circ 1 = A, \qquad 3 \circ 2 = B.$$

So we get

$\circ$	I	A	B	1	2	3
I	I	A	B	1	2	3
A	A	B	I			
B	B	I	A			
1	1			I	$\boxed{A}$	B
2	2			$\boxed{B}$	I	A
3	3			A	B	I

Figure 19.5: Partial multiplication table #3.

Composition of reflection and rotation

For the remaining entries in the table, we need only calculate two more and we get the rest by (19.1). We don't have to go back to the triangle, we can use tricks. We use the associative law (18.1) and some previous values that we know:

$$A \circ 1 = (3 \circ 1) \circ 1 = 3 \circ (1 \circ 1) = 3 \circ I = 3$$

and

$$1 \circ A = 1 \circ (1 \circ 2) = (1 \circ 1) \circ 2 = I \circ 2 = 2.$$

They are unequal, which does not surprise us. They follow the pattern

$$\text{reflection} \circ \text{rotation} = \text{reflection (and vice-versa).}$$

After filling in these two values, the rest of the values are forced by (19.1), and we get

$\circ$	I	A	B	1	2	3
I	I	A	B	1	2	3
A	A	B	I	$\boxed{3}$	1	2
B	B	I	A	2	3	1
1	1	$\boxed{2}$	3	I	A	B
2	2	3	1	B	I	A
3	3	1	2	A	B	I

Figure 19.6: Multiplication table.

The fact that multiplication does not commute corresponds to the fact that the table is not symmetric about the main diagonal.

We also notice that the table falls naturally into four subtables with:

- rotation $\circ$ rotation = rotation (blue)
- reflection $\circ$ rotation = reflection (and vice-versa) (red)

- reflection ∘ reflection = rotation (green)

Here we always count the identity as a rotation, albeit a trivial one.

Inspecting the table, we see that the set G is closed with respect to composition and taking inverses. This proves Claim 1.

Towards Claim 2

Now we must complete the last step: we must show that the table is complete, that is $\mathrm{Sym}(\Delta) = G$. Here $G = \{I, A, B, 1, 2, 3\}$.

Proof We have $G \subseteq \mathrm{Sym}(\Delta)$ by construction. We only need to know that G contains *all* the symmetries of Δ.

Let $\phi \in \mathrm{Sym}(\Delta)$. That means that ϕ is an isometry of $\mathbb{R}^2$ and $\phi(\Delta) = \Delta$. Then ϕ takes vertices to vertices, that is, ϕ permutes the corners. Since Δ has 3 vertices, there are 6 ways to do this. But all 6 ways are already effected by elements of G. That is, for every $\phi \in \mathrm{Sym}(\Delta)$, there is $\psi \in G$ that agrees with ϕ on the vertices:

$$\phi(A_1) = \psi(A_1) \tag{19.2}$$
$$\phi(A_2) = \psi(A_2) \tag{19.3}$$
$$\phi(A_3) = \psi(A_3). \tag{19.4}$$

Since the vertices are not collinear, Proposition 17.1 implies that

$$\phi = \psi$$

so

$$\phi \in G.$$

Since ϕ was arbitrary, $\mathrm{Sym}(\Delta) \subseteq G$. So $\mathrm{Sym}(\Delta) = G$. This verifies the completeness of the list.

□

Exercise 19.1 *Construct the multiplication table of the unit quaternions Q_8*[1]

$$\{\pm 1, \pm i, \pm j, \pm k\}$$

and the multiplication table of $D_4 = \mathrm{Sym}(P_4)$. Both sets have 8 elements. Can one convert the first table into the second one through clever symbol substitution?

[1] See Section 51 or Saracino 47-49.

Chapter 6

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20 Groups

Group axioms, examples, symmetry groups, uniqueness of identity, uniqueness of inverse, powers, cancellation law

References

- Knörrer, 20-26.
- Saracino, 10-15,16-24,25-32,33-34

We'll define the abstract concept of a group and derive its basic properties. It formalizes our observations about symmetries.

Definition A group is a pair $(G, \circ)$ where G is a set and $\circ$ is a function

$$\circ \colon G \times G \to G, \qquad (a, b) \mapsto a \circ b,$$

such that

$$\begin{array}{lcl} \forall\, a, b, c \in G: & (a \circ b) \circ c = a \circ (b \circ c) & \text{(associativity)} \\ \exists\, e \in G\ \forall\, a \in G: & e \circ a = a \circ e = a & \text{(neutral element)} \\ \forall\, a \in G\ \exists\, b \in G: & a \circ b = b \circ a = e & \text{(inverses)} \end{array}$$

It is customary to write $a \circ b$, $a \cdot b$, or even ab instead of $\circ(a, b)$. We often denote the group $(G, \circ)$ simply by G when the operation is understood from the context.

Example The multiplication table of $\mathrm{Sym}(\Delta)$ defines a group.

Example The composition rules for orienation define a group:

∘	E	U
E	E	U
U	U	E

Example The symmetries $\mathrm{Sym}(F)$ of any figure F, equipped with composition $\circ$ as the operation, form a group.

Example Groups from algebra:

- $(\mathbb{Z}, +)$, $(\mathbb{R}, +)$, $(\mathbb{R} \setminus \{0\}, \cdot)$
- ($m \times n$ matrices, $+$)
- (invertible $n \times n$ matrices, $\cdot$)
- ($\mathbb{R}^n$, $+$)

Exercise 20.1 *Are the integers a group? For what operations? What about the even numbers?*

Exercise 20.2 *Find a set and an operation that fulfills all the group axioms except for the existence of inverses.*

Simple properties

Because of the associativity axiom we can leave out the parentheses:

$$(ab)c = a(bc) = abc$$

This is also true for more than three elements.

Proposition 20.1 *The neutral element of a group is unique.*

Proof Let e and $e' \in G$ satisfy the second axiom of **??**. Then

$$e = ee' = e'$$

□

The neutral element e of a group can be written as e_G or 1_G. It is also called the *identity*. If the group is clear from the context, it can be written as e or 1.

Proposition 20.2 *Let $a \in G$. Then a has a unique inverse.*

Proof Fix $a \in G$. Let b and $b' \in G$ be inverses of a. Then

$$\begin{aligned} e = b'a \quad \& \quad ab = e \quad &\Longrightarrow \quad eb = (b'a)b \\ &\Longrightarrow \quad eb = b'(ab) \\ &\Longrightarrow \quad eb = b'e \\ &\Longrightarrow \quad b = b' \end{aligned}$$

□

Definition Let $a \in G$. We denote the unique inverse of a by a^{-1}.

Definition Let G be a group. The j'th power of an element $a \in G$ is defined by

$$
\begin{aligned}
a^j &:= \underbrace{a\,a \cdots a}_{j \text{ times}} \text{ for } j > 0 \\
a^{-j} &:= \underbrace{a^{-1}a^{-1} \cdots a^{-1}}_{j \text{ times}} \text{ for } j > 0 \\
a^0 &:= 1.
\end{aligned}
$$

Exercise 20.3 *(Saracino, p 34) Show*

(a) $a^j a^k = a^{j+k}$.
(b) $(a^j)^{-1} = a^{-j}$.
(c) $(a^j)^k = a^{jk}$.

for all integers j, k.

Proposition 20.3 (Cancellation law) *Let* G *be a group and* $a, b, c \in G$. *Then*

$$
\begin{aligned}
ab = ac \qquad &\Longrightarrow \qquad b = c \\
ba = ca \qquad &\Longrightarrow \qquad b = c.
\end{aligned}
$$

Proof We prove the first formula.

$$
\begin{aligned}
ab = ac &\Longrightarrow a^{-1}(ab) = a^{-1}(ac) \\
&\Longrightarrow (a^{-1}a)\; b = (a^{-1}a)c \\
&\Longrightarrow eb = ec \\
&\Longrightarrow b = c
\end{aligned}
$$

□

Proposition 20.4 *Each element of the group appears exactly once in each row and in each column of the multiplication table of the group.*

Proof

Proof for rows: Fix a row a and an element c. The element c appears in row a one time for each solution x of the equation

$$ax = c.$$

Indeed, x specifies the column that c is in. But this equation has exactly one solution, namely

$$x = a^{-1}c,$$

because of inverses and the cancellation law. So row a has exactly one occurence of c.

Proof for columns: Similar.

□

21 Isomorphisms

Isomorphisms, homomorphisms, automorphisms. Automorphism group.

References

- Saracino, 109-114 (but only read about isomorphisms)

Definition Let $(G, \circ_G)$ und $(H, \circ_H)$ be groups. An *isomorphism* between $(G, \circ_G)$ and $(H, \circ_H)$ is a bijection

$$f\colon G \to H,$$

that preserves the group operation:

$$f(a \circ_G b) = f(a) \circ_H f(b) \tag{21.1}$$

for all $a, b \in G$.

Let G and H be groups and $f\colon G \to H$ an isomorphism between G and H. We say that G and H are *isomorphic* and write

$$G \cong H$$

or

$$G \underset{f}{\cong} H.$$

Proposition 21.1 *Let G and H be groups and*

$$f\colon G \to H$$

an isomorphism between G and H. Then

(1) $f(e_G) = e_H$.

(2) $f(a^{-1}) = f(a)^{-1}$ for all $a \in G$.

This is obvious, because f losslessly preserves group multiplication, so f must preserve all constructions that are built upon it. But we'll give a calculational proof.

Proof 1. Compute

$$f(e_G)f(e_G) = f(e_Ge_G) = f(e_G) = f(e_G)e_H.$$

The desired result follows by the cancellation law.

2. Compute

$$f(a)f(a^{-1}) = f(aa^{-1}) = f(e_G) = e_H = f(a)f(a)^{-1}.$$

The desired result follows by the cancellation law.

□

The next proposition:

Proposition 21.2 $\cong$ *is an equivalence relation.*

Exercise 21.1 *Prove this.*

The properties of an equivalence relation correspond nicely to the grouplike axioms satisfied by isomorphisms.

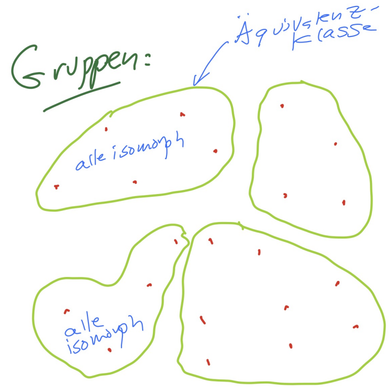


Figure 21.1: Isomorphism classes.

Homomorphisms

We'll also define "homomorphism" now, though we won't investigate it until later.

Definition Let G, H be groups. A *homomorphism* is a map $f\colon G \to H$ such that

$$f(ab) = f(a)f(b)$$

for all $a, b \in G$.

It's just like an isomorphism, but it doesn't have to be surjective or injective. In particular, it can "forget" information from G. The image of G in H can much smaller than G. It can also be much smaller than H.

Automorphisms

Here's a self-referential usage of a group. It shows how mathematics builds on itself.

Definition An automorphism of a group is an isomorphism of the group with itself. The set of automorphisms of G is written $\mathrm{Aut}(G)$.

Exercise 21.2 *Let G be a group. Show that $(Aut(G), \circ)$ is a group.*

There is more information on automorphisms in Section 33.

Exercise 21.3 *How many automorphisms does D_3 have?*

22 Order of elements and groups

Order of a group. Order of an element. D_3 example. Elements of a finite group have finite order.

References

- Knörrer, 47.
- Saracino, 34-35, 40.

Definition (Order of a group) Let G be a group. The *order* of G is the number of elements in it:

$$\text{order}(G) = \big|G\big| = \#G. \tag{22.1}$$

Definition (Order of an element) Let G be a group and $a \in G$. The *order* $\text{ord}(a)$ of a is the smallest positive number n such that $a^n = e$, if there is such an n. Otherwise, the order is infinity. We include both of these cases by defining[1]

$$\text{ord}(a) \coloneqq \inf\{n > 0 \bigm| a^n = e\} \tag{22.2}$$

Example In $D_3 = Sym(\Delta) = \{I, A, B, 1, 2, 3\}$, we have

$$I^1 = I, \qquad A^3 = B^3 = I, \qquad 1^2 = 2^2 = 3^2 = I,$$

and

$$\text{ord}(I) = 1, \qquad \text{ord}(A) = \text{ord}(B) = 3, \qquad \text{ord}(1) = \text{ord}(2) = \text{ord}(3) = 2.$$

Example There are at least three groups of order 8 – the cyclic group $\mathbb{Z}_8$, the dihedral group D_4, and the unit quaternions Q_8 – or are there?

Exercise 22.1 *Prove that these three groups are nonisomorphic. (Hint: Calculate the orders of the elements.)*

Proposition 22.1 *Let G be a finite group. Then every element has finite order.*

Proof Let $a \in G$. Consider the sequence $1, a, a^2, a^3, \ldots$ There must be a repetition, say $a^k = a^l$ with $0 \leq k < l$. By the cancellation law, $a^{l-k} = 1$.

□

[1] $\inf(\varnothing) = \infty$ and $\sup(\varnothing) = -\infty$.

23 Cyclic groups

Cyclic groups, examples.

References

- Knörrer, 44.
- Saracino, 39-40.

A group G is *cyclic* if there is an element $a \in G$ such that every element has the form a^j for some $j \in \mathbb{Z}$. a is called a *generator* of G.

Let $m \geq 1$. The set

$$\mathbb{Z}_m \coloneqq \{\overline{0}, \overline{1}, \overline{2}, ..., \overline{m-1}\}$$

with the addition rule

$$\bar{i} + \bar{j} \coloneqq \begin{cases} \overline{i+j} & \text{if } 0 \leq i+j < m \\ \overline{i+j-m} & \text{if } m \leq i+j < 2m \end{cases}$$

is a finite cyclic group. The addition rule is called *addition modulo m*.

Another way to present $\mathbb{Z}_m$ is

$$\mathbb{Z}'_m \coloneqq \{1, \sigma, \sigma^2, \dots, \sigma^{m-1}\}$$

with the rule $\sigma^m = 1$.

A third realization of $\mathbb{Z}_m$ is as the set of n'th roots of unity

$$\begin{aligned} \mathbb{Z}''_m &\coloneqq \{z \in \mathbb{C} \bigm| z^m = 1\} \\ &= \{e^{2i\pi k/m} \bigm| k = 0, 1, 2, \dots, m-1\}, \end{aligned}$$

with complex multiplication.

A finite cyclic group is often compared to a clock.

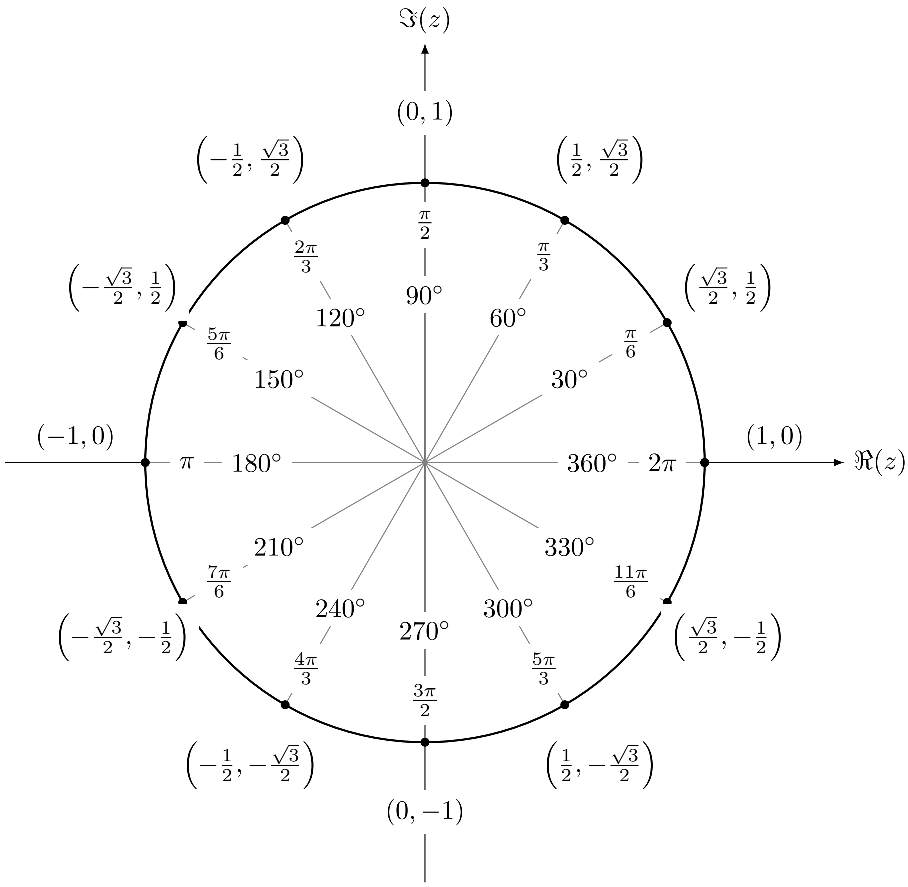


Figure 23.1: They don't make clocks like they used to.

Evidently $\mathbb{Z}_m \cong \mathbb{Z}'_m \cong \mathbb{Z}''_m$ with $\overline{k} \sim \sigma^k \sim\ e^{2i\pi k/m}$.

Indeed, there is just one cyclic group of each order – finite or infinite. The obvious example of an infinite cyclic group is $(\mathbb{Z}, +)$, generated by 1.

When $\mathbb{Z}_m$ is realized as the rotations

$$\{I, R_{2\pi/m}, R^2_{2\pi/m}, \ldots, R^{n-1}_{2\pi/m}, \}$$

acting on $\mathbb{R}^2$ or $\mathbb{R}^3$, it is called C_m.

Exercise 23.1 *Find a figure in $\mathbb{R}^3$ whose symmetry group is C_m .*

Exercise 23.2

(a) Find cyclic groups that possess exactly 1, 2, or 4 distinct generators.

(b) Prove that no cyclic group has exactly 3 generators.

(c) Show that an infinite cyclic group has exactly 2 generators.

Exercise 23.3

(a) How many automorphisms does $\mathbb{Z}_4$ *have?* $\mathbb{Z}_5$*?* $\mathbb{Z}_{28}$*?*

(b) Show that any automorphism of $\mathbb{Z}_n$ *is given by multiplication by some integer.*

Chapter 7

—

24 Permutations

Permutations. Symmetric group. The n-simplex in $\mathbb{R}^{n+1}$. $\mathrm{Sym}(\Delta^n)$.

References

- Knörrer, 26, 35-36, 59-60.
- Saracino, 66-69.
- Rotman, 2-7.

Let X be a set.

Definition A *permutation* of X is a bijection $f\colon X \to X$. The set of all permutations of X is written $\mathrm{Perm}(X)$:

$$\mathrm{Perm}(X) := \{\text{all permutations of } X\} = \{f : X \to X \mid f \text{ is bijective}\}$$

The set $\mathrm{Perm}(X)$ becomes a group with composition as the group operation. It is called the permutation group of X. The permutations of X are the symmetries of X, where X is viewed as a pure set with no structure.

Definition The *symmetric group on n letters* is the permutation group of $\{1, 2, \ldots, n\}$:

$$S_n := \mathrm{Perm}(\{1, 2, \ldots, n\})$$

The order of S_n is $n! = n(n-1)\cdots 2 \cdot 1$. (Knörrer 35-36)

We saw in Section 19, Claim 2 that

$$\mathrm{Sym}(\Delta) \cong \mathrm{Perm}(\{\underbrace{A_1, A_2, A_3}_{\text{corners of } \Delta}\}) \cong S_3.$$

With a similar proof,

Proposition 24.1 *(Knörrer p 60, problem 4) We have*

$$\mathrm{Sym}(T) \cong S_4.$$

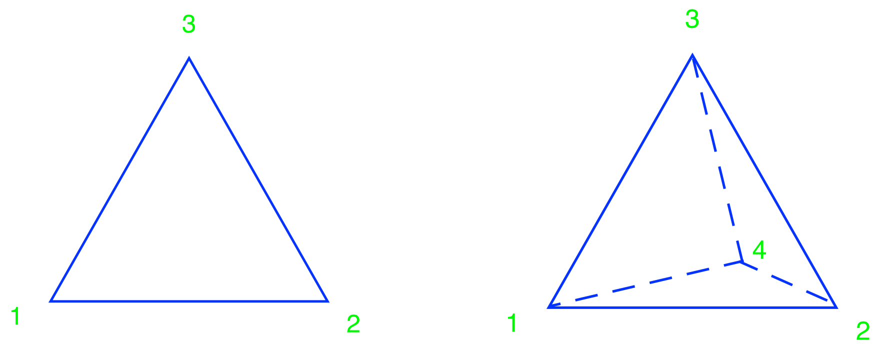


Figure 24.1: $\#S_3 = 3! = 6$; $\#S_4 = 4! = 24$.

We write Δ^n for the regular n-simplex in $\mathbb{R}^n$. It generalizes a line segment, an equilateral trangle, and a regular tetrahedron. All edges of Δ^n have length 1 and Δ^n has $n+1$ vertices.

A geometric realization of Δ^n in the space $\mathbb{R}^{n+1}$ (with edge-lengths $\sqrt{2}$) is the convex figure with the $n+1$ vertices

$$(1, \ldots, 0),\ \ldots,\ (0, \ldots, 1),$$

in $\mathbb{R}^{n+1}$. This figure lies in the hyperplane P defined by

$$x_1 + \cdots + x_{n+1} = 1$$

in $\mathbb{R}^{n+1}$. It is the intersection of P with the positive orthant defined by the inequalities

$$x_1 \geq 0, \quad \ldots, \quad x_{n+1} \geq 0.$$

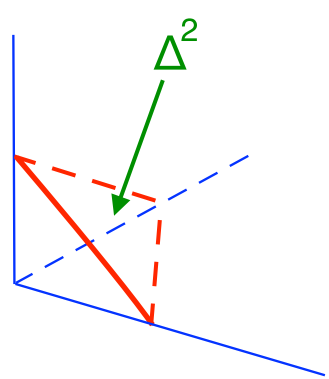


Figure 24.2: Δ^2 realized within the hyperplane $x_1 + x_2 + x_3 = 1$.

Exercise 24.1 *Show that*

$$\mathrm{Sym}(\Delta^n) \cong S_{n+1}.$$

Exercise 24.2 *Let P_n be the regular polygon with n sides. Compute the order of*

$$D_n = \mathrm{Sym}(P_n)$$

and

$$\mathrm{Perm}(\{\text{vertices of } P_n\}),$$

where P_n is a regular n-gon. By pure counting, can every permutation of the vertices of P_n be achieved by an isometry of the plane?

25 Notation for permutations

Function table. Downward arrow diagrams. Cycle diagrams. Cycle notation. Cycles. Transpositions. Disjoint cycles.

References: Same as last section.

In this section we explain at least four kinds of permutation notation.

Recall that the symmetric group S_n is the group of all permutations of an n-element set:

$$\begin{aligned} S_n &= \mathrm{Perm}(\{1,2,\dots,n\}) \\ &= \{\text{all bijections } \sigma : \{1,2,\dots,n\} \to \{1,2,\dots,n\}\}. \end{aligned}$$

Consider the following example for $n = 5$:

$$\sigma(1) = 2, \qquad \sigma(2) = 3, \qquad \sigma(3) = 1, \qquad \sigma(4) = 5, \qquad \sigma(5) = 4.$$

There are several ways to present this. First, the function table:

k	1	2	3	4	5
$\sigma(k)$	2	3	1	5	4

which is often abbreviated

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}.$$

Next, there's downward arrow diagrams:

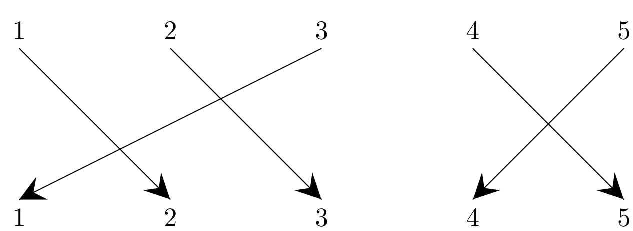


Figure 25.1: Downward arrow diagram.

Next, cycle diagrams:

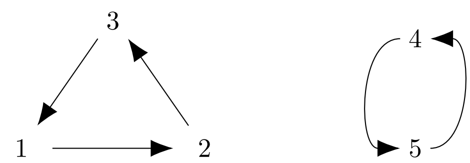


Figure 25.2: Cycle diagram.

Finally, in cycle notation:

$$\sigma = (123)(45)$$

Definition Let $k \geq 0$. A *k-cycle* is a permutation $\sigma \colon X \to X$ that moves only finitely many elements $x_1, x_2, \ldots, x_k$ of X (we assume they are distinct) and cyclically permutes them:

$$\begin{aligned}
\sigma(x_i) &= x_{i+1} && \text{for } i = 1, 2, \ldots, k-1 \\
\sigma(x_k) &= x_1 \\
\sigma(x) &= x && \text{for } x \notin \{x_1, x_2, \ldots, x_k\}
\end{aligned}$$

A 2-cycle is also called a *transposition.* 1-cycles and 0-cycles are all equal to the identity permutation. k-cycles move k elements, except for 1-cycles which move 0 elements.

We write

$$\sigma = (x_1 x_2 \ldots x_k),$$

for a k-cycle (no commas). Note that the entries of a cyclic permutation can be cyclically permuted, without affecting its value:

$$(12345) = (23451) \neq (12354).$$

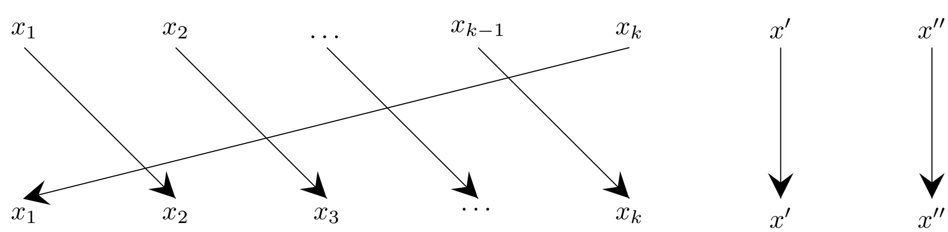


Figure 25.3: Visualizing a k-cycle.

When composing cycles, we use the usual convention of reading from right to left. For example, set

$$\sigma_3 = (123), \qquad \sigma_2 = (45),$$

and obtain

$$\sigma_3\sigma_2 = (123)(45),$$

where (45) is applied *first*.

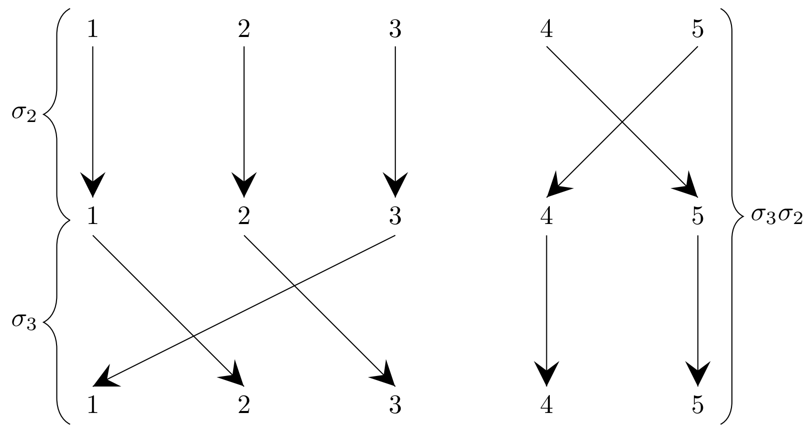


Figure 25.4: Visualizing $\sigma_3\sigma_2$.

Definition

(1) Let $k, l \geq 2$. Two cycles $(x_1 \ldots x_k)$ and $(y_1 \ldots y_l)$ are called *disjoint* if

$$\{x_1, \ldots, x_k\} \cap \{y_1, \ldots, y_l\} = \varnothing.$$

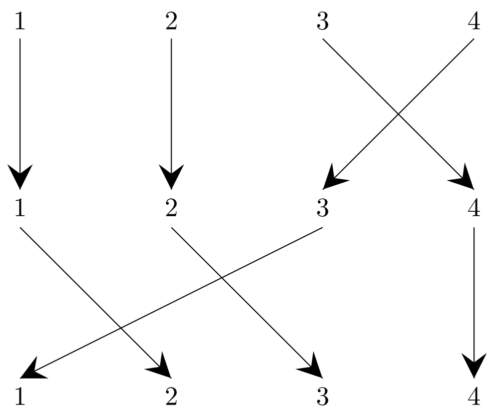
(2) Cycles of length 1 (or zero) are disjoint from everything by definition.

We observe that disjoint cycles commute:

$$(123)(45) = (45)(123).$$

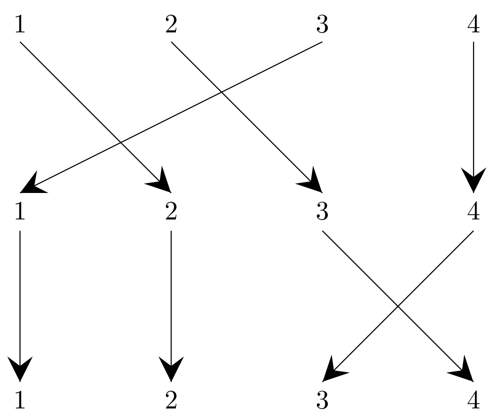
This is obvious in the diagram above: the threads slide past one another. But not all permutations commute. Let's visualize why

$$(123)(34) \neq (34)(123).$$



$= (123)(34) = (1234)$

Figure 25.5: It computes...



$= (34)(123) = (1243) \neq (1234)$

Figure 25.6: ...but it doesn't commute.

Indeed, elements in S_n often don't commute.

We end the section with the *disjoint cycle representation*:

Theorem 25.1 *Every permutation of a finite set can be represented as a product of disjoint cycles. The factorization is unique up to the order of the factors, provided that no* 0 *or* 1*-cycles appear.*

Since the 0-cycle () and all 1-cycles are equal to the identity map, we usually strip them off to reduce eyestrain: $(123)(45)(6)(7) = (123)(45)$.

The proof of the theorem is obvious: just look at the cycle diagram and read off the disjoint cycles.

Chapter 8

—

26 Subgroups

Subgroups. Subgroups of D_3. Subsets closed under group operations. Criterion for a subgroup. Intersection of subgroups. Union of subgroups.

References

- Saracino, 43-54.

We introduce subgroups.

Definition Let G be a group. A *subgroup* of G is a subset H of G that becomes a group when it is equipped with the inherited group operations.

We write $H \leq G$ to indicate that H is a subgroup of G.

Example G and $\{e_G\}$ are always subgroups of G.

Example Consider $D_3 = S_3 = \mathrm{Sym}(\Delta) = \{I, A, B, 1, 2, 3\}$. You can recognize $\{I, A, B\}$ as a subgroup because it forms a cocooned-off 3×3 subtable of the group table (in blue):

$\circ$	I	A	B	1	2	3
I	I	A	B	1	2	3
A	A	B	I	3	1	2
B	B	I	A	2	3	1
1	1	2	3	I	A	B
2	2	3	1	B	I	A
3	3	1	2	A	B	I

Figure 26.1: D_3 multiplication table.

Here are all the subgroups of D_3:

- $\{I\}$
- $\{I, A, B, 1, 2, 3\}$
- $\{I, A, B\}$
- $\{I, 1\}$
- $\{I, 2\}$
- $\{I, 3\}$

The inclusion relations are as follows:

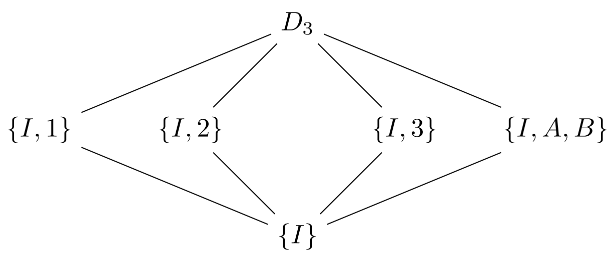


Figure 26.2: The lattice of subgroups of D_3.

Examples $\mathbb{Q}$ is an additive subgroup of $\mathbb{R}$. $\mathbb{R}_+$ is a multiplicative subgroup of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Sym$(F)$ is a subgroup of Isom$(\mathbb{R}^3)$.

□

Example Recalling (7.1), we see that Sym(T) is a subgroup of Sym(W):

$$\mathrm{Sym}(T) \leq \mathrm{Sym}(W).$$

Criterion for a subgroup

When is a subset of a group a subgroup?

Definition If

$$a, b \in H \quad \Longrightarrow \quad a \circ b \in H, \quad a^{-1} \in H,$$

then we say that H is *closed with respect to group operations.*

Example In $D_3 = \{I, A, B, 1, 2, 3\}$, the set

$$\{I, A, B, 1\}$$

is not closed with respect to multiplication because $A \circ 1 = 3$.

Being closed with respect to group operations is obviously a necessary condition to be a subgroup. The following convenient proposition says that for a nonempty set, it is also sufficient.

Proposition 26.1 *Let G be a group and H a subset of G. Assume*

- $H \neq \varnothing$
- *H is closed with respect to group operations.*

Then H is a subgroup of G with respect to the induced operation $\circ_H$.

Proof Define

$$\circ_H := \underbrace{\circ\big|H \times H}_{\substack{\text{restriction of}\\ \circ \text{ to } H\times H}} : H \times H \to G$$

$$\mathrm{inv}_H := \mathrm{inv}_G\big|H : H \to G,$$

$$e_H = e_G,$$

where inv_G is the inverse operation for G. Because H is closed with respect to group operations,

$$\circ_H : H \times H \to H$$

and

$$\mathrm{inv}_H : H \to H$$

are well-defined. Because H contains at least one element a, we have

$$e_H = e_G = a \circ a^{-1} \in H.$$

The group axioms are then automatically inherited by $(H, \circ_H)$, with inverses given by inv_H and the identity element by e_H.[1]

□

We abbreviate $\circ = \circ_H$ and $H = (H, \circ)$.

Proposition 26.2 *Let H and K be subgroups of a group G. Then $H \cap K$ is a subgroup of G.*

Proof $H \cap K$ contains e_G, so it is not empty. Since each of H and K is closed under group operations, so is $H \cap K$. The result now follows from the proposition.

□

A similar result holds for an arbitrary intersection of subgroups, without limitation.

On the other hand, the union of two subgroups need not be a subgroup. An example is $D_3 = \{I, A, B, 1, 2, 3\}$ which has the subgroups

$$H = \{I, A, B\}, \qquad K = \{I, 1\}.$$

As we noted above, the union $H \cup K$ is not closed under multiplication, so it can't be a subgroup. The smallest subgroup that contains $H \cup K$ is D_3 itself.

[1]Once the neutral element and the inverse operation have been given names, the group axioms become universal statements. Universal statements are always inherited by substructures.

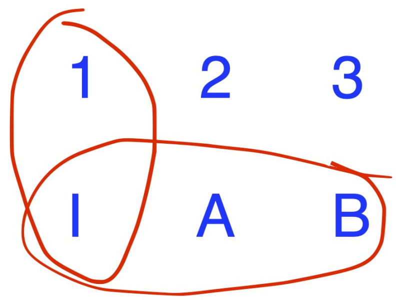


Figure 26.3: Union of two subgroups.

Exercise 26.1 *Show that the union of two subgroups is never a subgroup unless one is contained in the other.*

Example Consider the infinite chessboard:

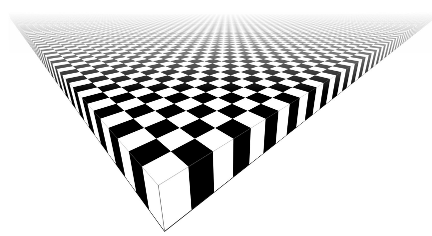


Figure 26.4: One-quarter of an infinite chessboard. (J. D. Hamkins)

Exercise 26.2

(a) Observe that the symmetries of the chessboard (respecting the colors) form a subgroup of the symmetries of the lattice $\mathbb{Z} \times \mathbb{Z}$ *(not respecting the colors):*

$$\mathrm{Sym}(\mathit{chessboard}) \leq \mathrm{Sym}(\mathit{lattice}).$$

(b) Are these two groups isomorphic?

27 Generators

Generators and generating sets. Examples. Products of powers.

References

- Brieskorn III, 20.
- Knörrer, 127.

Definition Let G be a group and S a subset of G. The *subgroup of G generated by S*, written

$$\langle S \rangle,$$

is the smallest subgroup of G that contains S.[2]

How can we be sure that such a "smallest subgroup" exists? We can construct $\langle S \rangle$ as

$$\langle S \rangle = \bigcap \{H \mid H \supseteq S,\ H \text{ is a subgroup of } G\}.$$

Then

(1) The RHS is a subgroup of G.

(2) The RHS contains S.

(3) The RHS is contained in any subgroup of G that contains S.

Therefore the RHS is the smallest subgroup of G that contains S. So we have constructed $\langle S \rangle$.

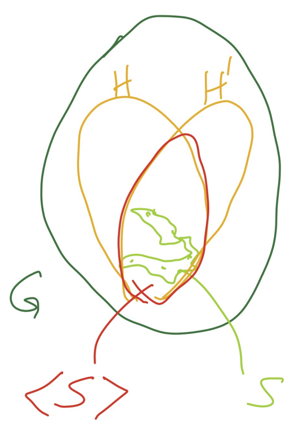


Figure 27.1: The intersection of all subgroups containing S.

[2]Note that $\langle S \rangle = \{e\}$ if $S = \varnothing$.

Example $D_3 = \{I, A, B, 1, 2, 3\}$ is generated by the subset

$$S = \{A, 1\},$$

because $\langle S \rangle$ is forced to contain $A^2 = B$, $1A = 2$, $A1 = 3$, $AA^{-1} = I$. Or see Figure 26.2.

Alternate construction: Observe that $\langle S \rangle$ can be built "from inside" as the set of all possible products of (positive and negative) powers of elements g of S:

$$\langle S \rangle = \{g_1^{\varepsilon_1} \dots g_k^{\varepsilon_k} \bigm| k \geq 0,\, g_1, \dots g_k \in S,\, \varepsilon_1, \dots, \varepsilon_k \in \{\pm 1\}\}, \tag{27.1}$$

where, by convention, the product is I for $k = 0$.

Verification: The RHS is nonempty and closed under group operations, so it is a subgroup containing S. All of its elements must appear in any subgroup containing S. So it is the smallest subgroup containing S. Q.E.D.

Notation: If $x_1, x_2, \dots, x_n \in G$ we write

$$\langle x_1, x_2, \dots x_n \rangle$$

for

$$\langle \{x_1, x_2, \dots, x_n\} \rangle.$$

We say that $x_1, \dots x_n$ are the *generators* of $H = \langle x_1, \dots, x_n \rangle$.

Example

$$\begin{aligned} \langle A, 1 \rangle &= \{I, A, A^{-1}, 1, 1^{-1}, AA, AA^{-1}, A1, A1^{-1}, A^{-1}A, A^{-1}A^{-1}, A^{-1}1, \dots\} \\ &= \{I, A, B, 1, 1, B, I, 3, 3, I, A, 2, \dots\} \\ &= D_3. \end{aligned}$$

Example S_5 is generated by the interlocking transpositions

$$(12), (23), (34), (45).$$

Example (Cyclic groups) The cyclic group $\mathbb{Z}_n = \{1, \sigma, \sigma^2, \dots, \sigma^{n-1}\}$ is generatated by σ:

$$\mathbb{Z}_n = \langle \sigma \rangle.$$

The infinite cyclic group can be written $\mathbb{Z} = \langle 1 \rangle$. For any group G and any element $g \in G$, $\langle g \rangle$ is a cyclic group. The order of g (as an element) equals the order of $\langle g \rangle$ (as a group).

Exercise 27.1 *Which elements in $\mathbb{Z}_n$ generate $\mathbb{Z}_n$?*

Exercise 27.2 *Show that the dihedral group D_n cannot be generated by a single element. It is generated either by two reflections, or by one reflection and one rotation.*

Exercise 27.3 *What is the smallest generating set you can find for the cube group? The icosahedron group?*

28 Cosets

Left and right cosets. Examples in D_3. Properties of the examples. Normal and non-normal subgroups.

References

- Saracino, 82(bottom)-85.
- Rotman, 24-25.

Let G be a group and H a subgroup.

Definition

(1) The sets

$$gH := \{gh \,\big|\, h \in H\}, \qquad g \in G,$$

are called *left cosets* of H in G.

(2) The sets

$$Hg := \{hg \,\big|\, h \in H\}, \qquad g \in G,$$

are called *right cosets* of H in G.

Example Let $G = D_3 = \{I, A, B, 1, 2, 3\}$. Let $H = \{I, 1\}$. The left cosets are

$$IH, AH, BH, 1H, 2H, 3H,$$

which, using the rules

$$\begin{gathered} A^2 = B, \quad B^2 = A, \quad A^3 = B^3 = I, \quad 1^2 = 2^2 = 3^2 = I, \\ 12 = 23 = 31 = A, \quad 21 = 13 = 32 = B, \\ A1 = 1B = 3, \quad A2 = 2B = 1, \quad A3 = 3B = 2, \\ 1A = B1 = 2, \quad 2A = B2 = 3, \quad 3A = B3 = 1, \end{gathered}$$

become

$$\{I, 1\}, \{A, 3\}, \{B, 2\}, \{1, I\}, \{2, B\}, \{3, A\},$$

and after removing duplicates, the left cosets become

$$\{I, 1\}, \{A, 3\}, \{B, 2\}.$$

The right cosets are

$$HI, HA, HB, H1, H2, H3,$$

which become

$$\{I, 1\}, \{A, 2\}, \{B, 3\}, \{1, I\}, \{2, A\}, \{3, B\},$$

and after removing duplicates, the right cosets become

$$\{I, 1\}, \{A, 2\}, \{B, 3\}.$$

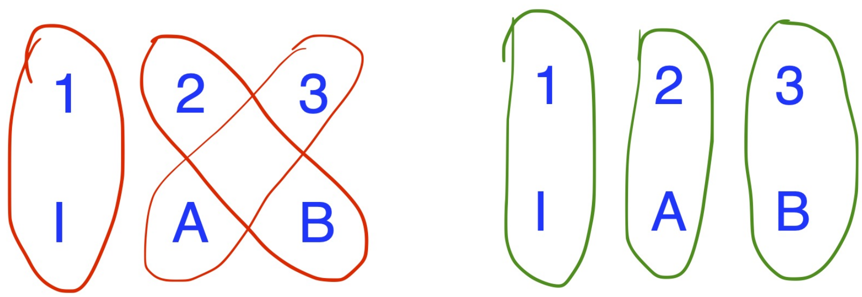


Figure 28.1: Left and right cosets.

Observe in this example:

(1) The six expressions yield 3 left cosets and 3 right cosets.

(2) The cosets are disjoint (after removing duplicates), and their union is G.

(3) $gH \neq Hg$ (usually).

(4) The collection of left cosets and the collection of right cosets are different.

We contrast this with the following example

Example Let $G = D_3 = \{I, A, B, 1, 2, 3\}$. Let $K = \{I, A, B\}$. The left cosets are

$$IK, AK, BK, 1K, 2K, 3K,$$

which become

$$\{I, A, B\}, \{I, A, B\}, \{I, A, B\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\},$$

and after removing duplicates, the left cosets are

$$\{I, A, B\}, \{1, 2, 3\}.$$

The right cosets are

$$KI, KA, KB, K1, K2, K3,$$

which become

$$\{I, A, B\}, \{I, A, B\}, \{I, A, B\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\},$$

and after removing duplicates, the right cosets are

$$\{I, A, B\}, \{1, 2, 3\}.$$

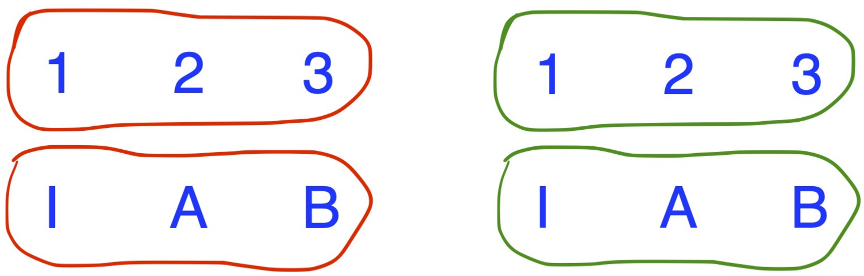


Figure 28.2: Left and right cosets.

Observe in this example:

(1) The six expressions yield 2 left cosets and 2 right cosets.

(2) The cosets are disjoint (after removing duplicates), and their union is G.

(3) $gK = Kg$ (always).

(4) The collection of left cosets equals the collection of right cosets. (This follows from (3)).

A subgroup for which $gH = Hg$ for every g (as in the second example) is called *normal*. Normal subgroups will be important later.

29 Divisibility of order

Coset theorem. Divisibility of order. Index of a subgroup.

References

- Knörrer, 36-37.
- Saracino, 88-91.

Let G be a group and H a subgroup. A set of elements

$$g_1, g_2, g_3, \cdots$$

is called a *system of representatives* for the left cosets of H if each left coset contains exactly one of the g_i's.

Theorem 29.1 (Coset Theorem) *Let G be a group and H a subgroup.*

(a) The left cosets gH, $g \in G$ of H are all the same size.

(b) The left cosets gH, $g \in G$ of H form a partition of G (with repetitions). That is, they cover G, and they are disjoint or equal. It follows that

$$G = \bigsqcup_{i \in I} g_i H$$

is a bona-fide partition, where $(g_i)_{i \in I}$ is a system of representatives for the left cosets of H.

The symbol $\bigsqcup$ means that it is a disjoint union, i.e. differently indexed sets are actually disjoint.[3] An analogous theorem holds for the right cosets.

[3]See `https://math.stackexchange.com/questions/1631396/what-is-the-difference-between-disjoint-union-and-union` for two possible meanings of this symbol. We are using the first one.

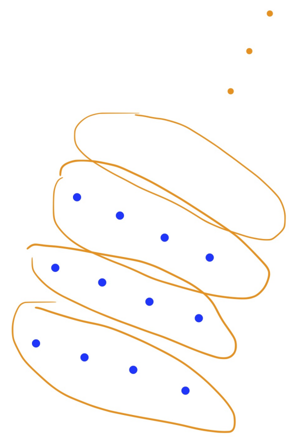


Figure 29.1: Cosets all the same size.

We will prove Theorem 29.1 in the next section. Let us present three corollaries.

Corollary 29.2 *(Lagrange's Theorem) Let G be a finite group. Then the order of H divides the order of G:*

$$\#H \mid \#G \tag{29.1}$$

Proof This follows from the fact that the cosets of H are all the same size and form a partition of G.

□

Corollary 29.3 *Let G be a group. Then there are the same number of left cosets as right cosets.*

Proof This follows from the Theorem and the Corollary in the case of a finite group, but to cover the infinite case, we give an independent proof.

The function

$$X \mapsto X^{-1} := \{g^{-1} \mid g \in X\}$$

takes left cosets to right cosets and right cosets to left cosets. Indeed, it takes

$$gH \mapsto (gH)^{-1} = H^{-1}g^{-1} = Hg^{-1}$$

and

$$Hg \mapsto (Hg)^{-1} = g^{-1}H^{-1} = g^{-1}H.$$

The function is bijective (it is its own inverse). So there are as many right cosets as left cosets.

□

We define the *index* of H in G to be the number of left or right cosets of H in G. The notation is $[G : H]$. If G is finite, then by the reasoning of Corollary 29.2,

$$[G : H] = \#G/\#H.$$

Examples

- $\{I, A, B\}$ has index 2 in D_3. Note $2 = 6/3$.
- $\{I, a\}$ has index 3 in D_3. Note $3 = 6/2$.
- $\langle n \rangle$ has index n in $\mathbb{Z}$, $n \neq 0$.
- $\langle 0 \rangle$ has index ∞ in $\mathbb{Z}$.

□

Corollary 29.4 *The order of an element divides the order of the group.*

Proof $\langle g \rangle$ is a subgroup, and

$$\operatorname{ord}(g) = \#\langle g \rangle \bigm| \#G.$$

□

Exercise 29.1 *Is the converse of Corollary 29.2 true? That is, if m divides $\#G$, must G have a subgroup of order m?*

30 Proof of coset theorem

Right and left multiplication. Proof of coset theorem.

References

- Rotman, p 15, Exercise 1.33
- Same as last section.

Let G be a group, $g \in G$. We define an action of G on G by

$$L_g : G \to G, \qquad h \mapsto gh,$$

called *left multiplication*. Similarly, we have *right multiplication*

$$R_g : G \to G, \qquad h \mapsto hg.$$

Both L_g and R_g are bijective maps, by the cancellation rule.

Exercise 30.1

(a) L_g and R_g are not group isomorphisms.

(b) $L_g \circ R_h = R_h \circ L_g$ (they commute).

(c) $C_g := L_g \circ R_{g^{-1}}$ is a group isomorphism.

Exercise 30.2

(a) $g \mapsto L_g$ is a homomorphism from G to $\mathrm{Perm}(G)$.

(b) $g \mapsto R_g$ is an anti-homomorphism from G to $\mathrm{Perm}(G)$ (it reverses the order of multiplication). $g \to R_{g^{-1}}$ is a homomorphism from G to $\mathrm{Perm}(G)$.

(c) $g \mapsto C_g$ is a homomorphism from G to $\mathrm{Aut}(G)$.

Proof of Coset Theorem 1. Let G be a group, H a subgroup. Then gH is a bijective image of H, because

$$gH = L_g(H),$$

and

$$L_g|H : H \to gH$$

is a bijection. Therefore,

$$\#(gH) = \#H.$$

So all left cosets of H have the same size.

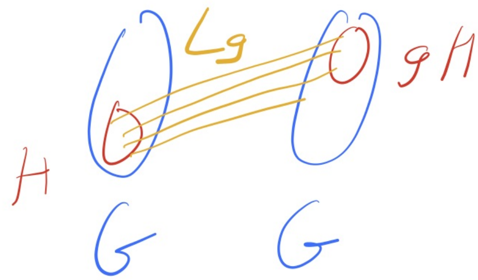


Figure 30.1: L_g is bijective from H to gH.

2. We will show that left cosets either coincide or are disjoint. Suppose sH and tH are left cosets. Assume sH and tH are not disjoint. We will show that $sH = tH$.

Fix an element k in $sH \cap tH$. Then there are $h_1, h_2 \in H$ such that

$$k = sh_1 = th_2.$$

Then

$$s^{-1}t = (h_2)^{-1}h_1 \in H,$$

so

$$s^{-1}tH \subseteq H,$$

so applying L_s to both sides and using the fact that $L_s L_{s^{-1}} = \mathrm{id}_G$, we get

$$tH \subseteq sH.$$

Similarly

$$sH \subseteq tH.$$

So

$$tH = sH,$$

as promised.

3. We will show that the left cosets cover G. We always have

$$g = g \cdot 1 \in gH.$$

So the union of the cosets gH, $g \in G$, covers G.

Together with step 2, this shows that the left cosets form a partition of G, as claimed.

□

Indeed, the above proof shows that

$$sH = tH$$

if and only if

$$s^{-1}t \in H.$$

We write

$$s \sim_L t$$

to denote this relation. It is not hard to prove that $\sim_L$ is an equivalence relation whose equivalence classes are the left cosets of H in G.

Chapter 9

—

31 Conjugate elements

Conjugacy. Conjugacy classes. Preservation of order.

References

- Saracino, 93-94.
- Rotman, 43-44

Definition Let G be a group. Elements g, h of G are *conjugate* in G is there is an element $c \in G$ such that

$$h = cgc^{-1}.$$

We write

$$a \sim b,$$

or $a \sim_G b$.

We have:

Proposition 31.1 *Conjugacy is an equivalence relation.*

Proof

$$\begin{aligned}
&a = eae^{-1} \\
&a = xbx^{-1} \quad \Longrightarrow \quad b = x^{-1}a(x^{-1})^{-1} \\
&(a = xbx^{-1} \text{ and } b = ycy^{-1}) \quad \Longrightarrow \quad a = (xy)c(xy)^{-1},
\end{aligned}$$

so

$$\begin{aligned}
&a \sim a \\
&a \sim b \quad \Longrightarrow \quad b \sim a \\
&(a \sim b \text{ and } b \sim c) \quad \Longrightarrow \quad a \sim c.
\end{aligned}$$

□

Definition The equivalence classes of the conjugacy relation are called *conjugacy classes.*

Example $\{e\}$ is always a conjugacy class by itself.

Example A group is called *abelian* or *commutative* if

$$ab = ba,$$

for all $a, b \in G$. In an abelian group, every element is its own conjugacy class.

Example These are the conjugacy classes in D_3:

$$\{I\}, \{A, B\}, \{1, 2, 3\}.$$

For

$$1A1^{-1} = B, \quad 2A2^{-2} = B, \quad 3A3^{-2} = B,$$

and

$$B2B^{-1} = 1, \quad B1B^{-1} = 3, \quad B3B^{-1} = 2,$$

but A and B are not conjugate to 1, 2, or 3.

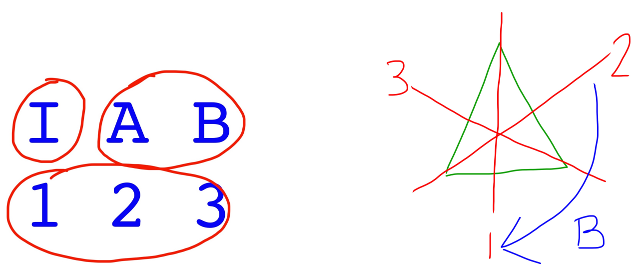


Figure 31.1: Conjugacy classes of D_3; B conjugates 2 to 1.

Preservation of properties

Proposition 31.2 *Conjugate elements have the same order.*

Proof Let G be a group. Let a and cac^{-1} be typical conjugate elements. Then

$$a^k = e \qquad \Longleftrightarrow \qquad (cac^{-1})^k = e$$

for every k. So a and cac^{-1} have the same order.

□

Conjugate elements don't just share their order. Indeed, they share all their algebraic properties. And – as we shall see in the next section – if they are isometries, they share their geometric properties.

Note that the converse of Proposition 31.2 is false:

Exercise 31.1 *Find as many elements as possible of* Isom$(\mathbb{R}^3)$ *that have the same finite order m, but are not conjugate. How does it depend on m?*[1]

[1]See Section 50 for the order 2 case.

32 Conjugate isometries

Reflections. Meaning of conjugacy. Reflections of the cube. Reflections of the dodecahedron.

References: Same as last section.

What does it mean to speak of symmetries, or subgroups of symmetry groups, as being "the same kind"?

Our thesis: *conjugate* implies *same kind*. But not necessarily the converse.

Reflections

Consider reflections in planes in $\mathbb{R}^3$.

These are obviously all the "same kind" of isometry. Below is a picture of a reflection. The point x is reflected through the plane E to a point $\overline{x}$. In symbols:

$$\overline{x} = \sigma_E(x).$$

The plane E cuts the segment $[x, \overline{x}]$ at the midpoint.[2]

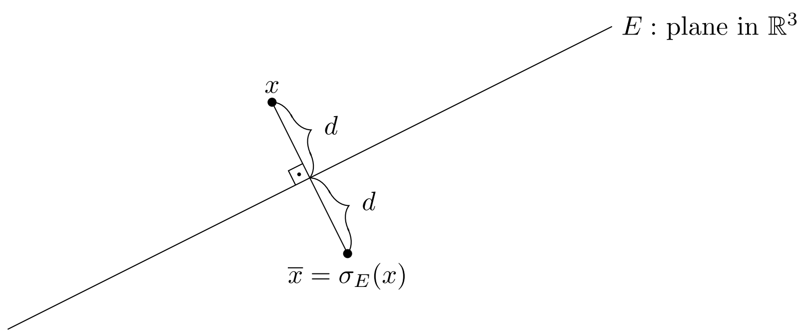


Figure 32.1: Reflection in a plane.

This geometric description *characterizes* the reflection operation. Therefore when we (isometrically) carry the picture somewhere else, we still have a reflection operation, as shown in the picture.

[2]The segment $[x, y]$ is defined by $[x, y] := \{tx + (1-t)y \mid 0 \le t \le 1\}$.

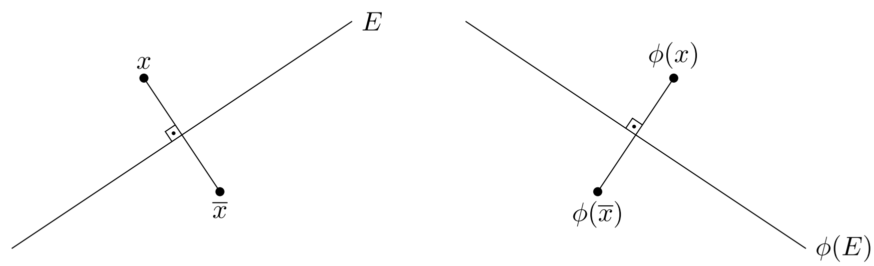


Figure 32.2: Identical pictures.

So $\phi(\overline{x})$ is the reflection of $\phi(x)$ across $\phi(E)$. That is,

$$\phi(\overline{x}) = \overline{\phi(x)}.$$

Using σ_E and $\sigma_{\phi(E)}$ to denote the reflections, this is

$$\phi(\sigma_E(x)) = \sigma_{\phi(E)}(\phi(x))$$

This implies $\phi \circ \sigma_E = \sigma_{\phi(E)} \circ \phi$ or

$$\sigma_{\phi(E)} = \phi \circ \sigma_E \circ \phi^{-1} \tag{32.1}$$

We have proven:

Proposition 32.1 *Let σ be a plane reflection. Then any conjugate of σ is a plane reflection.*

For reflections, the converse is true as well: “Same kind” implies conjugate.

Proposition 32.2 *All plane reflections are conjugate.*

Proof Let σ_E and $\sigma_F \in \mathrm{Isom}(\mathbb{R}^3)$ be plane reflections. Select an isometry $\phi \in \mathrm{Isom}(\mathbb{R}^3)$ so that $\phi(E) = F$. Then

$$\sigma_F = \sigma_{\phi(E)} = \phi \circ \sigma_E \circ \phi^{-1},$$

by equation (32.1).

□

For translations, it is a bit different. They satisfy (32.1), but not the converse. More about that below. First let us interpret (32.1).

A general principle

Equation (32.1) is a special case of a general principle. Suppose that an isometry ψ is specified by giving certain geometric data X. X could be planes, lines, angles or whatever. The principle states:

$$\boxed{\psi_{\phi(X)} = \phi \circ \psi_X \circ \phi^{-1}} \tag{32.2}$$

We can read this formula as follows. In order to compute $\psi_{\phi(X)}$, go back by ϕ^{-1}, pick up the action of ψ_X, and come back forward with ϕ.[3]

The formula can be expressed in words:

$$\boxed{\text{If you map forward the data, you conjugate the isometry.}} \tag{32.3}$$

Or as a commuting diagram:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\psi_X} & \mathbb{R}^n \\ \phi\Big\downarrow & & \Big\downarrow\phi \\ \mathbb{R}^n & \xrightarrow{\psi_{\phi(X)}} & \mathbb{R}^n \end{array}$$

Isometries that are conjugate share their geometric properties. But we classify isometries according to their geometric properties. This means calculating conjugacy classes.

$$\boxed{\begin{array}{l}\text{classification}\\\text{of isometries}\end{array}} \quad \Longrightarrow \quad \boxed{\begin{array}{l}\text{computation of}\\\text{conjugacy classes}\end{array}}$$

Something more subtle

When we named the ten types of rigid motion of Euclidean space in Section 12, we did something more subtle than conjugacy classes. A comparison will allow us to illustrate this point.

Example

(a) "Reflections in planes" constitute one big conjugacy class in $\mathrm{Isom}(\mathbb{R}^n)$. They're all the same.

(b) Translations are conjugate only if they translate by the same distance t. So translations are split up into a lot of little conjugacy classes.

These translation classes are geometrically different, since the distance moved is different. Yet nonzero translations all look broadly similar. They really should

[3] This formula motivated our choosing the convention cgc^{-1} over $c^{-1}gc$.

be grouped together. Brieskorn[4] gives a theoretical basis for this grouping – but beyond the scope of this lecture. See Appendix 48 for a quick explanation and a chart.

Dependence on the ambient group

Here is another quirk of the conjugacy relation: Conjugacy depends strongly on the containing group.

Example (Reflections of the cube)

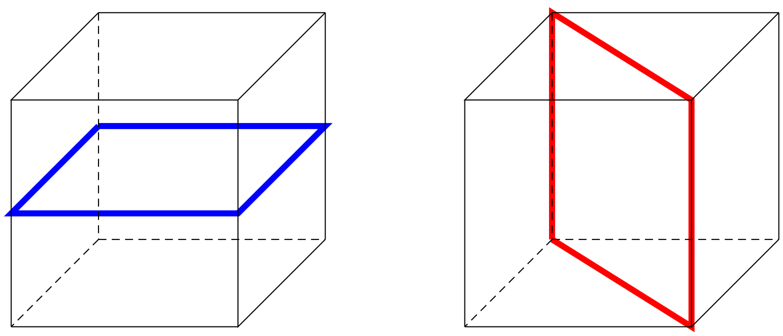


Figure 32.3: Two reflections of the cube.

The reflections σ_C and σ_D are the same kind of isometry, because they are conjugate in $\mathrm{Isom}(\mathbb{R}^3)$. Yet σ_C und σ_D are not the same kind of symmetry of the cube, reflecting the fact that they are not conjugate in $\mathrm{Sym}(W)$:

$$\exists\ \phi \in \mathrm{Isom}(\mathbb{R}^3): \ \sigma_C = \phi\,\sigma_D\phi^{-1}$$
$$\nexists\ \phi \in \mathrm{Sym}(W): \ \sigma_C = \phi\,\sigma_D\phi^{-1}.$$

The last statement follows from the fact that there is no symmetry of the cube that takes D to C.

Example (Reflections of the dodecahedron)

For the icosahedron and dodecahedron, there is only one conjugacy class of reflections. There are 15 reflection planes! Each one is orthogonal to two oppositely placed edges of the 30 edges. They can be visualized in the following two figures.

[4]Brieskorn III, Chap. 5, esp p 215 ff., p 236 Satz 5.11, and the chart on p 240. He calls them "Z-classes".

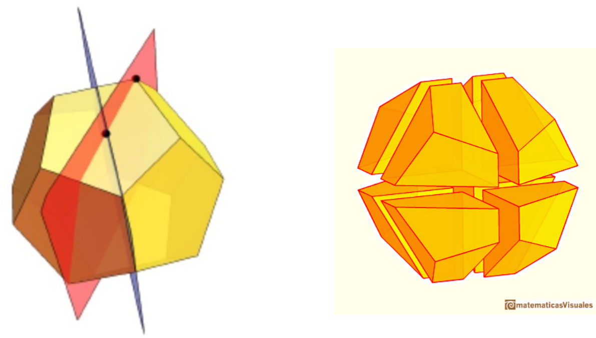


Figure 32.4: Mirror planes for the dodecahedron. (S. Wolfram; matematicasvisuales.com)

In the second figure, note the cubic symmetry, yielding three mirror planes at right angles to each other.

Exercise 32.1 *Show that the 15 reflection planes can be grouped in 5 sets of 3 planes each, where each triplet of planes is a rotated copy of the coordinate planes. All are conjugate.*

The reflections of the cube versus the icosahedron are analogous to the reflections of the square versus the pentagon in Section 5, where the square had two kinds of reflection and the pentagon only one. It seems to be 4's versus 5's.

33 Conjugate subgroups

Inner and outer automorphisms. Conjugate subgroups. Examples: D_3, cube group, icosahedral group. Conjugacy and isomorphism.

References: Same as last section.

Inner and outer automorphisms

Let G be a group and $c \in G$. We define the *inner automorphism* Φ_c as follows:

$$\Phi_c \colon G \to G, \qquad a \mapsto cac^{-1}. \tag{33.1}$$

Proposition 33.1 *Φ_c is an automorphism of G.*

Proof Let $a, b, c \in G$. Then

$$\Phi_c(a)\Phi_c(b) = (cac^{-1})(cbc^{-1}) = cabc^{-1} = \Phi_c(ab), \tag{33.2}$$

so Φ_c is a homomorphism. Bijectivity is clear, because the inverse of Φ_a is $\Phi_{a^{-1}}$.

□

Exercise 33.1 *Show that the map $\Phi \colon G \to Aut(G)$, $c \mapsto \Phi_c$ is a homomorphism.*[5]

The set of all inner automorphisms is written $\mathrm{Inn}(G)$. Being the image of the homomorphism Φ, it is easily seen to be a subgroup of the group of all automorphisms of G:[6]

$$\mathrm{Inn}(G) \le \mathrm{Aut}(G).$$

Automorphisms that are not inner automorphisms are called *outer automorphisms*.

Exercise 33.2 *What can happen if you consider the sequence*

$$G, \mathrm{Aut}(G), \mathrm{Aut}(\mathrm{Aut}(G)), \dots ?$$

Can it keep growing? Must it stabilize or go to the trivial element? Could it go to a nontrivial limit cycle?

[5] This provides a second motivation for our cdc^{-1} convention. Otherwise, this would be an anti-automorphism.

[6] See Exercise 39.2.

It may seem crazy to build these towers of self-referential definitions, but the beauty of mathematics is that it's possible to do this kind of fantasy stuff without generating "fake news".

Exercise 33.3

(a) *Find an outer automorphism of* C_4.
(b) *Find an outer automorphism of* C_5 *that cannot be realized by conjugation in* $\mathrm{Isom}(\mathbb{R}^2)$.
(c) *Find an outer automorphism of* D_n *that acts trivially on* C_n.

Conjugate subgroups

Now let $X \subseteq G$. Then cXc^{-1} is a synonym for $\Phi_c(X)$:

$$cXc^{-1} = \Phi_c(X) = \{cxc^{-1} \,\big|\, x \in X\} \tag{33.3}$$

If H is a subgroup, then cHc^{-1} is also a subgroup (because Φ_c is an automorphism). If $K = cHc^{-1}$ for some $c \in G$, we say that H and K are *conjugate*. We write

$$H \sim K,$$

or $H \sim_G K$. We interpret this to mean "H and K sit in G in the same way". Conjugacy is an equivalence relation on the set of all subgroups of G.

Example (Dihedral group D_3)

In $D_3 = \{I, A, B, 1, 2, 3\}$ we have $\{I, 1\} \sim \{I, 2\} \sim \{I, 3\}$. For example,

$$\{I, 1\} = B\,\{I, 2\}\,B^{-1}.$$

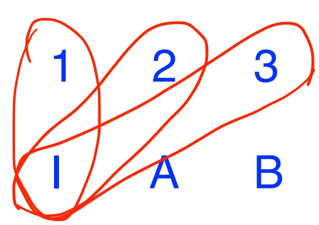


Figure 33.1: Conjugate subgroups in D_3.

Example (Order 4 rotations of the cube) Let

$$R_x, R_y, R_z$$

be 90°-rotations about the x-axis, y-axis and z-axis. Select the rotation direction by the right-hand rule.

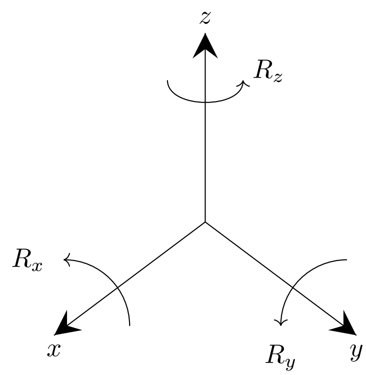


Figure 33.2: Axis rotations in $\mathbb{R}^3$.

For example, $R_x\colon (x,y,z) \mapsto (x,-z,y)$,

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

We have the conjugation relations

$$R_y = R_z R_x R_z^{-1}, \qquad R_z = R_x R_y R_x^{-1}, \qquad R_x = R_y R_z R_y^{-1}.$$

For example, R_z takes the x-axis to the y-axis, so conjugating by R_z takes R_x to R_y. So

$$R_x \sim R_y \sim R_z \qquad\qquad \text{in } \mathrm{Sym}(W).$$

Now let C_x, C_y, C_z be the groups generated by R_x, R_y, R_z:

$$C_x \coloneqq \langle R_x \rangle = \{I, R_x, R_x^2, R_x^3\}, \qquad \text{etc.}$$

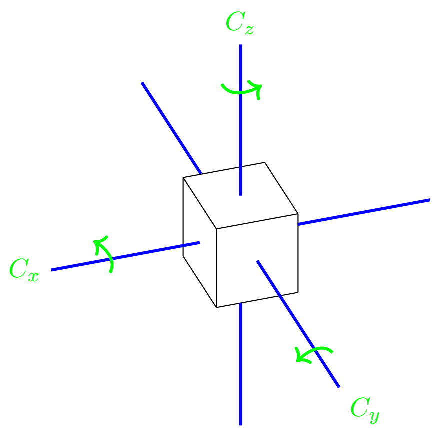


Figure 33.3: The groups C_x, C_y, C_z.

These groups are likewise conjugate in $\mathrm{Sym}(W)$:

$$C_x \sim C_y \sim C_z \qquad\qquad \text{in } \mathrm{Sym}(W).$$

If we forgot the names of the axes, we wouldn't be able to tell these subgroups apart. "Conjugate subgroups can be thought of as the same sets of symmetry operations acting in different locations." (Senechal, p 30)

Example (Icosahedral group) Next we turn to the icosahedral group. Recall from Section 7 that there are five cubes inscribed in the dodecahedron. We had trouble turning this into subgroups.

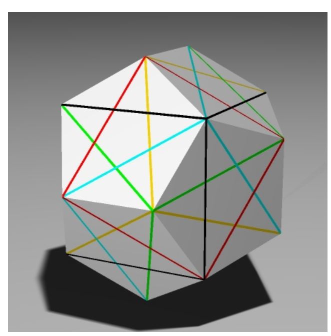


Figure 33.4: Five cubes in the dodecahedron. (S. Tatham)

Exercise 33.4 *Find 5 conjugate subgroups of order 24 in* $\mathrm{Sym}(D)$*.*

This will be answered in Section 37.

Conjugacy and isomorphism

Proposition 33.2 *Conjugate subgroups are isomorphic.*

Proof We already know that $\Phi_c\colon G \to G$ is an isomorphism. Clearly $\Phi_c\big|K\colon K \to \Phi_c(K)$ is also an isomorphism.

□

Example Find two subgroups of a group that are isomorphic, yet not conjugate.

We've already done this. Let σ_C be the reflection of the cube in a coordinate plane, and σ_D the reflection in a diagonal plane. Then

$$(\sigma_C)^2 = (\sigma_D)^2 = I$$

so

$$\langle \sigma_C \rangle \cong \langle \sigma_D \rangle \cong \mathbb{Z}_2,$$

yet these groups are not conjugate in $\mathrm{Sym}(W)$, because as noted above, the elements aren't conjugate in $\mathrm{Sym}(W)$.

Exercise 33.5 *Find two subgroups of* $\mathrm{Isom}(\mathbb{R}^3)$ *that are isomorphic to* $\mathbb{Z}_2 \times \mathbb{Z}_2$*, but not conjugate.*

Chapter 10

—

34 Symmetries of the cube

Symmetry elements. Reflection points, lines, planes. Rotation and roto-reflection axes. Elements of orders 1,2,3,4,6. Missing orders. Summary table. Conjugacy classes.

References

- Knörrer, 17-20.
- S. Zunic, *The Cubic Groups*.

Task: *List the symmetry operations of the cube by conjugacy class.*

The defining structures, or data, that give rise to symmetry operations are called "symmetry elements" in crystallography. We'll organize the symmetry operations according to their symmetry elements.

According to (32.3), this is the same as putting them into conjugacy classes. Two symmetry elements are equivalent if there is a cube symmetry that takes one to the other.

An interesting sub-question: $\#\mathrm{Sym}(W) = 48$. The divisors of 48 are

$$1, 2, 3, 4, 6, 8, 12, 16, 24, 48.$$

How many of these numbers appears as orders of elements of $\#\mathrm{Sym}(W)$?

Answer: 1,2,3,4,6.

Not: 8,12,16,24,48.

Many orders are "missing".[1]

Symmetry elements

Let's list the possible symmetry elements.

The cube has 2 kinds of reflection plane, 3 kinds of rotation axis, 2 kinds of roto-reflection axis, and an inversion center.[2]

We start with pictures. There are 2 types of reflection plane:

[1] The fact that the only possible orders are 1,2,3,4,6, is related to the so-called *crystallographic restriction*. (Senechal pp 18-19, Burns-Glazer pp 31-33, Knörrer pp 61-63)

[2] In addition there are reflection axes, but we count them as rotation axes.

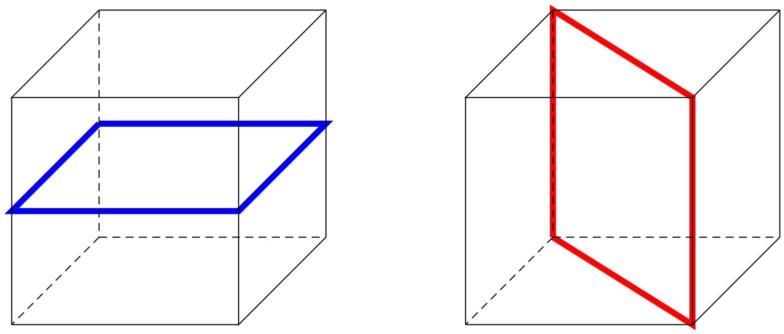


Figure 34.1: Coordinate plane. Diagonal plane.

In 9 variations:

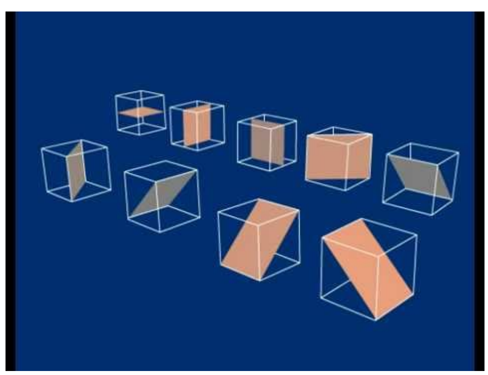


Figure 34.2: Nine reflection planes. (meritnation.com)

There are 3 types of axis in 13 variations.

This includes 3 types of rotation axis (red, blue, green) and 2 types of roto-reflection axis (red, blue). But one of the rotation groups (blue) get subsumed in the corresponding roto-reflection group. The other (red) does not. This will be explained below.

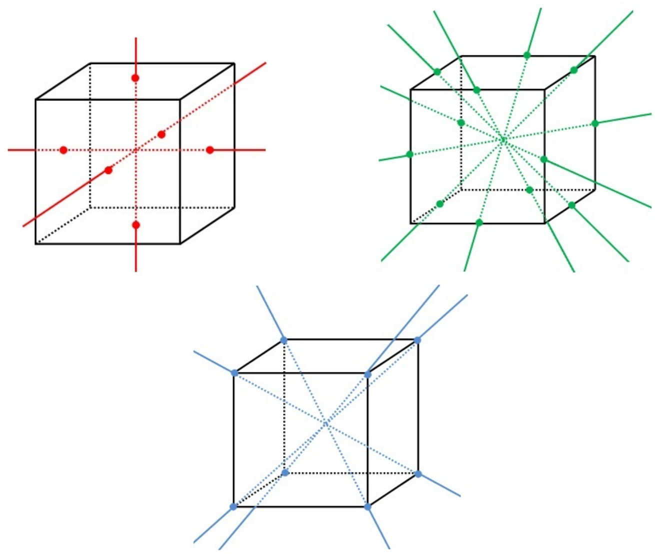


Figure 34.3: Three types of rotation axis. (public.iutenligne.net)

List of symmetries by symmetry elements

All of space:

- 1 identity map

Points:

- Center point : 1 inversion

Planes:

- 3 coordinate planes : 3 reflections
- 6 diagonal planes : 6 reflections

Axes:

Every axis type has rotation symmetries of the cube. Some of them have roto-reflection symmetries.

First:

- 6 short diagonal axes (2-fold rotation) : 6 2-fold rotations

There are no interesting roto-reflections for the diagonal axes. If you compose an order 2 rotation with a reflection, you get only the inversion.

Next:

- 3 coordinate axes
  - (4-fold rotation) : 3 2-fold rotations, 6 4-fold rotations

   ◦ (4-fold roto-reflection) : 6 4-fold roto-reflections

There is a 4-fold reflection R. There is also a 4-fold roto-reflection T. The groups intersect as follows:

$$\begin{tikzcd}
 & R \arrow[dl, dash] \arrow[dr, dash] & & R^3 \arrow[dl, dash] \arrow[dr, dash] & \\
1 \arrow[dr, dash] & & T^2 = R^2 \arrow[dl, dash] \arrow[dr, dash] & & 1 \arrow[dl, dash] \\
 & T & & T^2 &
\end{tikzcd}$$

The 4-fold rotations are R, R^3. The 4-fold roto-reflections are T, T^3. The operation $T^2 = R^2$ is a 2-fold rotation.

Next:

- 4 catty-corner axes (3-fold rotation/6-fold roto-reflection) : 8 3-fold rotations, 8 6-fold roto-reflections

There is a 6-fold roto-reflection S. It generates a group

$$\begin{tikzcd}
 & S \arrow[dl, dash] \arrow[dr, dash] & & S^3 \arrow[dl, dash] \arrow[dr, dash] & & S^5 \arrow[dl, dash] \arrow[dr, dash] & \\
1 & & S^2 & & S^4 & & 1
\end{tikzcd}$$

The 6-fold roto-reflections are S, S^5. The 3-fold rotations are S^2, S^4. S^3 is the inversion, which we omit to avoid double-counting.

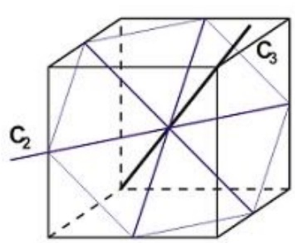


Figure 34.4: Six-fold roto-reflection axis. (S. Zunic)

To get S, rotate by 60 degrees about the catty-corner axis, then reflect across a plane E normal to it at the origin. It takes the cube to itself. Hard to visualize; it is worth observing that E intersects the cube in a *regular hexagon*.

Summary

There is a total of 1+1+3+6+6+3+6+6+8+8 = 48 operations.

You can verify there are 1+6+3+6+8 = 24 proper rotations and 1+3+6+6+8 = 24 improper rotations.

Exercise 34.1 *Why is the collection of numbers the same for the proper and improper rotations?*

Having listed the operations by symmetry elements, we have identified the conjugacy classes in Sym(W). Their sizes are the addends in the sums.

Here it is in a table:

order	#	# in conj class	form	
1	1	1	id	E
2	19	1	inversion	U
		3	coord 180	E
		3	coord refl	U
		6	diag 180	E
		6	diag refl	U
3	8	8	catty 120	E
4	12	6	coord 90	E
		6	coord 90 roto	U
6	8	8	catty 60 roto	U

Too bad about the missing orders 8,12,16,24,48.

Exercise 34.2 *Construct a large group where all elements have order* 1 *or* 2*.*

35 Subgroups of the cube group

Subgroups of orders 1, 2, 3, 4, 6, 8, 12, 16, 24, 48. Three subgroups of order 24. Isomorphism between $\mathrm{Sym}_+(W)$ *and* $\mathrm{Sym}(T)$. *Pyritohedral group. Subgroups of order 12 and 16.*

References: Same as last section.

Task: *List the subgroups of the cube group.*

We won't quite complete this, but we'll answer the sub-question: What are all the orders of subgroups of $\mathrm{Sym}(W)$?

Recall Theorem 29.2: *The order of any subgroup H of G divides the order of G:*

$$\#H \mid \#G. \tag{35.1}$$

We will discover that $Sym(W)$ has subgroups of all theoretically possible orders:

$$1, 2, 3, 4, 6, 8, 12, 16, 24, 48.$$

These are all the factors of 48.

Let us construct a subgroup realizing each of these orders.

Orders 1,2,3,4,6

Subgroups of orders 1,2,3,4, and 6 are easy. We just take the cyclic groups generated by elements of those orders – found in the last section.

There is another interesting subgroup of order 6, besides the cyclic group.

Namely, we should find 3 things to permute. One possibility is to permute the three axes X, Y, Z. That is effected by the six permutation matrices

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}, \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}.$$

Clearly this is isomorphic to $S_3 \cong \{(), (XYZ), (XZY), (XY), (XZ), (YZ)\}$!

Another possibility is to look at the cube along a diagonal. You will see three rays with 120 angles. The symmetry group of what you see is D_3. It consists of the three rotations around the diagonal, supplemented by three reflections.

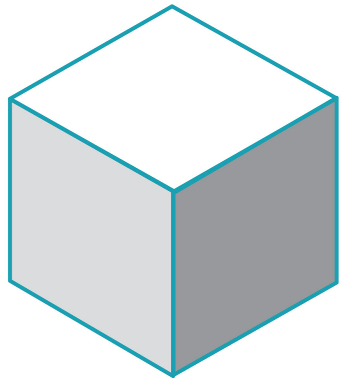


Figure 35.1: A cube viewed along the diagonal. (www.smartgames.eu)

These two noncommutative subgroups (the S_3 and the D_3) are actually identical. I just said it differently.

Order 8

A subgroup K of order 8 is generated by the three commuting reflections

$$\sigma_x, \sigma_y, \sigma_z$$

where $\sigma_x : (x, y, z) \to (-x, y, z)$, for example. This interesting subgroup will be the topic of a later lecture.[3]

Order 24 - three distinct groups.

1) Our first example is the group of orientation-preserving isometries

$$\mathrm{Sym}_+(W) \leq \mathrm{Sym}(W).$$

Exactly half of the symmetries of W are orientation-preserving, so $\mathrm{Sym}_+(W)$ has order 24.

We can prove that

$$\mathrm{Sym}_+(W) \cong S_4 \tag{35.2}$$

by considering the action of $\mathrm{Sym}_+(W)$ on the four catty-corner diagonals of the cube. (Knörrer, p 59, Satz 1.9)

Exercise 35.1 *Verify this.*

2) A second way to get a subgroup of order 24 is the construction in Section 7, where we inscribed a tetrahedron T in the cube to obtain

$$\mathrm{Sym}(T) \leq \mathrm{Sym}(W).$$

[3]There are several other subgroups of order 8 as well.

Recall from Proposition 24.1 that

$$\mathrm{Sym}(T) \cong S_4. \tag{35.3}$$

Therefore $\mathrm{Sym}(T)$ also has order 24.

Indeed, by combining these isomorphisms, we obtain

$$\mathrm{Sym}(T) \cong \mathrm{Sym}_+(W) \tag{35.4}$$

even though the groups sit differently in $\mathrm{Sym}(W)$.

Exercise 35.2 *Prove this isomorphism directly by composing each orientation-reversing element of* $\mathrm{Sym}(T)$ *with the antipodal map* $-I$*.*

3) There is yet a third subgroup of order 24 in $\mathrm{Sym}(W)$. It is called the *pyritohedral group*[4]

$$\mathrm{Sym}(P) \leq \mathrm{Sym}(W). \tag{35.5}$$

It is the symmetry group of a volleyball, or of the Borromean rings. It is clearly a subgroup of $\mathrm{Sym}(W)$. We leave it to the reader to verify that it has order 24, and that it is distinct from the previous two subgroups of order 24.

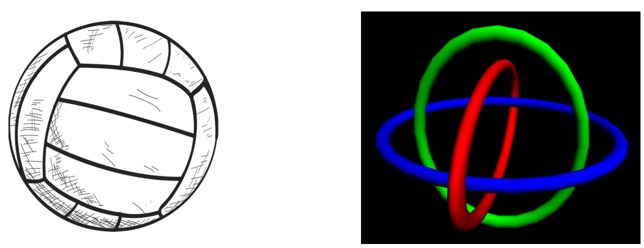


Figure 35.2: Pyritohedral symmetry. (www.oogazone.com; Ronbennett2001, Wikipedia)

Exercise 35.3 *What are the color-preserving symmetry groups of the following three so-called "perfect" 2-colorings of the cube?*

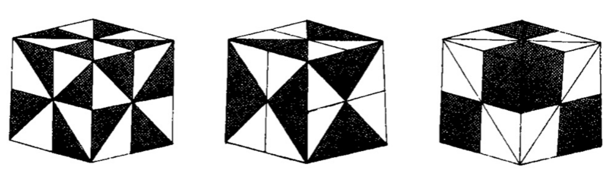


Figure 5.5. The perfect two-colorings of the plate and the cube.

Figure 35.3: Three two-colorings of the cube. (Senechal, p 78)

[4] Knörrer, pp 55-56.

Order 12

We can find a subgroup of order 12 of $\mathrm{Sym}(W)$, namely the orientation-preserving symmetries of the tetrahedron $\mathrm{Sym}(T)$. We have

$$\mathrm{Sym}_+(T) = \mathrm{Sym}(T) \cap \mathrm{Sym}_+(W),$$

the intersection of our two previous examples.[5] In fact

Exercise 35.4 *Show that* $\mathrm{Sym}_+(T)$ *is the intersection of any two of the three groups*

$$\mathrm{Sym}(T), \quad \mathrm{Sym}_+(W), \quad \mathrm{Sym}(P).$$

Exercise 35.5 *Find an order 16 subgroup of* $\mathrm{Sym}(W)$.

[5] But there are also other subgroups of order 12.

36 Tables of subgroups of the cube group

Conjugacy classes of subgroups of Sym(T) *and* Sym(W)

References

- *Octahedral symmetry*, Wikipedia

How close are we to finding all the subgroups of the cube group?

Here is a table of the subgroups of Sym(T), arranged by conjugacy class.[6]

order	#	# in conj class	type	Schoenflies symbol	description
1	1	1			identity
2	9	3	$\mathbb{Z}_2$	C_2	2-fold coord axis
		6	$\mathbb{Z}_2$	$C_s = C_{1h}$	diag reflection plane
3	4	4	$\mathbb{Z}_3$	C_3	3-fold catty axis
4	7	1	$\mathbb{Z}_2 \times \mathbb{Z}_2$	D_2	three 2-fold coord axes
		3	$\mathbb{Z}_2 \times \mathbb{Z}_2$	C_{2v}	two perpendicular diag reflection planes
		3	$\mathbb{Z}_4$	S_4	4-fold coord roto-reflection axis
6	4	4	S_3	C_{3v}	3-fold catty axis three diag reflection planes
8	3	3	$\mathbb{Z}_4 \rtimes \mathbb{Z}_2$	D_{2d}	4-fold coord roto-reflection axis three 2-fold coord axes two vertical reflection planes
12	1	1	A_4	T	orientation-preserving
24	1	1	S_4	T_d	all

Figure 36.1: Conjugacy classes of subgroups of Sym(T).

I expect that there is at least one mistake in the above table.

Exercise 36.1 *Color a tetrahedron to reduce its symmetry groups to each of the above subgroups.*

Exercise 36.2 *Recall that* Sym(T) $\cong S_4$*, acting on the vertices (Proposition 24.1). Write out the subgroups*

$$C_2, C_s, D_2, C_{2v},$$

in cycle notation. Observe how conjugacy relates to the cycle structure.

[6] Observe the notation conflict: the symmetric groups S_3 and S_4 under "type" versus the roto-reflection group S_4 under "Schoenflies symbol".

Here is a table of the subgroups of $\mathrm{Sym}(W)$.

Subgroups of full octahedral symmetry [edit]

Schoe.	Coxeter		Orb.	H-M	Structure	Cyc.	Order	Index
O_h	[4,3]		*432	$m\bar{3}m$	$S_4 \times S_2$		48	1
T_d	[3,3]		*332	$\bar{4}3m$	S_4		24	2
D_{4h}	[2,4]		*224	4/mmm	$Dih_1 \times Dih_4$		16	3
D_{2h}	[2,2]		*222	mmm	$Dih_1^3 = Dih_1 \times Dih_2$		8	6
C_{4v}	[4]		*44	4mm	Dih_4		8	6
C_{3v}	[3]		*33	3m	$Dih_3 = S_3$		6	8
C_{2v}	[2]		*22	mm2	Dih_2		4	12
$C_s = C_{1v}$	[ ]		*	$\bar{2}$ or m	Dih_1		2	24
T_h	$[3^+,4]$		3*2	$m\bar{3}$	$A_4 \times S_2$		24	2
C_{4h}	$[4^+,2]$		4*	4/m	$Z_4 \times Dih_1$		8	6
D_{3d}	$[2^+,6]$		2*3	$\bar{3}m$	$Dih_6 = Z_2 \times Dih_3$		12	4
D_{2d}	$[2^+,4]$		2*2	$\bar{4}2m$	Dih_4		8	6
$C_{2h} = D_{1d}$	$[2^+,2]$		2*	2/m	$Z_2 \times Dih_1$		4	12
S_6	$[2^+,6^+]$		3×	$\bar{3}$	$Z_6 = Z_2 \times Z_3$		6	8
S_4	$[2^+,4^+]$		2×	$\bar{4}$	Z_4		4	12
S_2	$[2^+,2^+]$		×	$\bar{1}$	S_2		2	24
O	$[4,3]^+$		432	432	S_4		24	2
T	$[3,3]^+$		332	23	A_4		12	4
D_4	$[2,4]^+$		224	422	Dih_4		8	6
D_3	$[2,3]^+$		223	322	$Dih_3 = S_3$		6	8
D_2	$[2,2]^+$		222	222	$Dih_2 = Z_2^2$		4	12
C_4	$[4]^+$		44	4	Z_4		4	12
C_3	$[3]^+$		33	3	$Z_3 = A_3$		3	16
C_2	$[2]^+$		22	2	Z_2		2	24
C_1	$[\,]^+$		11	1	Z_1		1	48

Figure 36.2: Subgroups of $\mathrm{Sym}(W)$. (*Octahedral symmetry*, Wikipedia)

Exercise 36.3

(a) Find three subgroups of order 8 in $\mathrm{Sym}(T)$ *that are conjugate.*

(b) Find three subgroups of order 16 in $\mathrm{Sym}(W)$ *that are conjugate.*

Exercise 36.4 *Up to what equivalence relation are subgroups being classified in Figures 36.1 and 36.2?*

Exercise 36.5 *Does* $\mathrm{Sym}(\Delta^n)$ *have any outer automorphisms? What about* $\mathrm{Sym}(W)$*? If so, can they be realized by conjugation by elements of* $\mathrm{Isom}(\mathbb{R}^3)$*?*

Finally...

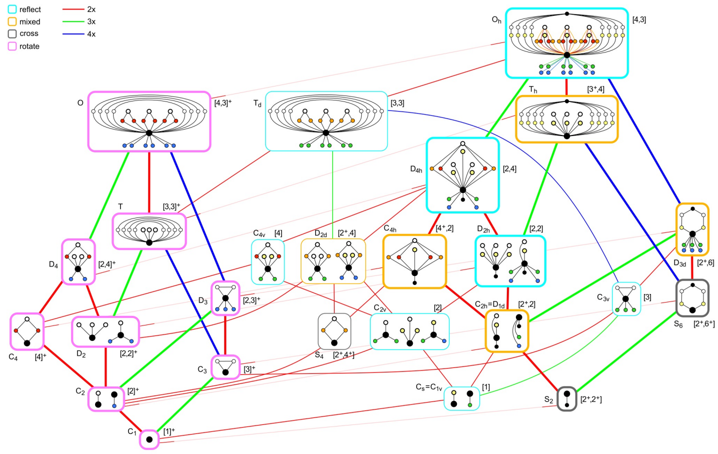


Figure 36.3: Inclusion relations for subgroups of Sym(W). (Watchduck, Wikipedia)

37 Subset relationships among Platonic groups

The pyritohedral group in the icosahedral group. Subset relationships.

References: None.

Now that we know about the pyritohedral group, we can complete our list of inclusions between the symmetry groups of the Platonic solids.

Recall the picture of the five cubes inscribed in the dodecahedron:

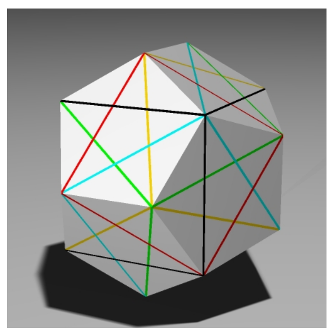


Figure 37.1: Five cubes in the dodecahedron. (S. Tatham)

At the time, we despaired of finding a relationship between the cube group and the icosahedral group. Consider the black cube in the figure. Not all the symmetries of the black cube preserve the dodecahedron.

But look carefully at the black-outlined cube. The reader will see that the dodecahedron differs from the cube in having "tentlike structures" that raise above each face of the cube. These tents must be respected by any cube symmetry that also preserves the dodecahedron. But the cube symmetries that respect the tents are exactly the pyritohedral symmetries of (35.5). Therefore, if P is the black cube equipped with the tents – a figure we call a "pyritohedron" – we find

$$\mathrm{Sym}(P) = \mathrm{Sym}(W) \cap \mathrm{Sym}(D). \tag{37.1}$$

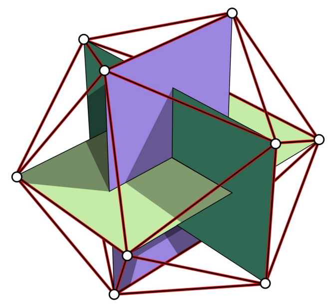


Figure 37.2: Pyritohedron in the icosahedron. (Fropuff & Mysid, Wikipedia)

But we can go further – there are actually five pyritohedra $P_1, \ldots, P_5$ labelled by five colors in Figure 37.1. Since they can be moved to one another by symmetries of the dodecahedron, principle (32.2) implies that the five groups are all conjugate in $\mathrm{Sym}(D)$:

$$\mathrm{Sym}(P_1) \sim \cdots \sim \mathrm{Sym}(P_5).$$

Collecting (7.1), (35.4), (35.5), (37.1) and Exercises 35.2 and 35.4, we get the following theorem. It summarizes what we know about inclusions and isomorphisms of Platonic groups.

Theorem 37.1 *We have in a natural way (once the figures are positioned correctly)*

$$\mathrm{Sym}_+(W),\ \mathrm{Sym}(T),\ \mathrm{Sym}(P) \subseteq \mathrm{Sym}(W), \qquad \mathrm{Sym}(P) \subseteq \mathrm{Sym}(D),$$

and

$$\mathrm{Sym}_+(P) = \mathrm{Sym}_+(T), \qquad \mathrm{Sym}(D) = \mathrm{Sym}(I), \qquad \mathrm{Sym}(W) = \mathrm{Sym}(O),$$

as well as the natural isomorphism

$$\mathrm{Sym}_+(W) \cong \mathrm{Sym}(T).$$

38 Product relationships among Platonic groups

Direct product of groups. Four product relationships with $\mathbb{Z}_2$. Review of subgroup relationships.

References

- Saracino, 55-58.

Definition We define the *direct product* $A \times B$ of two groups A and B with the obvious multiplication:

$$(a, b) \cdot (a', b') := (aa', bb'), \qquad a, a' \in A, \quad b, b' \in B.$$

Example

$$\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6, \qquad \mathbb{Z}_2 \times \mathbb{Z}_2 \ncong \mathbb{Z}_4.$$

To see the equality, write

$$\mathbb{Z}_2 = \{1, a\}, \qquad \mathbb{Z}_3 = \{1, b, b^2\},$$

and observe that $c = (a, b)$ is an order 6 element that generates $\mathbb{Z}_2 \times \mathbb{Z}_3$.

To see the non-equality, observe that every element of $\mathbb{Z}_2 \times \mathbb{Z}_2$ has order 2, whereas $\mathbb{Z}_4$ has an element of order 4. $\mathbb{Z}_2 \times \mathbb{Z}_2$ is called the *Klein four-group*.[7]

The following product relationships among Platonic symmetry groups are fundamental:

Theorem 38.1 *We have*

(a) $\mathrm{Sym}(I) = \mathrm{Sym}_+(I) \times \{\pm I\}$.

(b) $\mathrm{Sym}(W) = \mathrm{Sym}_+(W) \times \{\pm I\}$.

(c) $\mathrm{Sym}(W) = \mathrm{Sym}(T) \times \{\pm I\}$.

(d) $\mathrm{Sym}(P) = \mathrm{Sym}_+(T) \times \{\pm I\}$, *where P is a pyritohedron.*

The reader may compare these new relationships to the inclusions and isomorphisms in Theorem 37.1. They are fully compatible with them and partly explain them.

Proof of Theorem The same idea works for all four of these identities. Namely, $-I$ commutes with every isometry that fixes 0. In addition, $-I$ lies in $\mathrm{Sym}(I)$, $\mathrm{Sym}(W)$ and $\mathrm{Sym}(P)$. We can use $-I$ to double the size of the group.

[7] Saracino p 40, p 115.

We will prove only statement (b). Write

$$A = \mathrm{Sym}_+(W), \qquad B = \{\pm I\}, \qquad C = \mathrm{Sym}(W).$$

Note that $-I \notin A$, but every orientation-reversing element of $\mathrm{Sym}(W)$ can be expressed as an orientation-preserving one times $-I$. It follows that

$$A \cap B = \{I\}, \qquad AB = C, \tag{38.1}$$

and in addition:

$$\textit{Every element of } A \textit{ commutes with every element of } B. \tag{38.2}$$

We claim that (38.1) and (38.2) imply

$$A \times B = C,$$

which will complete our proof. Define $\Phi : A \times B \to C$ by

$$\Phi((a, b)) = ab.$$

This is a homomorphism because

$$\Phi((a, b))\,\Phi((a', b')) = aba'b' = aa'bb' = \Phi((aa', bb')) = \Phi((a, b) \cdot (a', b')),$$

using the fact that elements of A commute with elements of B. It is surjective because $AB = C$. It is injective because

$$\begin{aligned} \Phi((a, b)) = I \quad &\Longrightarrow \quad ab = I \\ &\Longrightarrow \quad b = a^{-1} \in A \cap B = \{I\} \\ &\Longrightarrow \quad (a, b) = (I, I), \end{aligned}$$

which is the identity element of $A \times B$. So Φ is bijective. So $A \times B = C$, as claimed.

□

A direct product isomorphism of this kind, where both factors actually live inside the product group, is called an *internal direct product*. It is generally written with an equality sign, as we have done.

Exercise 38.1 *Why don't we get*

$$Sym(T) = Sym_+(T) \times \{\pm I\}?$$

Chapter 11

—

39 Homomorphisms

Homomorphisms, examples, kernel. Image and preimage of subgroups.

References

- ◦ Knörrer, pp 28-30.
- ◦ Saracino, 109-117 (there is more here than you need).

The objective of this section is to investigate structure-preserving maps between groups. Recall

Definition Let G, H be groups. A *homomorphism* is a map $f\colon G \to H$ such that

$$f(ab) = f(a)f(b)$$

for all $a, b \in G$.

It preserves the multiplication, but can "forget" or "crush down" some of the structure of G. H can be much smaller than G.

An isomorphism is precisely a bijective homomorphism.

Example The canonical homomorphism from $\mathbb{Z}$ to a cyclic group is

$$f\colon \mathbb{Z} \to \mathbb{Z}_m, \qquad j \mapsto \bar{j}.$$

Proposition 39.1 *Let G, H be groups and $f\colon G \to H$ a homomorphism. Then*

$$f(e_G) = e_H, \qquad f(a^{-1}) = f(a)^{-1}.$$

Proof Same as the proof given for isomorphisms. See Proposition 21.1.

□

Definition Let G, H be groups and $f\colon G \to H$ a homomorphism. The *kernel* of f, written $\ker(f)$, is the subset of G defined by

$$\ker(f) := f^{-1}(e_H) = \{g \in G \,|\, f(g) = e_H\} \tag{39.1}$$

Proposition 39.2 *Let G, H be groups and $f\colon G \to H$ a homomorphism. f is injective iff $\ker(f) = \{e_G\}$.*

Proof The $\Longrightarrow$ direction is immediate. We'll prove $\Longleftarrow$. Assume $\ker(f) = \{e_G\}$. Then

$$\begin{aligned} f(a) = f(b) \quad &\Longrightarrow \quad f(a)f(b)^{-1} = e_H \\ &\Longrightarrow \quad f(ab^{-1}) = e_H \\ &\Longrightarrow \quad ab^{-1} \in \ker(f) \\ &\Longrightarrow \quad ab^{-1} = e_G \\ &\Longrightarrow \quad a = b \end{aligned}$$

So f is injective.

□

Example Consider the homomorphism of

$$\mathrm{Sym}(\Delta) = D_3 = \{\underbrace{I, A, B,}_{OE} \underbrace{1, 2, 3}_{OU}\}$$

given by

$$\pi : \mathrm{Sym}(\Delta) \to \{E, U\} \cong \mathbb{Z}_2,$$

defined by $I, A, B \mapsto E$, $1, 2, 3 \mapsto U$. Then the group of orientation-preserving symmetries of Δ is

$$\mathrm{Sym}_+(\Delta) = \ker(\pi) = \{I, A, B\}.$$

Exercise 39.1 *The same construction works for D_n. We get $\#D_n = 2n$, $(D_n)_+ = \ker(\pi) = C_n$, $\#(D_n)_+ = n$.*

$$\{1, a, ..., a^{n-1}, b, ba, ..., ba^{n-1}\} \to \{E, U\}.$$

Figure 39.1: Orientation homomorphism for D_n.

These examples come from the *orientation homomorphism*

$$\pi : \mathrm{Isom}(\mathbb{R}^n) \to \{E, U\}$$

where $U^2 = E$, with kernel precisely the orientation-preserving isometries $\mathrm{Isom}_+(\mathbb{R}^n)$.

Subgroups and homomorphisms

Recall that the *image* of f is defined by

$$\mathrm{im}(f) = f(G) = \{f(g) \,\big|\, g \in G\} \tag{39.2}$$

Proposition 39.3 *Let G, H be groups and $f\colon G \to H$ a homomorphism. Then*

(a) $\ker(f)$ *is a subgroup of G.*

(b) $\mathrm{im}(f)$ *is a subgroup of H.*

Proof Let $f : G \to H$ be a homomorphism.

1. We will show that $\ker(f)$ is a subgroup. First we show that $\ker(f)$ is closed with respect to multiplication and inversion. We have

$$\begin{aligned} a, b \in \ker(f) \quad &\Longrightarrow \quad f(a) = f(b) = e_H \\ &\Longrightarrow \quad f(ab) = f(a)f(b) = e_H e_H = e_H \\ &\Longrightarrow \quad ab \in \ker(f). \end{aligned}$$

So $\ker(f)$ is closed under multiplication. Furthermore

$$\begin{aligned} a \in \ker(f) \quad &\Longrightarrow \quad f(a) = e_H \\ &\Longrightarrow \quad f(a^{-1}) = f(a)^{-1} = e_H \\ &\Longrightarrow \quad a^{-1} \in \ker(f). \end{aligned}$$

So $\ker(f)$ is closed under inversion. Also $e_G \in \ker(f)$, so $\ker(f)$ is not empty. So by Proposition 26.1, $\ker(f)$ is a subgroup.

2. We will show that $\mathrm{im}(f)$ is a subgroup. First we show that $\mathrm{im}(f)$ is closed with respect to multiplication and inversion. We have

$$\begin{aligned} u, v \in \mathrm{im}(f) \quad &\Longrightarrow \quad \exists\, a, b \in G : f(a) = u, f(b) = v \\ &\Longrightarrow \quad uv = f(a)f(b) = f(ab) \in \mathrm{im}(f) \end{aligned}$$

and

$$\begin{aligned} u \in \mathrm{im}(f) \quad &\Longrightarrow \quad \exists\, a \in G : f(a) = u \\ &\Longrightarrow \quad u^{-1} = f(a)^{-1} = f(u^{-1}) \in \mathrm{im}(f). \end{aligned}$$

So $\ker(f)$ is closed under multiplication and inversion. Also $e_H \in \mathrm{im}(f)$, so $\mathrm{im}(f)$ is not empty. So by Proposition 26.1, $\mathrm{im}(f)$ is a subgroup.

□

The following exercise generalizes what we have just proven.

Exercise 39.2 *Let G, H be groups, $f\colon G \to H$ a homomorphism.*

(a) If K is a subgroup of G, then $f(K)$ is a subgroup of G.

(b) If L is a subgroup of H, then $f^{-1}(L)$ is a subgroup of G.

Exercise 39.3 *Find a homomorphism of the unit quaternions onto the Klein four-group.*

40 The kernel is normal

Definition of normal subgroup. The kernel is normal. The cosets are preimages. The nonempty preimages are all the same size.

References

- Saracino, 99-102.

Sketch

Let $f : G \to H$ be a homomorphism. Let $N = \ker(f)$. The kernel has a remarkable property, namely that left cosets are right cosets:

$$cN = Nc,$$

for all $c \in G$. We call a subgroup with this property a *normal subgroup*. We will also establish that the cosets of $\ker(f)$ are precisely the preimages of elements of H:

$$cN = Nc = f^{-1}(u),$$

where $u = f(c)$. It follows from this that all preimages $f^{-1}(u)$ are the same size as $\ker(f)$ (when they are not empty).

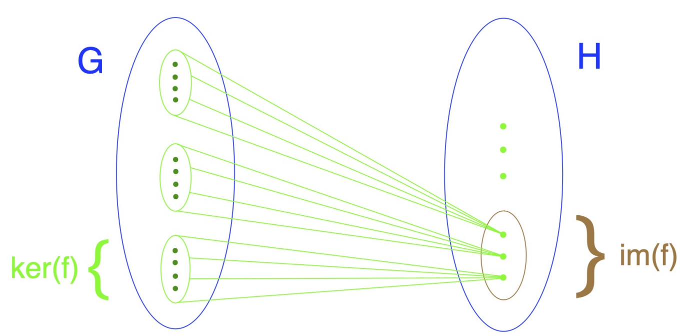


Figure 40.1: All the same size.

Implementation

We will now carry out these promises. Here is the formal definition of a normal subgroup.

Definition Let G be a group, $N \leq G$ a subgroup. N is called *normal* in G if

$$cNc^{-1} = N,$$

or equivalently

$$cN = Nc,$$

for all $c \in G$. (If you like, c commutes past N.)

So N is not conjugate to any other subgroup of G. The two definitions of normality can be converted into each other by applying the bijections R_c and $R_{c^{-1}}$.

Intuitively, a normal subgroup is "recognizable". It is an invariant, self-conjugate, "nameable" subgroup of G, at least in the language of G itself. It cannot be displaced by inner automorphisms![1]

Our main theorem is that the kernel is normal.

Theorem 40.1 *Let G, H be groups and $f: G \to H$ a homomorphism. Then $N = \ker(f)$ is normal in G.*

Proof Let $a \in N, c \in G$. We will show that $cac^{-1} \in N$. Compute

$$f(cac^{-1}) = f(c)f(a)f(c^{-1}) = f(c)e_H f(c^{-1}) = e_H.$$

It follows that $cac^{-1} \in N$ for all a. So $cNc^{-1} \subseteq N$ for all c. So $N \subseteq c^{-1}Nc$ for all c. So $N \subseteq cNc^{-1}$ for all c. So $N = cNc^{-1}$, as claimed. So $cN = Nc$.

□

Next we show that the cosets of the kernel are the preimages of elements of H.

Theorem 40.2 *Let G, H be groups and $f: G \to H$ a homomorphism. Let $N = \ker(f)$. Then*

$$cN = Nc = f^{-1}(u),$$

where $u = f(c)$. So every preimage $f^{-1}(u)$ is a left coset of $\ker(f)$ *or empty.*

Proof Fix $c \in G$. Set $u = f(c)$. Then

$$\begin{aligned} x \in cN &\iff c^{-1}x \in N \\ &\iff f(c^{-1}x) = e \\ &\iff f(x) = f(c) \\ &\iff x \in f^{-1}(u) \end{aligned}$$

[1] An even stronger notion is that of a *characteristic subgroup*, which cannot be displaced even by outer automorphisms.

so

$$cN = f^{-1}(u),$$

as claimed.

□

As a consequence together with Theorem 29.1(a), we get the following corollary.

Corollary 40.3 *All nonempty preimages $f^{-1}(u)$ are the same size as* $\ker(f)$*.*

41 Properties of normal subgroups

Properties of normal subgroups. Examples. Intersection property. Subset property (or not). Image and preimage of normal subgroups.

References: Same as previous section.

Let us explore further general properties of normal subgroups. We write

$$N \trianglelefteq G,$$

when N is normal in G.

Example $\{e\}$ und G are always normal in G.

Example Every subgroup of an abelian group is normal.

Example In $D_3 = \{I, A, B, 1, 2, 3\}$ the subgroup

$$\{I, A, B\}$$

is normal. The subgroups $\{I, 1\}, \{I, 2\}, \{I, 3\}$ are not normal.

Proposition 41.1 *The intersection of an arbitrary family of normal subgroups of G is a normal subgroup of G:*

$$\text{Each } N_\alpha \text{ is normal in } G \quad \Longrightarrow \quad \bigcap_{a \in \mathcal{A}} N_\alpha \ \text{ is normal in } G.$$

Proof Exercise.

□

Here are some exercises, which explore subgroups and homomorphisms for normal subgroups.

Exercise 41.1 *Let $A \subseteq B \subseteq C$.*

(a) If A is a subgroup of B and B is a subgroup of C, show that A is a subgroup of C.

(b) Find an example where A is normal in B and B is normal in C, yet A is not normal in C.

Recall that images and preimages of subgroups are subgroups. Is the same thing true for normal subgroups?

Exercise 41.2 *Let G, H be groups and $f\colon G \to H$ a homomorphism.*

(a) Show the preimage of a normal subgroup of H is normal in G.

(b) Must the image of a normal subgroup of G be normal in H? What can go wrong? What if f is surjective?

42 Quotient groups

Existence and properties of quotient groups. Isomorphism theorem. Examples.

References

- Saracino, 102(middle)-105, 121-126.

We've seen that a homomorphism

$$f : G \to H$$

yields a normal subgroup

$$N = \ker(f).$$

In this section we'll see the converse: any normal subgroup

$$N \trianglelefteq G$$

can be realized as the kernel of a homomorphism f from G to some group.

Indeed, if the homomorphism is surjective, then the image group is determined up to isomorphism.

The idea is to arrange the cosets of N to form a group. This is facilitated by the fact that for a normal subgroup, the left cosets are the same as the right cosets:

$$gN = Ng \qquad \text{for all } g \in G.$$

Theorem 42.1 *Let G be a group and let $N \trianglelefteq G$ be a normal subgroup.*

(1) Then there is a group, called the quotient of G by N and designated

$$G/N,$$

and a surjective homomorphism

$$\pi : G \to G/N$$

with

$$N = \ker(\pi).$$

(2) If

$$f : G \to A, \qquad g : G \to B,$$

are surjective homomorphisms with the same kernel, then there is a (unique) isomorphism

$$h : A \to B$$

such that this diagram commutes

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[dr, "g"'] & A \arrow[d, "h"] \\ & B \end{tikzcd}$$

that is, $g = h \circ f$.

Example $H = \{I, A, B\}$ is normal in $D_3 = \{I, A, B, 1, 2, 3\}$ and

$$D_3/H = \{[I], [1]\} \cong \mathbb{Z}_2.$$

We have written $[x]$ for the coset xH. So $[I] = \{I, A, B\}$ and $[1] = \{1, 2, 3\}$. The cosets of H are the elements of D_3/H.

$\circ$	I	A	B	1	2	3
I	I	A	B	1	2	3
A	A	B	I	3	1	2
B	B	I	A	2	3	1
1	1	2	3	I	A	B
2	2	3	1	B	I	A
3	3	1	2	A	B	I

$\longrightarrow$

$\circ$	$[I]$	$[1]$
$[I]$	$[I]$	$[1]$
$[1]$	$[1]$	$[I]$

Figure 42.1: Quotient map.

Example

$$\mathbb{Z}/n\mathbb{Z} = \{[0], [1], \ldots, [n-1]\} = \mathbb{Z}_n.$$

Here $[k]$ is the coset $k + n\mathbb{Z}$.

Proof of Theorem 1. Define G/H to consist of the cosets

$$H, gH, \ldots$$

of H. Write

$$[g] = gH$$

to make it easier to read. Recall that the cosets $[g]$ are nothing but equivalence classes of the equivalence relation

$$g \sim g' \qquad \iff \qquad g = g'h \qquad \text{for some } h \in H.$$

Define the group multiplication in G/H by

$$[a][b] := [ab]$$

for cosets $[a], [b]$ of H. To see that this is well defined, assume

$$a \sim a', \qquad b \sim b',$$

which means

$$a = a'h, \qquad b = b'k$$

for some $h, k \in H$. Then

$$ab = a'hb'k.$$

But

$$hb' = b'h'$$

for some $h' \in H$ because of the fact that

$$Hb' = b'H$$

for a normal subgroup. Then

$$ab = a'hb'k = a'b'h'k$$

and $h'k \in H$ so

$$ab \sim a'b'.$$

But this implies that $[ab] = [a'b']$, so

$$[a][b] = [ab] = [a'b'] = [a'][b']$$

and the product of two cosets is well defined independent of the representatives. So multiplication is well defined.

It is now trivial to verify the group axioms for G/H with

$$[g]^{-1} := [g^{-1}], \qquad e_{G/H} := [e] = H.$$

2. Suppose

$$f : G \to A$$

is a surjective homomorphism. The association

$$u_A : a \mapsto f^{-1}(a)$$

sets up a bijection between A and the cosets of $\ker(f)$ in G. It is obviously a group isomorphism from A to G/H as constructed above. Therefore

$$A \cong G/H.$$

So any two images A, B are isomorphic. This proves that the isomorphism h exists. It is clear that the only isomorphism that makes the diagram commute is the one we just constructed, namely $(u_B)^{-1} \circ u_A$. So h is unique.

□

43 A quotient of the cube group

An interesting order 8 subgroup H of Sym(W) (it's normal!). A homomorphism onto S_3. Two proofs. Determination of the quotient.

References

- ∘

Our task : Find a normal subgroup of order 8 in $\mathrm{Sym}(W)$ and take the quotient.

Let K be the group generated by the reflections

$$\begin{aligned}\sigma_x &:= \text{ reflection across the } (x=0)\text{-plane}\\ \sigma_y &:= \text{ reflection across the } (y=0)\text{-plane}\\ \sigma_z &:= \text{ reflection across the } (z=0)\text{-plane}\end{aligned}$$

(we mentioned this group before). They commute, and

$$(\sigma_x)^2 = (\sigma_y)^2 = (\sigma_z)^2 = I.$$

It follows that K has order 8 and

$$\begin{aligned}K &= \{I, \sigma_x, \sigma_y, \sigma_z, \sigma_y\sigma_z, \sigma_x\sigma_z, \sigma_x\sigma_y, \sigma_x\sigma_y\sigma_z = -I\}\\ &= \{I, \sigma_x\} \times \{I, \sigma_y\} \times \{I, \sigma_z\}\\ &\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\end{aligned}$$

an 8-element commutative subgroup of $\mathrm{Sym}(W)$.

Note that K takes each coordinate axis to itself, either fixing it or reversing it.

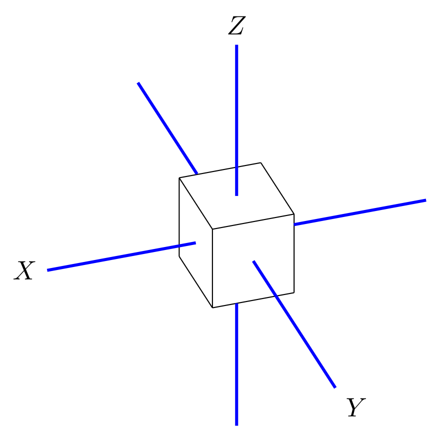


Figure 43.1: Axes.

We will now show that K is normal in $\mathrm{Sym}(W)$. I'll do it in two ways.

Normality - first proof

The first proof is conceptual. The idea is that a normal subgroup of Sym(W) is one that is "recognizable" using data from the cube, even after relabeling the cube and correspondingly renaming the elements of the group (by conjugation).

Let X, Y, Z denote the coordinate axes. Clearly K is the group of isometries that preserve each axis, i.e.

$$\phi(X) = X, \qquad \phi(Y) = Y, \qquad \phi(Z) = Z.$$

That is,

$$K = \bigcap_{A \in T} \mathrm{Sym}(A)$$

where $T = \{X, Y, Z\}$. So K is determined by the set T of axes: $K = K_T$. But T is an intrinsic geometric object that is invariant under symmetries of the cube: $\phi(T) = T$. So by principle 32.2,

$$\phi K_T \phi^{-1} = K_{\phi(T)} = K_T,$$

for $\phi \in \mathrm{Sym}(W)$. So K is normal in the cube group.

Normality - second proof

Now a more concrete, computational proof. It suffices to show

$$aKa^{-1} \subseteq K,$$

for each a in Sym(W). Now Sym(W) is generated by

$$R_x, R_y, R_z, \sigma_x, \sigma_y, \sigma_z,$$

where R_x is a rotation, σ_x is a reflection, etc. And K is generated by

$$\sigma_x, \sigma_y, \sigma_z.$$

So it suffices to prove

$$aua^{-1} \in K,$$

for each generator a of Sym(W) and each generator u of K. (Why?) The cases $a = \sigma_x$, σ_y, σ_z, are trivial since they lie in K. The cases $a = R_x, R_y, R_z$, are similar to one another. So it suffices to show

$$R_x u R_x^{-1} \in K,$$

for $u = \sigma_x$, σ_y, σ_z. We have the commutation rules

$$R_x \sigma_x R_x^{-1} = \sigma_x, \quad R_x \sigma_y R_x^{-1} = \sigma_z, \quad R_x \sigma_z R_x^{-1} = \sigma_y,$$

all of which lie in K. So K is normal in Sym(W).

Example Using K, we can find groups such that[2]

$$A \trianglelefteq B, \qquad B \trianglelefteq C, \qquad \text{yet} \qquad A \trianglelefteq C \text{ fails.}$$

Indeed,

$$\langle \sigma_x \rangle \trianglelefteq K, \qquad K \trianglelefteq \mathrm{Sym}(W),$$

yet

$$\langle \sigma_x \rangle \ntrianglelefteq \mathrm{Sym}(W).$$

For

$$R_y \langle \sigma_x \rangle R_y^{-1} = \langle \sigma_z \rangle \neq \langle \sigma_x \rangle.$$

The quotient

Now that we have a normal subgroup

$$K \trianglelefteq \mathrm{Sym}(W),$$

we can ask: what is the quotient

$$Q := \mathrm{Sym}(W)/K?$$

We have

$$\#Q = \#\mathrm{Sym}(W)/\#K = 48/8 = 6.$$

So Q is probably S_3, although it might be $\mathbb{Z}_6$. Above, we observed that each $\phi \in \mathrm{Sym}(W)$ induced a permutation $\tilde{\phi} \in \mathrm{Perm}\{X, Y, Z\}$. It is easily seen that the function

$$\Phi : \mathrm{Sym}(W) \to \mathrm{Perm}\{X, Y, Z\}, \qquad \phi \mapsto \tilde{\phi},$$

is a homomorphism. Its kernel is clearly K ! Because K is precisely the set of symmetries that leaves each axis invariant. And Φ is surjective. So by Theorem 42.1(2) concerning the uniqueness of the quotient, we get

$$\begin{aligned} \mathrm{Sym}(W)/K &\cong \mathrm{Perm}\{X, Y, Z\} \\ &= \{I, (XYZ), (XZY), (XY), (YZ), (XZ)\} \\ &\cong S_3. \end{aligned}$$

as predicted.[3]

Exercise 43.1 *(Recall Exercise 9.2.) Express the elements of* $\mathrm{Sym}(W)$ *as matrices. Identify the subgroup* K *as a set of matrices. What do the cosets of* K *look like?*

[2] An even simpler example can be found in $A_4 \cong \mathrm{Sym}_+(T)$. See the table in Figure **??**

[3] Note that R_x, R_y, R_z act as transpositions (YZ), (XZ), (XY) and the long-diagonal 3-fold rotations yield (XYZ) and (XZY).

Figure 43.2: Mysterious ottoman. (www.onekingslane.com)

Exercise 43.2 *What is the largest possible symmetry group of the mysterious ottoman, if we are allowed to color the unseen sides as we choose?*

Chapter 12

—

44 Pyrite

Symmetry group, crystal structure, crystal habit. Wide variety of shapes.

References

- *Pyrite*, Wikipedia.
- C. Arrouvela, J-G. Eon, *Understanding the Surfaces and Crystal Growth of Pyrite FeS_2*.
- J. Baez, `https://math.ucr.edu/home/baez/tcu/5_tcu.pdf`.
- Burns-Glazer, 25-26, 345.
- S. Dutch, Miller indices, `https://stevedutch.net/symmetry/millerdx.htm` (contains an error)
- *Miller indices*, Wikipedia.
- B. Morgan, many Peruvian pyrites, `https://www.rasny.org/pyrite`.
- R. Van Dommelen, Nova Scotia pyrite blog, very readable, `http://nsminerals.atspace.com/pyrite.html`.
- *Wulff shape of crystals*, Scholarpedia, `http://www.scholarpedia.org/article/Wulff_shape_of_crystals`.

Figure 44.1: Pyrite. (R. Lavinsky via Wikipedia)

Pyrite is the sulfide mineral FeS_2. It's called pyrite because it makes a usable spark when struck, "pyr" meaning fire in Greek. It's also known as fool's gold because of the color, which has led many a prospector astray.[1]

[1] *Pyrite*, Wikipedia.

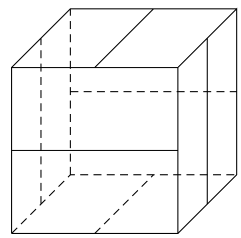


Figure 44.2: Pyritological markings on the cube.

The crystal structure of pyrite is a cubic lattice (so-called "primitive cubic") decorated with iron and sulfur atoms in a certain way. The iron and sulfur atoms are placed in a slightly asymmetrical way, so that the point group is not the full cube group, but the pyritohedral subgroup $\mathrm{Sym}(P)$ of index 2 in $\mathrm{Sym}(W)$. It is the symmetry group of the marked cube shown in Figure 44.2.

The space group, among the 230 possible, is called $\mathrm{Pa}\overline{3}$ (T_h^6).[2]

The crystal structure is shown in Figure 44.3, taken from Arrouvela-Eon. It is generated by their computer studies of the energy and thermodynamics of various cleavage surfaces. Unfortunately I haven't found a picture where the cubic lattice and pyritohedral symmetry is more clearly visible.

Crystal habit

So far we have described an infinite crystal. But real-life crystals come in finite sizes, and they have a definite shape. The typical macroscopic shape or shapes of a particular mineral is called the *crystal habit*.

What determines the crystal habit? It has something to do with the microscopic point group of the crystal structure, but what?

[2] Burns and Glazer, p 345, group no. 205.

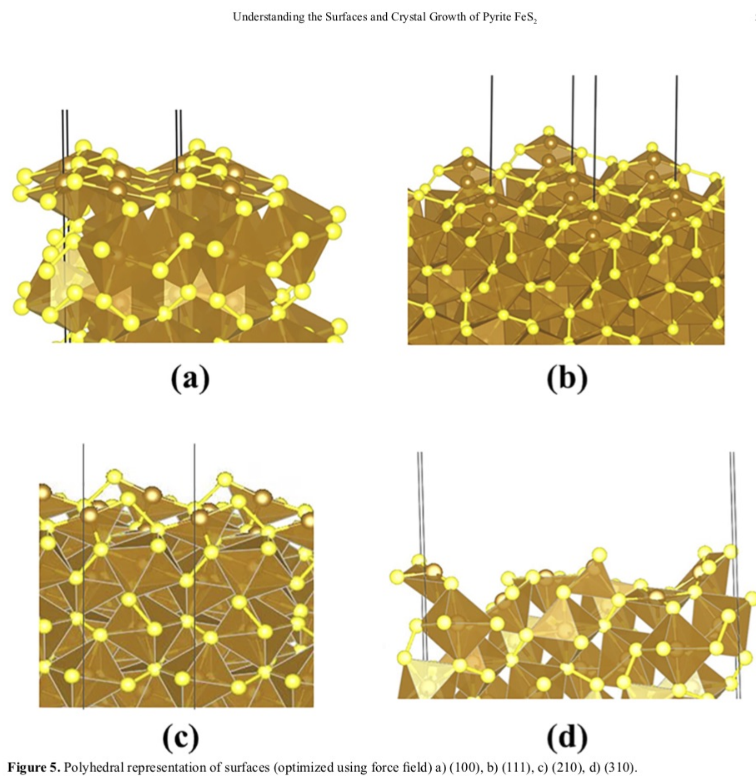


Figure 44.3: Pyrite crystal surfaces with various Miller indices. (Arrouvela-Eon)

Pyrite crystals have several *different* morphologies, notably

cubic, octahedral, non-regular dodecahedral,

and many exotic combinations of these. The dodecahedral one is shown above in Figure 44.1. Below we see a cube and an octahedron.

Figure 44.4: Cube (CarlesMillan, Wikipedia); octahedron (R. Lavinsky via gemstones-list.com).

Here are some rather exotic theoretical polyhedra with the same Sym(P) group symmetry.

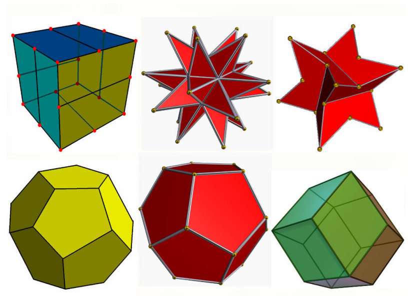


Figure 44.5: Pyritohedral symmetry. (T. Ruen, R. Webb, Cyberpunk, Wikipedia via tumblr.)

In fact, there is a multi-parameter family of compatible polyhedra. Which ones do we see in nature?

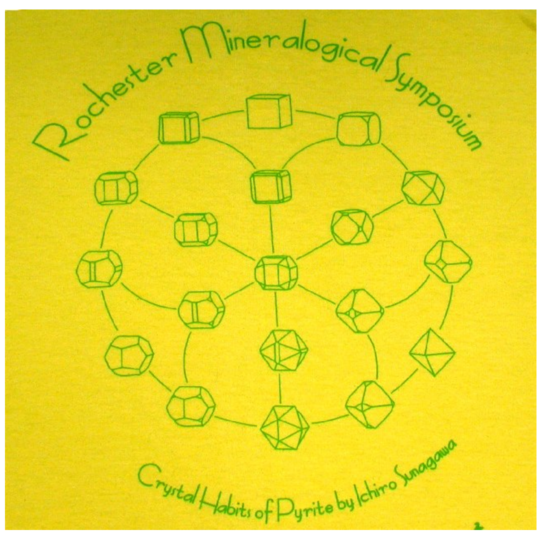


Figure 44.6: Pyritohedral symmetry. (I. Sunagawa)

They all do! But some are more common than others. A great variety of habits is illustrated on B. Morgan's blog `https://www.rasny.org/pyrite`. With many pictures of real pyrite crystals.

45 Pyrite - cleavage planes

Cleavage planes, growth planes. Miller indices. Surface energy. Cubic, octagonal, and 210 (pyritohedron) shapes. Irregular dodecahedron.

References: Same as last section.

The planes you see in a crystal are determined by two mechanisms:

1) Cleavage planes (statics) - determined by energy and geometry alone.

2) Growth planes (kinetics) - determined by thermodynamic variables as well: chemical environment and temperature.

We will give only a very vague overview, intended to stimulate thinking; the observant reader will notice that we don't say much at all.

Cleavage planes

Energy is released when crystals form. Once the energy is released, all other things being equal, it tends to wander away. It is then no longer available to unmake the crystal. Therefore, there is a general tendency for crystals to form. Afterwards, the iron and sulfur atoms are held together by chemical bonds.

If you cut the infinite crystal along some surface, you must break these bonds, and that costs energy. The energy depends on which types of bonds are broken, also slightly including longer-range interactions. It is approximately proportional to the surface area of the cleavage surface, but with a weighting factor that depends on the angle that the surface makes with respect to the geometry of the crystal lattice.

It turns out that the most efficient – and therefore the most likely – cleavage surfaces are planes that have a "slope" consisting of small integer numbers. That is, the crystal gets cut along a plane of the form

$$Q = \{(x, y, z) : mx + ny + pz = \text{const}\},$$

where m, n, p are small integers.

Here (x, y, z) is not the standard (metric) coordinate of a point X, but its expression

$$X = xv_1 + yv_2 + zv_3$$

in terms of a basis v_1, v_2, v_3 for the crystal lattice. As a result, x, y, and z count lattice steps, not distances. Now for pyrite, the underlying lattice is primitive cubic, so we may take v_1, v_2, v_3 to be the standard orthonormal basis for $\mathbb{R}^3$, so the coordinates become standard coordinates, and the distinction disappears. This does not work for "schief" crystal structures.

The triple mnp is written without punctuation in crystallography. The triple is called *Miller indices*. The fact that it is small integers gives a simple step structure to the cleavage of the crystal lattice.

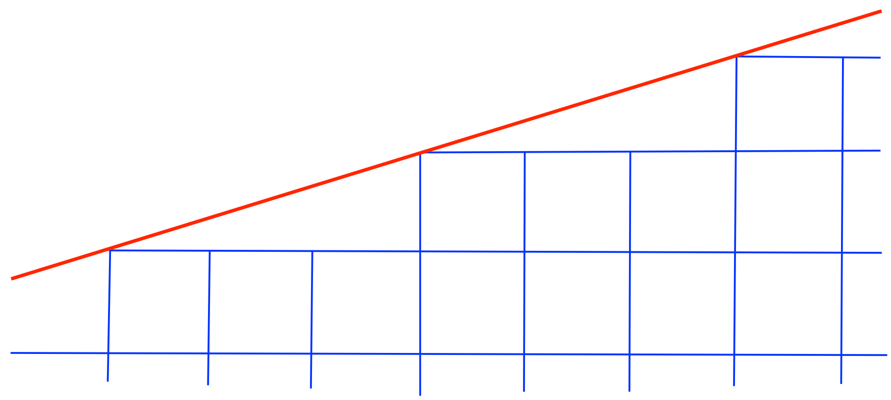


Figure 45.1: Step structure.

The very simplest set of cleavage planes is

$$\pm 100, \quad 0\pm 10, \quad 00\pm 1,$$

which is consistent with a cube. These are six Miller triples, for the six faces of a cube. By symmetry, these planes all have the same energy density, since they stand in the same geometric relation to the underlying lattice.

Another simple set of planes is

$$\pm 1 \pm 1 \pm 1,$$

which corresponds to the faces of an octahedron. There are eight Miller vectors.

The characteristic energy density (per area) will be different for a 111 plane than for a 100 plane. Which one is favored depends on complicated realities, but you might be able to approximate it with a physical model, such as energy-minimizing techniques. That is what Arrouvela-Eon do in their paper.

For pyrite, the 210 series is also important. Here we start with 210 and apply the pyritohedral group $\mathrm{Sym}(P)$. We get

$$210, \quad 021, \quad 102$$

but not

$$120, \quad 012, \quad \text{or} \quad 201,$$

because the needed reflections don't lie in the group. The full set of Miller directions is

$$2\pm 10, \quad 02\pm 1, \quad \pm 102, \quad -2\pm 10, \quad 0{-2}\pm 1, \quad \pm 10{-2},$$

arranged here in twin pairs. There are 12 of them.

Why not 24? After all, the pyrite group $T_h = \mathrm{Sym}(P)$ has 24 elements. But the 0 in the Müller indices 210 causes the stabilizer subgroup of each Miller triple to have size 2. So the orbit of 210 under T_h has only 12 elements.

The omitted 12 plane directions are not totally forbidden, but they lie in a different geometric relation to the detailed lattice of sulfur and iron atoms than the first set of 12, so the characteristic energy density will be different. We assume that we are interested only in the lower energy of the two, so we ignore the "dark 12".

We can form a lot of figures with planar faces modeled on the 210 family of indices, but the most symmetrical one is a dodecahedron with pentagonal faces called a pyritohedron.

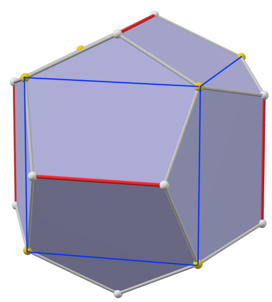


Figure 45.2: Pyritohedron. (Watchduck, Wikipedia, modified)

This is the shape of the real-life pyrite crystal in Figure 44.1 above.

It can be constructed from a cube by erecting a "tent" on each face. The sheets of the tent have slope 1/2 with respect to the nearest cube face. The pentagonal sheets of a given tent form "twin pairs". The tent ridges are shown in red; the gray sloping edges define a hip roof structure. The added blue lines show the cube structure.

It can be visualized via a gif that transitions from the cube to the pyritohedron:

`http://argos.vu/wp-content/uploads/2016/04/ezgif.com-gif-maker.gif`

Here is another nice gif. It transitions from the cube, to the pyritohedron, to the rhombic dodecahedron. All share the pyritohedral symmetry.

`https://fr.wikipedia.org/wiki/Fichier:Pyritohedron_animation.gif`

This pyritohedron is labelled with the Miller indices:

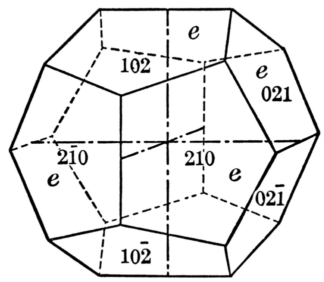


Figure 45.3: Pyritohedron with Miller directions. (W. Dana, *Manual of Mineralogy*)

Here is a picture of a pyrite crystal that shows the "steps" – one step up for every two over. The little cubes can be viewed as the unit cell of the underlying primitive cubic lattice. Remarkably, a similar picture (!!) appears in the crystallographic work of R. J. Haüy, 1822.[3] It's notable that such an accurate picture was produced at such an early point, before the atomic theory was established.

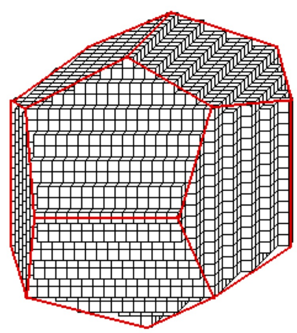


Figure 45.4: Pyrite crystal with steps. (Provenance unknown, via J. Baez)

Incidentally, the step structure is quite real, and is reflected by the appearance of striations on real-life pyrites. The striations go in the 100 direction (Arrouvela-Eon, pp 3, 8).

[3] Burns and Glazer claimed this (pp 25-26), but I couldn't find this picture in Haüy's 2795-page work. What was Haüy trying to depict? No way to be sure without the original context.

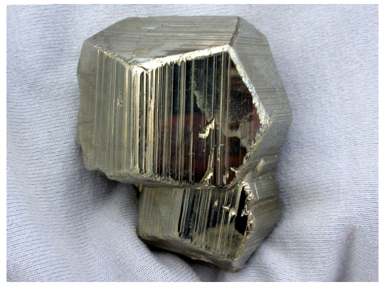


Figure 45.5: Pyrite crystal with striations. (B. Morgan)

Observe that the pentagons are not regular, as shown in the following exercise. So it is not a *regular* dodecahedron.

Exercise 45.1 *Calculate the 20 edge-lengths of pyrite as shone in the picture, assuming that the slope of the "tent" planes is 1/2. Is it a regular dodecahedron?*

Partial Solution: Find the dihedral angles about a pentagon.

The 12 faces of the pyrotohedron are described by the following 3 twin pairs

$$210,\ \ 2{-}10; \qquad 021,\ \ 02{-}1; \qquad 102,\ \ {-}102,$$

and their negatives.

Calculate the angles between face 210 and its five adjacent faces, namely

$$2{-}10(\text{twin}), \qquad 021, \qquad 02{-}1, \qquad 102, \qquad 10{-}2.$$

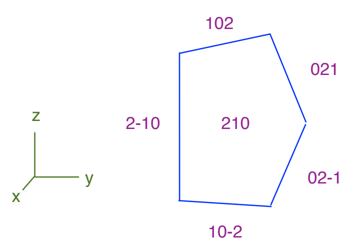


Figure 45.6: Five pentagons around a pentagon.

These angles are

$$180 - \arccos((2,1,0)\cdot(2,-1,0)/5) = 180 - \arccos(3/5) = 126.9 \text{ degrees},$$

$$180 - \arccos((2,1,0)\cdot(0,2,1)/5) = 180 - \arccos(2/5) = 113.6 \text{ degrees}.$$

The remaining three angles are also 113.6. So the dihedral angles around a face are

$$126.9, \qquad 113.6, \qquad 113.6, \qquad 113.6, \qquad 113.6.$$

This already shows the dodecahedron is not regular.[4]

[4]Note that we used the fact that the Miller vectors are orthogonal to the physical faces, which only works in a primitive cubic system.

46 Pyrite - growth planes

Growth planes. Surface energy, thermodynamics, kinetics. Octahedral salt.

References: Same as last section.

Now we come to a snag – and something more interesting. Natural crystals form by growth, not by cleavage. Cleavage can only happen to a crystal that already exists. So what counts is kinetics –

Fundamentally, crystals grow because a crystal has lower energy than the dissociated atoms, but the nature of the growth surface and its environment determine the rate of growth.

At first, the sulfur and iron atoms are swimming around in a solution at random. By accident, a few stick together in an irregular blob. But by accident, certain surfaces of that very small blob resemble some of the low-integer planes described above.

Such planes are stable for energy reasons – some more stable than others, as we have seen. But they are also capable of growing outwards by laying down a new layer of iron and sulfur atoms.

Then the question becomes, not which planes cleave, but which ones grow the fastest? For that determines which planes predominate in the population and are seen to reach macroscopic size.

The relationship of speed to size is rather counterintuitive – the fastest growing planes will go the farthest but be the smallest.[5]

This is illustrated by a picture of a hypothetical crystal in the shape of a non-square rectangle, where the long side grows slowly and the short side grows quickly. It maintains an oblong shape as it grows. The fastest-growing sides are the farthest out, and the shortest. The slowest-growing sides are the nearest, and the longest. The ratio of the side lengths tends to a stable limit. In this way, the shape persists to macroscopic size.[6]

[5] Senechal p 57.

[6] This may be compared to the theory of the Wulff shape; see *Wulff shape of crystals*, Scholarpedia, and references therein.

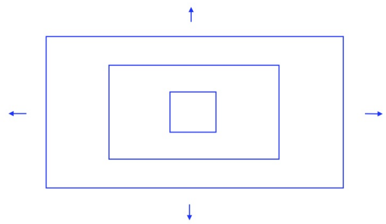


Figure 46.1: Each side grows at a constant rate.

The speed of growth of a face will depend not only on the geometric slope as above (i.e. 210 vs 110 etc) but on chemical and thermodynamic variables such as the relative concentration of sulfur and iron, as well as catalysts, and the temperature.

The reason is that for different slopes mnp, different proportions and angles of sulfur versus iron atoms will be exposed on the outer surface, with geometry conducive to different reaction mechanisms, and the speed of growth will be facilitated by different ratios of ingredients and/or different catalysts. This is alluded to in Arrouvela-Eon, who investigate it analytically and with computer models.

One result: For pyrite, extra sulfur favors the pyritohedral and octahedral shapes, and extra iron favors cubes.

This is an experimental observation, but it is at least partially confirmed in theoretical models (Arrouvela-Eon, p 7).

More exotic environments presumably explain all the mixed shapes in-between, described in Section 44.

Salt

This phenomenon – the dependence of crystal habit on the thermochemical environment – is discussed on random internet threads. And not just for pyrite.

For example, salt (NaCl) that crystallizes from water comes out as cubes, but in the presence of urea, octahedra are favored.[7]

But apparently also Thai fish sauce will do the trick as well:

[7] D. M. Többens citing J. S. T. Gehler, *Physikalisches Wörterbuch*, 1787, `https://www.researchgate.net/post/what_are_the_possible_forms_of_sodium_chloride_crystals`.

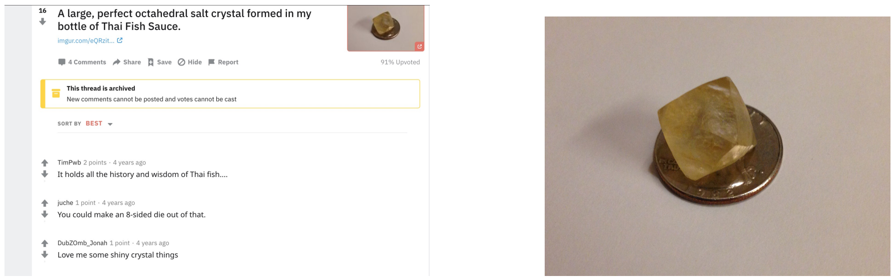


Figure 46.2: Octahedral salt in fish sauce. (Capt_Triskal, reddit)

Chapter 13

Appendices

47 Metric spaces

Metric spaces. Examples. Isometries, symmetries. Fractal examples. Infinite-dimensional examples.

References

- ∘

Let X be a set.

Definition A function

$$f\colon X \times X \to \mathbb{R}$$

is a *metric* on X if for all $x, y, z \in X$ we have

$$\begin{aligned} &f(x,y) \geq 0 && \text{(positivity)} \\ &f(x,y) = 0 \iff x = y && \text{(definiteness)} \\ &f(x,y) = f(y,x) && \text{(symmetry)} \\ &f(x,z) \leq f(x,y) + f(y,z) && \text{(triangle inequality)} \end{aligned}$$

The pair (X, d) is called a *metric space*. We often use X as an abbreviation for (X, d).

The idea of a metric space is to enthrone a notion of *distance* in a fully abstract setting. Distance is the essence of geometry. Notice that metric spaces don't have angles, lengths, or areas – at least not at first. We would have to work hard to define usable versions of these concepts, if it works at all – but we won't do that in this course.

Example This metric space has 3 points.

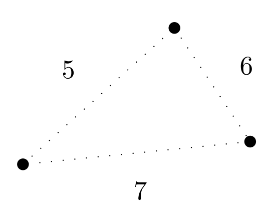


Figure 47.1: A three-point metric space.

Example (R^n,d) is a metric space, as well as any subset of R^n with the induced metric.

Indeed, if (X, d_X) is a metric space and $Y \subseteq X$ any subset, then Y inherits a metric space structure, called the *induced metric*, defined by

$$d_Y := d_X | Y \times Y.$$

That is, we just use the same distances in Y that we were already using in X. It is trivial to verify that

$$(Y, d_Y)$$

is a metric space. (Y, d_Y) is called a *metric subspace* of (X, d_X).

Isometries

We define isometries and symmetries for general metric spaces just as we do for $\mathbb{R}^n$. Namely: an isometry is a distance-preserving bijection, and

$$\mathrm{Isom}(X, Y) := \{\text{isometries } f : X \to Y)$$

$$\mathrm{Isom}(X) := \mathrm{Isom}(X, X)$$

The injectivity follows from the distance-preserving property. The surjectivity must be separately assumed.

Some fractal examples

We've already seen the Sierpinki space. It is a fractal.

Example The limit of the following process is called the Koch snowflake – another fractal space. Both the Sierpinski space and the Koch snowflake are metric subspaces of $\mathbb{R}^2$.

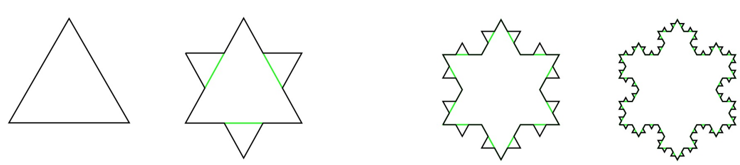


Figure 47.2: Koch snowflake. (Wxs, Wikipedia)

Example Another way to define a fractal metric space is by modifying the Euclidean metric. Define a metric on $\mathbb{R}$ by

$$d(x, y) = \sqrt{|x - y|}, \qquad x, y \in \mathbb{R}.$$

This is easily seen to be a metric. In particular, the triangle inequality follows from the well-known inequality

$$\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}, \qquad a, b \geq 0.$$

Let's call this metric space $(\mathbb{R}, d)$ the $\sqrt{}$-space. What's special about this metric is that it can't be embedded in $\mathbb{R}^n$ as a subset with the induced metric (I don't think). See below. But it has lots of symmetries – infact, it has the same set of isometries that the normal real numbers have!

An infinite-dimensional example

Let Z be the set of all unit vectors of the form

$$e_i = (0, \ldots, 0, 1, 0, \ldots),$$

where the 1 occurs in the i'th place. Note that

$$d(x, y) = \sqrt{2}, \qquad x, y \in Z.$$

Z is sometimes called the infinite-dimensional simplex. It is a so-called *discrete space*, since no point has other points arbitrarily close to it. To justify the "infinite-dimensional" appellation, consider the following exercise.

Exercise 47.1

Show that Z cannot be isometrically embedded in $\mathbb{R}^n$ for any n.

An *isometric embedding* is a distance-preserving injection; it need not be surjective. Equivalently, it is an isometry between the domain and a metric subspace of the target space.

But it gets better:

Exercise 47.2

Find a metric space with four points that cannot be isometrically embedded in $\mathbb{R}^n$ for any n.

Exercise 47.3

Prove or disprove: The $\sqrt{}$ space cannot be isometrically embedded in $\mathbb{R}^n$ for any n.

A useful tool for the previous two questions is Lemma 49.1 (generalized to sets of 4 points in $\mathbb{R}^3$).

Finally an easy one:

Exercise 47.4 *Find a metric on* $\mathbb{R}^2$ *so that every bijective map* $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ *is an isometry.*

48 Ten types of motion of $\mathbb{R}^3$

Listing the rigid motions of $\mathbb{R}^3$ in various ways.

References

- *Euclidean group*, Wikipedia.
- Senechal, pp 23-27.
- Brieskorn III, p 210.

How did we settle on ten types of rigid motion of $\mathbb{R}^3$?

M. Senechal says there are seven types (Senechal pp 23-27). Wikipedia, *Euclidean group*, says there are eight types. Here is the chart from Wikipedia.

Isometries of E(3)

Type of isometry	Degrees of freedom	Preserves orientation?	fixed point?
Identity	0	Yes	space
Translation	3	Yes	-
Rotation about an axis	5	Yes	line
Screw displacement	6	Yes	-
Reflection in a plane	3	No	plane
Glide plane operation	5	No	-
Improper rotation	6	No	point
Inversion in a point	3	No	point

Figure 48.1: Eight kinds of rigid motion of $\mathbb{R}^3$. (*Euclidean group*, Wikipedia)

The Wikipedia table also gives the degrees of freedom, or number of free variables, that are necessary to specify the data of each kind of isometry. For example, to specify an inversion, you have to give the 3 coordinates of the inversion point.

Exercise 48.1 *If you select an isometry of $\mathbb{R}^3$ at random, will it tend to have a fixed point? What about $\mathbb{R}^2$?*

So why did we list 10 types?

Brieskorn III has the ultimate table, based on his method for deciding just which conjugacy classes of isometries to group together. He calls them "Z-classes". There are ten Z-classes in dimension 3.

d	(n_+, n_-, ε)	Z	Typ	$\kappa\ \lambda\ \mu\ \nu$
		$n = 1$		
1	(1,0,0)	I(1)	Indentität	1 0 0 0
	(1,0,1)	$\mathrm{I}(1)^+$	Translation	2 0 1 1
-1	(0,1,0)	O(1)	Inversion	1 1 0 1
		$n = 2$		
1	(2,0,0)	I(2)	Indentität	1 0 0 0
	(2,0,1)	$\mathrm{I}(1)^+ \times \mathrm{I}(1)$	Translation	1 1 1 2
	(0,2,0)	O(2)	Inversion	1 2 0 2
	(0,0,0)	U(1)	Drehung	2 2 1 3
-1	(1,1,0)	$\mathrm{I}(1) \times \mathrm{O}(1)$	Spiegelung	1 2 0 2
	(1,1,1)	$\mathrm{I}(1)^+ \times \mathrm{O}(1)$	Gleitspiegelung	1 2 1 3
		$n = 3$		
1	(3,0,0)	I(3)	Indentität	1 0 0 0
	(3,0,1)	$\mathrm{I}(1)^+ \times \mathrm{I}(2)$	Translation	1 2 1 3
	(1,2,0)	$\mathrm{I}(1) \times \mathrm{O}(2)$	Geraden-Symmetrie	1 4 0 4
	(1,2,1)	$\mathrm{I}(1)^+ \times \mathrm{O}(2)$	Geraden-Gleitsymmetrie	1 4 1 5
	(1,0,0)	$\mathrm{I}(1) \times \mathrm{U}(1)$	Drehung	1 4 1 5
	(1,0,1)	$\mathrm{I}(1)^+ \times \mathrm{U}(1)$	Schraubung	2 4 2 6
-1	(2,1,0)	$\mathrm{I}(2) \times \mathrm{O}(1)$	Spiegelung	1 3 0 3
	(2,1,1)	$\mathrm{I}(1)^+ \times \mathrm{I}(1) \times \mathrm{O}(1)$	Gleitspiegelung	1 4 1 5
	(0,3,0)	O(3)	Inversion	1 3 0 3
	(0,1,0)	$\mathrm{U}(1) \times \mathrm{O}(1)$	Drehspiegelung	1 5 1 6

Zerlegungen von $\mathrm{I}(X)$, Tabelle zu Satz 5.11

Figure 48.2: Ten kinds of rigid motion of $\mathbb{R}^3$. (Brieskorn III, p 210)

The key to Z-classes is to calculate the *centralizer* of each isometry ϕ in $\mathrm{Isom}(\mathbb{R}^3)$. This is the set of isometries that commute with ϕ:

$$Z(\phi) := \{\psi \in \mathrm{Isom}(\mathbb{R}^3)\ :\ \phi \circ \psi = \psi \circ \phi\}.$$

Then two isometries are in the same Z-class, by definition, if their centralizers are conjugate.

To put it differently, ψ is in the centralizer of ϕ provided

$$\psi \circ \phi \circ \psi^{-1} = \phi.$$

But conjugating ϕ by ψ is just *moving* ϕ by ψ. That is, ψ is in the centralizer of ϕ if and only if ψ is a *symmetry* of ϕ. So the centralizer of ϕ in $\mathrm{Isom}(\mathbb{R}^3)$ is really the symmetry group of ϕ:

$$Z(\phi) = \mathrm{Sym}(\phi).$$

($Z(\phi)$ will also be the symmetry group of the data determining ϕ.)

So: Two symmetries are in the same Z-class if and only if their symmetry groups are conjugate in $\mathrm{Isom}(\mathbb{R}^3)$.

If two symmetries are conjugate, clearly their centralizers are conjugate. So Z-classes contain whole conjugacy classes. But Z-classes are in general coarser than conjugacy classes, as shown by the following exercise.

Exercise 48.2

(a) Prove that all plane reflections are in the same conjugacy class and Z-class.
(b) Prove that translations are in the same conjugacy class if and only if they translate by the same distance.
(c) Prove that all nontrivial translations are in the same Z-class.
(d) Why do glide line-reflections form a different Z-class than screw motions with rotation angle $\theta \neq 0, 180$?

Now this makes a lot of sense.

49 Second proof of coincidence

Characterizing points by distances. Direct proof of Proposition 17.1.

References

- Knörrer, pp 5-6, 9-10, 20 (Exercises (i),(ii)).

Recall Proposition 17.1:

> *Two isometries of $\mathbb{R}^2$ that take the same values on 3 non-collinear points must be equal everywhere.*

We give an alternate proof this Proposition. It is adapted from Knörrer pp 5-6, 9-10, 20. It uses the following Lemma, which generalizes observations inside the proof of Lemma 11.2.

Lemma 49.1 *Let A, B, C be noncollinear points in $\mathbb{R}^2$. Then each point X in $\mathbb{R}^2$ is uniquely determined by the three numbers*

$$d(X, A), \qquad d(X, B), \qquad d(X, C).$$

Proof (Lemma) Fix A, B, C. Suppose the numbers

$$a = d(X, A), \qquad b = d(X, B), \qquad c = d(X, C),$$

are known, and we have to find X. Where can X be? If we just know the first two distances a and b, then X must lie on the intersection of the two circles

$$C_a = \{X \ : \ d(X, A) = a\}, \qquad C_b = \{X \ : \ d(X, B) = b\}.$$

The circles intersect in 0, 1, or 2 points, or they coincide. Zero points is impossible. If they coincide, then we are done, because then $A = B$, so A, B, C are collinear. If they intersect in just 1 point, we are also done, because then X is already determined by a and b.

So we may assume that the two circles meet in two distinct points $X' \neq X''$:

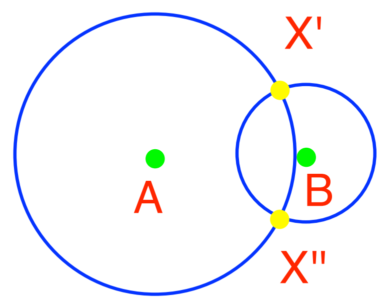


Figure 49.1: Two circles meeting in two points.

So X is nearly determined by just two of the distances. There is just one "bit" of information left to determine.

Is the knowledge of c enough to resolve this remaining ambiguity? Typically, yes, because the circle

$$C_c = \{X \ : \ d(X, C) = c\}$$

will usually pass through one of the points but not the other.

However, there is one situation where X is not determined. That is when C_a, C_b, C_c all three pass through the two points X', X''.

But then, let the line L be the perpendicular bisector of the segment $[X', X'']$. Then L is exactly the locus of points P that are equidistant to X' and X''. But A is equidistant to X' and X'' because

$$d(A, X') = d(A, X'').$$

So A lies on L. Similarly, B and C lie on L. So A, B and C are collinear. This proves the Lemma.

□

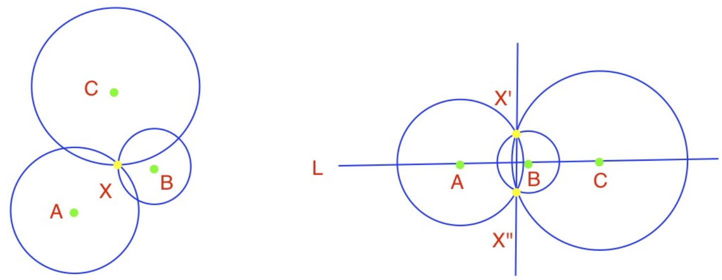


Figure 49.2: Three circles can have two distinct points in common only if the centers are collinear.

Now we prove the Proposition from the Lemma.

Second proof of Proposition 17.1 Let A, B, C be non-collinear points in $\mathbb{R}^2$. Let $\phi, \psi : R^2 \to \mathbb{R}^2$ be isometries that agree on A, B, C. Let X be any point of $\mathbb{R}^2$. It suffices to prove that $\phi(X) = \psi(X)$.

Set

$$a = d(X, A), \qquad b = d(X, B), \qquad c = d(X, C).$$

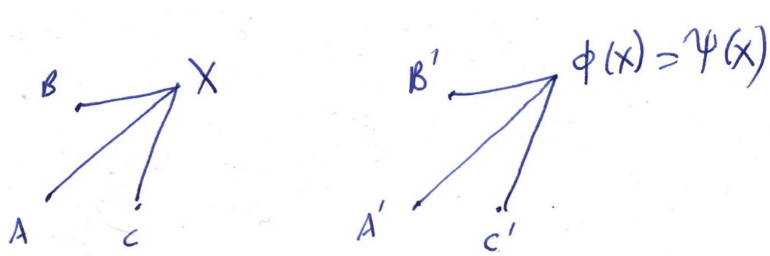


Figure 49.3: The image points $\phi(X)$, $\psi(X)$ are determined by their distances to the reference points A, B, C.

Since ϕ and ψ agree on A, B, C, we may define

$$A' = \phi(A) = \psi(A), \qquad B' = \phi(B) = \psi(B), \qquad C' = \phi(C) = \psi(C).$$

Because A, B, C are not collinear and ϕ and ψ are isometries, it follows that A', B', C' are not collinear.

Because ϕ and ψ are isometries, we have

$$d(\phi(X), A') = a, \qquad d(\phi(X), B') = b, \qquad d(\phi(X), C') = c$$

and

$$d(\psi(X), A') = a, \qquad d(\psi(X), B') = b, \qquad d(\psi(X), C') = c.$$

But by the Lemma, the points $\phi(X)$ and $\psi(X)$ are uniquely determined by these relations, that is,

$$\phi(X) = \psi(X).$$

Since X was arbitrary, $\phi = \psi$.

□

The method of proof can be generalized to higher dimensions – using four intersecting spheres in $\mathbb{R}^3$, for example. Look at Knörrer pp 5-6, 9-10, 20 for hints.

50 Involutions

Involutions are always reflections across affine subspaces.

References

- Brieskorn, p 15, Prop. 1.8.

Theorem 50.1 *An isometry of $\mathbb{R}^n$ that solves $\phi^2 = I$ is reflection in an affine subspace of some dimension $0 \le k \le n$.*

Anything whose square is the identity is called an *involution*.

Proof Let ϕ be an isometry of $\mathbb{R}^n$ that solves $\phi^2 = I$. Let E be the fixed-point set of ϕ. Recall from Theorem 17.3 that E is an affine subspace of $\mathbb{R}^n$.

We claim that $\phi = \sigma_E$, the reflection in E.

Let $x \in \mathbb{R}^n$. Assume

$$x \notin E.$$

Then $\phi(x) \neq x$, and the segment

$$S = [x, \phi(x)]$$

has positive length. We wish to prove that S meets E orthogonally at its midpoint.

Suppose E has dimension k. Let F be the $(n-k)$-dimensional affine subspace that contains x and is orthogonal to E. Suppose F meets E at m.

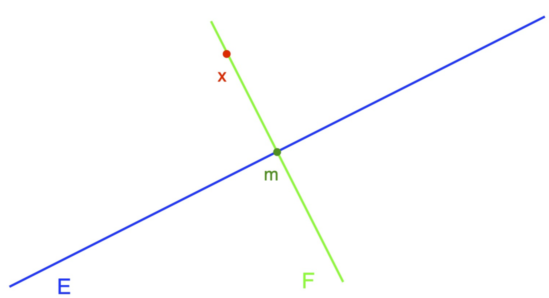


Figure 50.1: F is orthogonal to E.

Now

$$\phi(E) = E, \qquad \phi(m) = m,$$

so

$$\phi(F) = F.$$

So

$$\phi(x) \in F.$$

So

$$S \subseteq F.$$

Because $\phi^2(x) = x$, ϕ interchanges the endpoints of S. So

$$\phi(S) = S.$$

It follows that ϕ fixes the midpoint of S. But the only fixed point of ϕ in F is m. So m is the midpoint of S. Since F meets E orthogonally, we have proven

The segment $[x, \phi(x)]$ meets E orthogonally at its midpoint m.

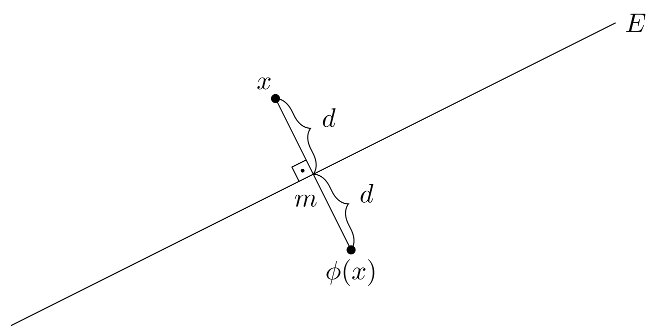


Figure 50.2: Geometric effect of ϕ.

But this is precisely the defining characteristic of the reflection $\sigma_E(x)$. So

$$\phi(x) = \sigma_E(x), \qquad x \notin E.$$

But also

$$\phi(x) = \sigma_E(x), \qquad x \in E.$$

So

$$\phi = \sigma_E.$$

□

Remark: This theorem can be proven using linear algebra. Using a fixed point of ϕ as the origin, ϕ becomes a linear map. It is self adjoint, so it has an eigenspace decomposition. The eigenvalues are ± 1. So $\mathbb{R}^n = E \oplus F$, where ϕ fixes E and acts as an inversion on F.

51 Quaternions

Quaternion multiplication.

References

- Saracino, 47-49 (for the 8-element group Q_8).

Definition The quaternions are an extension of the complex numbers. The underlying set is the vector space

$$\mathbb{H} := \{a1 + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\},$$

spanned by independent vectors 1, i, j, k. $a1$ is abbreviated a. Addition is defined by

$$\begin{aligned}(a_1 + b_1 i + c_1 j + d_1 k) &+ (a_2 + b_2 i + c_2 j + d_2 k) \\ &= (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k.\end{aligned}$$

Multiplication is defined by the following rules:

$$\begin{gathered} i^2 = j^2 = k^2 = -1 \\ ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j, \end{gathered}$$

together with the rule $1u = u1 = u$ and the distributive law. All of the field axioms hold except for commutativity. In particular, the inverse of

$$u = a + bi + cj + dk \neq 0$$

is

$$u^{-1} = (a - bi - cj - dk)/(a^2 + b^2 + c^2 + d^2).$$

The unit quaternion group is the 8-element set

$$Q_8 := \{\pm 1, \pm i, \pm j, \pm k\},$$

with quaternion multiplication. It is closed under multiplication and taking inverses, so it is a group.

52 Every finite group has a figure

Finite groups as symmetry groups of figures. Two lemmas. Cayley's theorem. Decorating a simply transitive orbit.

References: None.

Question: Is every finite group the symmetry group of some figure?

Theorem 52.1 *Let G be a finite group. Then there is a finite set F in some $\mathbb{R}^N$ and an embedding $G \subseteq O(N)$ such that*

$$G = \mathrm{Sym}(F).$$

In fact, we can take $N = \#G$ and $\#F = N(N+1)$.

The theorem follows from two Lemmas.

Lemma 52.2 *Let G be a finite group. Let $N = \#G$. There is an embedding $G \subseteq O(N)$ and a point x in $\mathbb{R}^N$ such that G acts simply transitively on $G \cdot x$.*

Proof Let G be a finite group. Now G acts simply transitively on itself by left multiplication. Let us make this into an action on some $\mathbb{R}^n$ by isometries.

Let $N = \#G$. Consider R^N with an orthonormal basis u_h, $h \in G$ labeled by elements of G. Let $g \in G$ permute the u_h's by left multiplication:

$$g \,:\, u_h \mapsto u_{gh}.$$

This extends by linearity to an orthogonal action of G on $\mathbb{R}^N$. G acts simply transitively on the orbit

$$G \cdot u_e = \{u_h \bigm| h \in G\}.$$

It follows that the action is faithful, that is, the associated homomorphism from G to $O(N)$ is injective. So we may identify G with a subgroup of $O(N)$.

□

Exercise 52.1 *Verify Cayley's Theorem: Any finite group can be realized as a subgroup of a permutation group.*

Exercise 52.2 *Improve the numbers in Theorem 52.1 to $N = \#G - 1$ with a corresponding improvement in $\#F$.*

It is clear that

$$G \subseteq \mathrm{Sym}(G \cdot x).$$

Could $G \cdot x$ be the figure we seek?

The problem is that $G \cdot x$ may have additional symmetries.

Example Let $G = C_m$ acting on $\mathbb{R}^2$. Let $x \neq 0$. Then $G \cdot x$ is a regular n-gon, and G acts simply transitively on $G \cdot x$. But $\mathrm{Sym}(G \cdot x) = D_n$, which is larger than C_n.

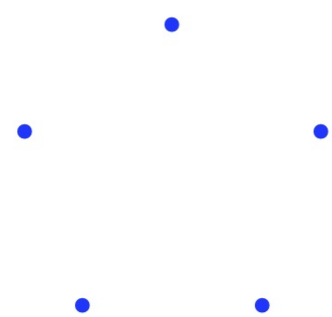


Figure 52.1: The orbit $G \cdot x$ for $G = C_5$.

We construct F from $G \cdot x$ by "marking" or "expanding" each point of $G \cdot x$ in an asymmetrical way. The markings follow the action of G and are compatible with it, but they eliminate any symmetries of $G \cdot x$ that are not in G.

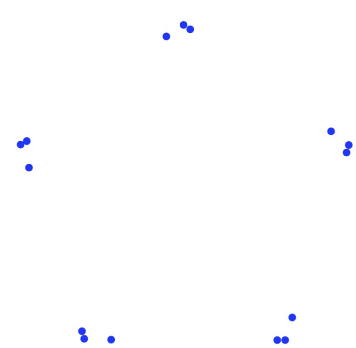


Figure 52.2: Decorating the orbit.

Lemma 52.3 *Let $G \subseteq O(N)$ be a finite group. Suppose G acts simply transitively on $G \cdot x$. Then there is a slight modification F of $G \cdot x$ such that*

$$G = \mathrm{Sym}(F).$$

We may take F to be finite of size $(\#G)(N+1)$.

The proof of this lemma completes the proof of the theorem.

Proof Set

$$d := \text{minimum distance between elements of } G \cdot x.$$

Let R be a small, asymmetrical figure in $\mathbb{R}^N$ such that

a) R lies in a ball of radius $d/10$ about 0,

b) $\mathrm{Sym}(R) = \{I\}$,

c) $\#R = N + 1$.

(See the Exercise below.) Define

$$F = \bigcup_g g(x + R).$$

All the "minifigures" $g(x + R)$, $g \in G$, are disjoint because G acts simply transitively and R is small. All of these minifigures are isometric. The number of points is $\#F = (N + 1)(\#G)$.

We now prove $G = \mathrm{Sym}(F)$.

1) $G \subseteq \mathrm{Sym}(F)$.

Let $g \in G$. Then

$$\begin{aligned} g(F) &= g\left(\bigcup_h h(x+R)\right) \\ &= \bigcup_h g(h(x+R)) \\ &= \bigcup_h h(x+R) \\ &= F, \end{aligned}$$

proving 1).

2) $\mathrm{Sym}(F) \subseteq G$.

Let $\phi \in \mathrm{Sym}(F)$. Because R is so much smaller than d, for each $g \in G$, ϕ must take each minifigure $g(x + R)$ isometrically to another minifigure $g'(x + R)$. In particular, there is h in G such that ϕ takes

$$x + R$$

isometrically to

$$h(x+R).$$

We claim: $\phi = h$.

We see that $h^{-1}\phi$ takes $x+R$ isometrically to $x+R$. So

$$\psi = T_{-x}\, h^{-1}\, \phi\, T_x$$

takes R isometrically to R. Since $\mathrm{Sym}(R) = \{I\}$, we have $\psi = \mathrm{id}$. So $\phi = h$. So $\phi \in G$. So

$$\mathrm{Sym}(F) \subseteq G,$$

as desired for 2).

□

Exercise 52.3 *Construct a set R in $\mathbb{R}^N$ with $\mathrm{Sym}(R) = \{I\}$ and $\#R = N+1$.*

Exercise 52.4 *Instead of a finite set, can we make F be a convex body?*

53 Exercises in search of a home

Exercise 53.1 *Let P_n be a regular n-gon in the plane. Let R be a rotation by $2\pi/2n$. Let Z be inversion through the origin.*

(a) Does Z belong to $\mathrm{Sym}(P_n)$?

(b) Does R belong to $\mathrm{Sym}(P_n)$?

(c) Do P_n and $R(P_n)$ have the same set of symmetries?

(d) Does a given reflection symmetry have the same "type" with respect to P_n as it does with respect to $R(P_n)$?

Exercise 53.2 *Let D be a triangle in the plane. Note that it has more symmetries in $\mathbb{R}^3$ than in $\mathbb{R}^2$.*

Exercise 53.3 *We can write D_4 as*

$$D_4 = \mathrm{Sym}(\square) = \{I, A, A^2, A^3, Z, ZA, ZA^2, ZA^3\}.$$

It has four rotations and four reflections. Here A is a 90-degree rotation and Z is any one fixed reflection.

Exercise 53.4

(a) Compute the number of symmetries of the dodecahedron and the icosahedron.

(b) Show that the dodecahedron (and likewise the icosahedron) has 15 mirror planes and they are all of the same type.

Exercise 53.5 *The coordinates of the vertices of a regular icosahedron of edge length 2 are:*

$$\begin{gathered}(0, \pm 1, \pm\phi)\\(\pm 1, \pm\phi, 0)\\(\pm\phi, 0, \pm 1)\end{gathered}$$

where $\phi = (1+\sqrt{5})/2$ is the golden ratio.

(a) Find the smallest central angle between two vertices of the icosahedron. Verify that it occurs 30 times.

(b) Find 5 cubes in the icosahedron.

(c) Find a few mirror planes and describe how to find the rest of them.

Exercise 53.6 *Consider the superposition of three cubes at the top of Escher's tower on the left.*

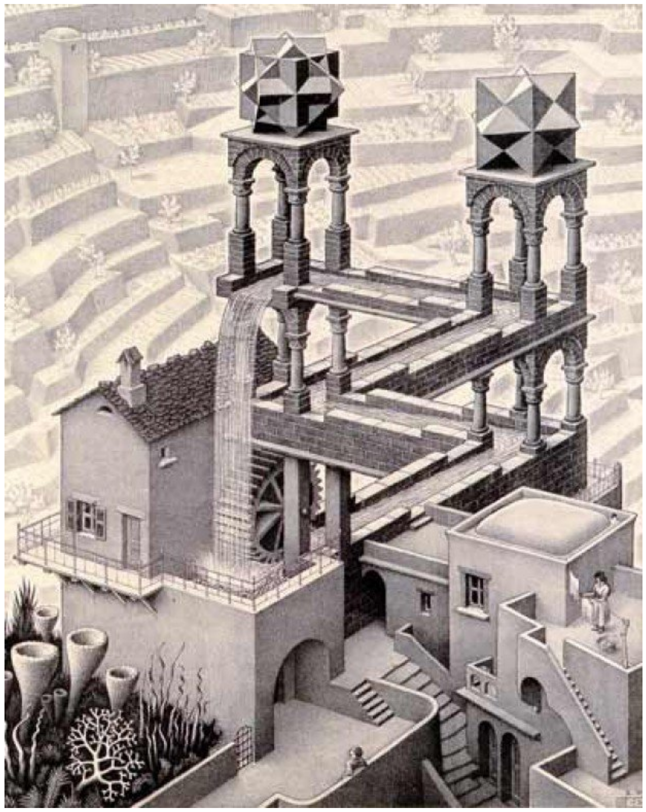


Figure 53.1: Waterfall with multi-cube atop it. (M. C. Escher via churchm.ag)

Let L be the collection of all lines that pass through the common center of the cubes as well as a vertex, face center, or edge midpoint of a cube.

(a) Verify that L has 33 elements.

(b) Verify the following obscure property of L. Suppose that we color each line in L either red or blue. It is impossible to do so in such a way that each set a, b, c of three orthogonal lines

$$a \perp b \perp c \perp a$$

contains two red lines and one blue line.

See `http://www.tum.dds.nl/polyh/ClassiC.htm`, `http://www.tum.dds.nl/polyh/Iris.htm` for models you can manipulate.

(This geometry problem has significance for the Bell-Kochen-Specker theorem in

quantum mechanics.[1]*)*

Figure 53.2: Flag cube. (Provenance unknown)

Exercise 53.7 *What is the maximum possible size of the symmetry group of the flag cube?*

Exercise 53.8 *Let H be a subgroup of G. For which g is the set $gH := \{gh \big| h \in H\}$ a subgroup of G?*

Exercise 53.9 *A proper subgroup of G is any subgroup $H \subseteq G, H \neq G$.*

- *(a) Let G be a group such that every proper subgroup is cyclic. Is G then cyclic?*
- *(b) Let G be a group such that every proper subgroup is abelian. Is G then abelian?*

Exercise 53.10 *Find all conjugacy classes in the symmetric group S_4 using cycle notation. You will discover that the conjugacy class depends only on the cycle structure of the permutation. See Rotman, p 47, Theorem 3.5.*

Exercise 53.11

(a) Find an outer automorphism of $\mathrm{Sym}(W)$*.*

(b) Can the outer automorphism be realized by conjugating by an element of $\mathrm{Isom}(\mathbb{R}^3)$*?*

[1] N. D. Mermin, *Hidden variables and the two theorems of John Bell*, Rev. Mod. Phys. 65, 1993; Errata Rev. Mod. Phys. 85, 2013; Rev. Mod. Phys. 88, 2016; Rev. Mod. Phys. 89, 2017, corrected version `https://arxiv.org/abs/1802.10119`.

Exercise 53.12

(a) Show that the half-order subgroups of a finite group G of even order are in one-to-one correspondence with homomorphisms of G onto $\mathbb{Z}_2$.

(b) Show that these homomorphisms (together with the trivial homomorphism) form a commutative group.

(c) What is the structure of this group in the case $G = \mathrm{Sym}(W)$?

Exercise 53.13 *Interpret and contrast the two following "cycle diagrams" for the dihedral group of order 8 and the unit quaternions.*

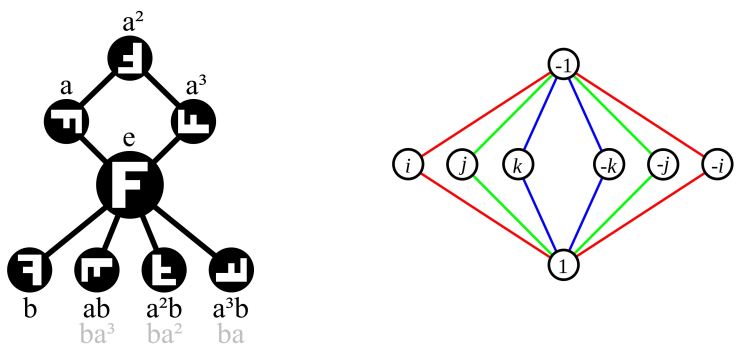


Figure 53.3: D_4 versus Q_8. (Watchduck, Dega180; Wikipedia)

54 Solution to a few exercises

References: None.

Solution to Exercise 33.1: We have

$$(\Phi_c \circ \Phi_d)(a) = \Phi_c \circ (\Phi_d(a)) = c(dad^{-1})c^{-1} = (cd)a(cd)^{-1} = \Phi_{cd}(a), \quad c, d, a \in G,$$

so

$$\Phi_c \circ \Phi_d = \Phi_{cd} \qquad c, d \in G.$$

This shows that $c \mapsto \Phi_c$ is a homomorphism from G to $\mathrm{Aut}(G)$. The image is $\mathrm{Inn}(G)$. QED

Solution to Exercise 35.2:

1. Let Z be the antipodal map $Z : x \mapsto -x$. Z commutes with everything. Using this fact, one verifies that the function f defined by

$$f : \phi \mapsto \begin{cases} \phi & \text{if } \phi \text{ is OE,} \\ Z\,\phi & \text{if } \phi \text{ is OU.} \end{cases}$$

is a homomorphism of $\mathrm{Sym}(W)$ into itself. (We have dropped the $\circ$ notation for composition, using juxtaposition instead.)

Why is this so? f is the product of two homomorphisms with values in $\mathrm{Sym}(W)$, namely $f(\phi) = g(\phi)h(\phi)$ where $g(\phi) = \phi$ is the identity map and

$$h(\phi) := \begin{cases} I & \text{if } \phi \text{ is OE,} \\ Z & \text{if } \phi \text{ is OU.} \end{cases}$$

Normally the product of two homomorphisms is not a homomorphism, but it is if the images commute:

$$\begin{aligned} f(\phi\chi) &= g(\phi\chi)h(\phi\chi) && (54.1) \\ &= g(\phi)g(\chi)h(\phi)h(\chi) && (54.2) \\ &= g(\phi)h(\phi)g(\chi)h(\chi) && (54.3) \\ &= f(\phi)f(\chi). && (54.4) \end{aligned}$$

2. f "corrects" every symmetry of W to an orientation-preserving symmetry. So it is a surjective homomorphism

$$f : \mathrm{Sym}(W) \to \mathrm{Sym}_+(W)$$

with kernel $f^{-1}(I) = \{I, Z\}$. Restricting it to $\mathrm{Sym}(T)$, we get a homomorphism

$$F = f|\mathrm{Sym}(T) : \mathrm{Sym}(T) \to \mathrm{Sym}_+(W).$$

Since Z does NOT lie in $\mathrm{Sym}(T)$, F has kernel $\{I\}$. But then F is injective. Since both groups have 24 elements, F is also surjective. So F is an isomorphism between $\mathrm{Sym}(T)$ and $\mathrm{Sym}_+(W)$. QED

Solution to Exercise **??***:*

1) *Observation:* Let H be a normal subgroup of G. Let G act on M. Then each $g \in G$ permutes the orbits of H. That is, if X is an orbit of H, then $g(X)$ is an orbit of H.

Proof: Let $X = H \cdot x$ be an orbit of H in M. Then

$$g \cdot (H \cdot x) = (gH) \cdot x = (Hg) \cdot x = H \cdot (g \cdot x),$$

which is an orbit of H. QED

In the present situation, $H = G_+ = T$ is a subgroup of index 2 in G. A subgroup of index 2 is always normal. So we get a homomorphism

$$G \to \mathrm{Perm}(\{X, Y\}) \cong \mathbb{Z}_2.$$

So either all elements of $G \setminus G_+$ preserve X and Y, or all elements of $G \setminus G_+$ reverse X and Y, as claimed.

2) σ_D preserves each tetrahedron, so T_d preserves each tetrahedron. σ_H reverses the tetrahedra, so every element of $T_h \setminus T$ reverses the tetrahedra.

QED

Chapter 14

Bibliography

55 Books

The book by Horst Knörrer is recommended:

- H. Knörrer, *Geometrie*, Springer `https://www.springer.com/de/book/9783322939807`, pp 1-60.

For group theory:

- D. Saracino, *Abstract Algebra: A First Course,* Waveland Press, 2008, `https://www.waveland.com/browse.php?t=483`, pp 1-132.

For crystallography:

- M. Senechal, *Crystalline Symmetries, An informal mathematical introduction*, 1990.

Very accessible, a little off-topic:

- E. A. Abott, *Flatland,* Dover Publications.
- J. R. Weeks, *The Shape of Space,* CRC press, 2019.

Background reading:

- A. F. Beardon, *The Geometry of Discrete Groups*, 1983. (Hyperbolic geometry)
- E. Brieskorn, *Lineare Algebra und Analytische Geometrie I*, 1983. (Brieskorn I)
- E. Brieskorn, *Lineare Algebra und Analytische Geometrie III*, 2018. (Brieskorn III)
- G. Burns, A. M. Glazer, *Space Groups for Solid State Scientists,* Third Edition, Academic Press, 2013.
- H. S. M. Coxeter, *Regular Polytopes,* Third Edition, Dover Publications, 1973.
- R. J. Haüy, *Traité de Crystallographie*, 1822.
- I. Lakatos, *Proofs and Refutations,* 1976.
- R. Pink, *Geometrie.* (Online at ETH. Chapters 2 and 3 treat geometry axiomatically, complementary to this lecture.)
- W. P. Thurston, *Three-dimensional Geometry and Topology,* Princeton University Press, 1997. (Pages 221-231 for the Euclidean group.)

56 Articles, blogs, and references

Articles:

- C. Arrouvela, J-G. Eon, Understanding the Surfaces and Crystal Growth of Pyrite FeS_2, `https://www.scielo.br/scielo.php?pid=S1516-14392019000100215&script=sci_arttext`
- Mark Hannah, *Classifying Finite Subgroups of SO(3)*, 2011, `http://www.math.uchicago.edu/~may/VIGRE/VIGRE2011/REUPapers/MarkH.pdf`
- S. Zunic, *The Cubic Groups*, BA thesis, Vienna Univ. Technology, 2010.

Blogs:

- J. Baez, *Platonic Solids in All Dimensions*, `http://math.ucr.edu/home/baez/platonic.html`.
- J. Baez, `https://johncarlosbaez.wordpress.com/2012/05/21/symmetry-and-the-fourth-dimension-par`
- J. Baez, `https://johncarlosbaez.wordpress.com/2012/05/27/symmetry-and-the-fourth-dimension-par`
- J. Baez, `https://johncarlosbaez.wordpress.com/2012/07/22/symmetry-and-the-fourth-dimension-par`
- J. Baez, `https://johncarlosbaez.wordpress.com/2012/07/26/symmetry-and-the-fourth-dimension-par`
- J. Baez, `https://johncarlosbaez.wordpress.com/2017/12/16/the-600-cell`
- J. Baez, `https://math.ucr.edu/home/baez/tcu/5_tcu.pdf`
- S. Dutch, Symmetry, Crystals and Polyhedra, `https://stevedutch.net/symmetry/symmetry.htm`
- S. Dutch, Miller indices, `https://stevedutch.net/symmetry/millerdx.htm` (contains an error)
- G. Hart, `http://www.georgehart.com/virtual-polyhedra/symmetry_axes.html` clean list of the number of axes, planes etc.
- B. Johnson, *Classifying regular polytopes in dimension 4 and beyond*, `https://digitalcommons.wou.edu/cgi/viewcontent.cgi?article=1116&context=aes`.
- B. Morgan, many Peruvian pyrites, `https://www.rasny.org/pyrite`.
- R. Van Dommelen, Nova Scotia pyrite blog, very readable, `http://nsminerals.atspace.com/pyrite.html`.
- *Wulff shape of crystals*, Scholarpedia, `http://www.scholarpedia.org/article/Wulff_shape_of_crystals`.
- L. Zefiro, `http://www.mi.sanu.ac.rs/vismath/zefiro/_polyhedra_colouring_2007_08_10old.html`, great pictures.

Wikipedia:

- *16-cell*
- *24-cell*
- *120-cell*
- *600-cell*
- *Chasles' theorem*
- *Euclidean group*
- *Fluorite structure*
- *Frieze group*

- *Group action*
- *Miller indices*
- *Octahedral symmetry*
- *Point group*
- *Point groups in three dimensions*
- *Pyrite*
- *Regular polyhedron*
- *Regular polytope*
- *Schoenflies notation*
- *Tesseract*
- *Vertex figure*

Names and properties of concrete groups:

- T. Dokchitser, interactive list of groups of small order, `https://people.maths.bris.ac.uk/~matyd/GroupNames/`
- J. Jones, interactive calculator for groups of small order, `https://hobbes.la.asu.edu/groups/groups.html`.
- X. Lee, wallpaper groups, `http://xahlee.info/Wallpaper_dir/c5_17WallpaperGroups.html`

Mathematical symbols:

- *Liste mathematischer Symbole,* `https://de.wikipedia.org/wiki/Liste_mathematischer_Symbole`

Mathematical dictionaries:

- G. Eisenreich, R. Sube, *Dictionary of Mathematics; Wörterbuch Mathematik*, Verlag Harry Deutsch, 1987.
- `https://archive.org/details/germanenglishmat00hyma` (unpleasant to use)

57 Software and visualization

Jeff Weeks geometry apps:

1) Crystal flight (iOS, macOS):
`http://www.geometrygames.org/CrystalFlight/index.html`

2) Kaleidotile (iOS, macOS, Windows):
`http://www.geometrygames.org/KaleidoTile/index.html`

3) Flying in curved space (iOS, macOS, Windows):
`http://www.geometrygames.org/CurvedSpaces/index.html`

4) Kaleidopaint (iOS):
`http://www.geometrygames.org/KaleidoPaint/index.html`

Greg Egan geometry apps (largely unexplored):

1) `https://www.gregegan.net/APPLETS/Applets.html`

International Tables of Crystallography (6250 pp, accessible online through the ETH library):

1) Home: `https://it.iucr.org/`

2) Tables: `https://symmdb.iucr.org/`

3) Point groups: `https://symmdb.iucr.org/point_groups/groups/select?w=vis`

More stuff. Please let me know which ones are good:

1) Various curvature tilings (web app):
`http://timhutton.github.io/hyperplay/`. Very simple. See Section 8.

2) Islamic tilings (Java app): `https://sourceforge.net/projects/taprats/`.
Girih patterns. Did not try.

3) Islamic tilings (Mac):
`https://itunes.apple.com/us/app/girih-polygon-pattern-design/id1400485589`.
Girih patterns. Works.

4) Islamic tilings (web app): `http://girihdesigner.com`. Girih patterns. Works.

5) Penrose tiling generator written in postscript, `http://www.math.ubc.ca/~cass/courses/m308-02b/projects/schweber/penrose.html`, `http://www.math.ubc.ca/~cass/courses/m308-02b/projects/schweber/penrose.ps`. Remarkably, you can reprogram the postscript by changing the variable N to make the tiling coarser or finer.

More advanced stuff (have not tried):

1) Java app for Penrose tiling: `https://sourceforge.net/projects/rhombi/`.

2) Java app for non-periodic tilings: `https://sourceforge.net/projects/quasitiler/`.

3) Java app for tilings: `https://sourceforge.net/projects/tilefarm/`.

4) Professional-level app for real crystals (free for students through ETH IT Shop, `https://itshop.ethz.ch/`, `http://crystalmaker.com`.

List of Figures

Figure credits

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Octahedron, Dodecahedron, Icosahedron. Cyp, Wikipedia, license https://creativecommons.org/licenses/by-sa/3.0.

NaCl crystal. Benjah-bmm27, Wikipedia, public domain.

Fluorite crystal. https://gotbooks.miracosta.edu/earth_science/images/fluorite_cube.jpg, https://geologycafe.com/class/chapter3.html.

Fluorite unit cell. ShutterWaves, Wikipedia, https://commons.wikimedia.org/wiki/File:Fluorite_Structure.jpg.

Diamond. strpeter,
https://tex.stackexchange.com/questions/141363/draw-realistic-3d-crystal-structures-di

Penrose tiling. Preshing on programming, 2011, https://preshing.com/20110822/penrose-tiling-in-obfuscated-python, https://preshing.com/20110831/penrose-tiling-explained.

Silicon crystal in a nanowire. D. Li, Y. Wu, P. Kim, L. Shi, P. Yang, A. Majumdara, *Thermal conductivity of individual silicon nanowires*, Applied Physics Letters, 2003.

Tetrahedron. Aldoaldoz, Wikipedia, license https://creativecommons.org/licenses/by-sa/3.0/deed.en.

Octahedron, Dodecahedron, Icosahedron. Cyp, Wikipedia, license https://creativecommons.org/licenses/by-sa/3.0.

The tetrahedron and its “opposite”. Watchduck, Wikipedia, license https://creativecommons.org/licenses/by/4.0/deed.en.

The tetrahedron in the dodecahedron. G. Egan, http://www.gregegan.net/SCIENCE/Origami/Origami.html.

Five tetrahedra inscribed in the dodecahedron. R. Webb, Wikipedia, created with Stella software, http://www.software3d.com/Stella.php.

Five cubes in the dodecahedron. S. Tatham, http://www.chiark.greenend.org.uk.

Hexagonal bipyramid. Quatrostein, Wikipedia, license https://commons.wikimedia.org/wiki/Commons:GNU_Free_Documentation_License,_version_1.2.

Jessen’s icosahedron. D. Eppstein, Wikipedia, public domain.

Hexagonal torus. T. Ruen, Wikipedia, public domain.

Klein bottle. Tttrung, Wikipedia, license https://creativecommons.org/licenses/by-sa/4.0/deed.en.

Why isn’t the small stellated dodecahedron a regular polyhedron? William & Robert Chambers Encyclopaedia - A Dictionary of Universal Knowledge for the People,, J. B. Lippincott & Co., Philadelphia, PA, 1881, via University of

Figures not otherwise attributed in the figure captions are due to T. Ilmanen and/or A. Veneluth.